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Matrix Representation of Recursive Sequences of Order 3 and Its Applications

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Abstract Here presented is a matrix representation of recursive number sequences of order 3 defined by $a_n = pa_{n-1} + qa_{n-2} + ra_{n-3}$ with arbitrary initial conditions a_0, $a_1 = 0$, and a_2 and their special cases of Padovan number sequence and Perrin number sequence with initial conditions $a_0 = a_1 = 0$ and $a_2 = 1$ and $a_0 = 3$, $a_1 = 0$, and $a_2 = 2$, respectively. The matrix representation is used to construct many well known and new identities of recursive number sequences as well as Pavodan and Perrin sequences.

Keywords recursive number sequence of order 3; matrix representation of recursive number sequences; Padovan number sequence; Perrin number sequence; Tribonacci polynomial sequence

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1. Introduction

The matrices associated with recursive sequences and their properties always attract the attentions of the people working in fields of combinatorics, linear algebra, approximation theory, and numerical analysis. In this paper, we focus our attention on those matrices related recursive sequences of order 3 defined by

$$a_n = pa_{n-1} + qa_{n-2} + ra_{n-3} \tag{1.1}$$

for $n \geq 3$ with some initial conditions a_0, a_1, and a_2. The admissible matrix associated with sequences was defined and studied by Aigner in [1]. The matrices associated with some well-known recursive sequences of order 3 with $p = 0$ and $q = r = 1$, such as Padovan sequence, Perrin sequence, and Van der Laan sequence, are discussed by Shannon, Anderson, and Horadam [2], Sokhuma [3], Stewart [4], Yilmaz and Bozkurt [5,6]. The matrices associated with recursive polynomial sequences can be found from Chen and Louck [7], Hoggatt, Jr., and Bicknel [8], etc.

Padovan number sequence $\{P_n\}$ defined by $P_n = P_{n-2}+P_{n-3}$ $(n \geq 3)$ with initial conditions $P_0 = P_1 = P_2 = 1$ was introduced by Dutch architect Hans var der Laan. Architect Richard Padovan attributed the discovery of the sequence to Hans var der Laan in a 1994 essay, and

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used the sequence in design. In 1996 Ian Stewart described the sequence as the plastic number sequence [4] because of a genesis similar to the golden ratio. Unlike the golden ratio, the plastic ratio does not seem to have interesting manifestations in nature. However, the sequence and its related materials have more and more connections with other mathematics. For instance, the sequence with the same recursion but initial conditions $\hat{P}_0 = 3, \hat{P}_1 = 0$, and $\hat{P}_2 = 2$, called the Perrin number sequence, has an interesting property noticed by Edouard Lucas in 1876: If n is a prime, n divides $\hat{P}_n$. This result provides a speedy test (in $\log n$ steps) for nonprimality.

In this paper we discuss a generalized recursive number and polynomial sequences of order 3 including the Padovan sequence, Perrin sequence, etc. as special cases. In next section, we shall give matrices associated with recursive number and polynomial sequences. The special cases of those matrices associated with Padovan sequence, Perrin sequence, Tribonacci sequence, and Tribonacci polynomial sequence will also be given. In Section 3, the matrices obtained in the previous section will be used to derive some new and well known identities for the recursive number and polynomial sequences.

2. Matrices associated with recursive number and polynomial sequences

We first give a matrix representation of a recursive number sequence of order 3.

Theorem 2.1 *Let $a_n = pa_{n-1} + qa_{n-2} + ra_{n-3}$, $a_0 = a_1 = 0$, $a_2 = a \neq 0$, and let*

$$\phi = \begin{bmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.1}$$

Then we have the following matrix representation of $\{a_n\}$:

$$\phi^n = \frac{1}{a}\begin{bmatrix} a_{n+2} & qa_{n+1} + ra_n & ra_{n+1} \\ a_{n+1} & qa_n + ra_{n-1} & ra_n \\ a_n & qa_{n-1} + ra_{n-2} & ra_{n-1} \end{bmatrix}. \tag{2.2}$$

Proof We prove (2.2) by using mathematical induction. First, for $n = 2$,

$$\phi^2 = \begin{bmatrix} p^2 + q & pq + r & pr \\ p & q & r \\ 1 & 0 & 0 \end{bmatrix} = \frac{1}{a}\begin{bmatrix} a_4 & qa_3 + ra & ra_3 \\ a_3 & qa & ra \\ a & 0 & 0 \end{bmatrix}.$$

Assume (2.2) holds for $n = k$. Thus

$$\begin{aligned} \phi^{k+1} = \phi^k\phi &= \frac{1}{a}\begin{bmatrix} a_{k+2} & qa_{k+1} + ra_k & ra_{k+1} \\ a_{k+1} & qa_k + ra_{k-1} & ra_k \\ a_k & qa_{k-1} + ra_{k-2} & ra_{k-1} \end{bmatrix}\begin{bmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &= \frac{1}{a}\begin{bmatrix} pa_{k+2} + qa_{k+1} + ra_k & qa_{k+2} + ra_{k+1} & ra_{k+2} \\ pa_{k+1} + qa_k + ra_{k-1} & qa_{k+1} + ra_k & ra_{k+1} \\ pa_k + qa_{k-1} + ra_{k-2} & qa_k + ra_{k-1} & ra_k \end{bmatrix} \\ &= \frac{1}{a}\begin{bmatrix} a_{k+3} & qa_{k+2} + ra_{k+1} & ra_{k+2} \\ a_{k+2} & qa_{k+1} + ra_k & ra_{k+1} \\ a_{k+1} & qa_k + ra_{k-1} & ra_k \end{bmatrix}. \end{aligned}$$

This completes the proof of theorem. □

Corollary 2.2 ([6]) *Let $\{P_n\}$ be the Padovan sequence, i.e., $P_n = P_{n-2} + P_{n-3}$ $(n \geq 3)$ with initial conditions $P_0 = P_1 = 0$ and $P_2 = 1$, and let*

$$\phi = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.3}$$

Then by substituting $p = 0$ and $q = r = 1$ into (2.2), we have

$$\phi^n = \begin{bmatrix} P_{n+2} & P_{n+3} & P_{n+1} \\ P_{n+1} & P_{n+2} & P_n \\ P_n & P_{n+1} & P_{n-1} \end{bmatrix}. \tag{2.4}$$

Remark 2.3 Formula (2.4) was derived in Yilmaz and Bozkurt [6] by using an approach associated with Hessenberg matrices shown in [5]. A similar result can be derived from the cordonnier number sequence defined in [2] by $P_n = P_{n-2} + P_{n-3}$ $(n > 3)$ with initial conditions $P_1 = P_2 = P_3 = 1$. Another similar result can be found from the third-order Pell number sequence studied in Shannon and Wong in [9]: $t_{m,n} = 2^m t_{m,n-2} + t_{m,n-3}$, which is a special case of Theorem 2.1 with $p = 0$, $q = 2^m$, and $r = 1$. Particularly, when $m = 1$, the corresponding sequence is the third-order Fibonacci sequence.

Corollary 2.4 *Let $\{T_n\}$ be the Tribonacci sequence, i.e., $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ $(n \geq 3)$ with initial conditions $T_0 = T_1 = 0$ and $T_2 = 1$, and let*

$$\phi = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.5}$$

Then by substituting $p = q = r = 1$ into (2.2), we have

$$\phi^n = \begin{bmatrix} T_{n+2} & T_{n+1} + T_n & T_{n+1} \\ T_{n+1} & T_n + T_{n-1} & T_n \\ T_n & T_{n-1} + T_{n-2} & T_{n-1} \end{bmatrix}. \tag{2.6}$$

We may establish a result similar to Theorem 2.1 for the recursive polynomial sequence of order 3.

Theorem 2.5 *Let $a_n(x) = px^2 a_{n-1}(x) + qx a_{n-2}(x) + r a_{n-3}(x)$, $a_0(x) = a_1(x) = 0$, $a_2(x) = a \neq 0$, and let*

$$\phi(x) = \begin{bmatrix} px^2 & qx & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.7}$$

Then we have the following matrix representation of $\{a_n(x)\}$:

$$\phi^n(x) = \frac{1}{a} \begin{bmatrix} a_{n+2}(x) & qx a_{n+1}(x) + r a_n(x) & r a_{n+1}(x) \\ a_{n+1}(x) & qx a_n(x) + r a_{n-1}(x) & r a_n(x) \\ a_n(x) & qx a_{n-1}(x) + r a_{n-2}(x) & r a_{n-1}(x) \end{bmatrix}. \tag{2.8}$$

Corollary 2.6 *Let $\{T_n(x)\}$ be the Tribonacci polynomial sequence [6], i.e., $T_n = x^2 T_{n-1} +$*

$xT_{n-2}+T_{n-3}$ $(n\geq 3)$ *with initial conditions* $T_0=T_1=0$ *and* $T_2=1$, *and let*

$$\phi(x)=\begin{bmatrix} x^2 & x & 1\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}. \tag{2.9}$$

Then by substituting $p=q=r=1$ *into (2.8) we have*

$$\phi^n(x)=\begin{bmatrix} T_{n+2}(x) & xT_{n+1}(x)+T_n(x) & T_{n+1}(x)\\ T_{n+1}(x) & xT_n(x)+T_{n-1}(x) & T_n(x)\\ T_n(x) & xT_{n-1}(x)+T_{n-2}(x) & T_{n-1}(x)\end{bmatrix}. \tag{2.10}$$

Theorem 2.7 *Let* $a_n=pa_{n-1}+qa_{n-2}+ra_{n-3}$ *with arbitrary initial conditions* a_0, a_1, *and* a_2, *and let*

$$\phi=\begin{bmatrix} p & q & r\\ 1 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}. \tag{2.11}$$

Then we have the following matrix representation of $\{a_n\}$*:*

$$\begin{bmatrix} a_{n+2}\\ a_{n+1}\\ a_n\end{bmatrix}=\phi^n\begin{bmatrix} a_2\\ a_1\\ a_0\end{bmatrix}. \tag{2.12}$$

Proof For $n=1$,

$$\phi\begin{bmatrix} a_2\\ a_1\\ a_0\end{bmatrix}=\begin{bmatrix} pa_2+qa_1+ra_0\\ a_2\\ a_1\end{bmatrix}=\begin{bmatrix} a_3\\ a_2\\ a_1\end{bmatrix}.$$

From the induction assumption for $n=k$

$$\phi^k\begin{bmatrix} a_2\\ a_1\\ a_0\end{bmatrix}=\begin{bmatrix} a_{k+2}\\ a_{k+1}\\ a_k\end{bmatrix},$$

there hold

$$\phi^{k+1}\begin{bmatrix} a_2\\ a_1\\ a_0\end{bmatrix}=\phi\begin{bmatrix} a_{k+2}\\ a_{k+1}\\ a_k\end{bmatrix}=\begin{bmatrix} pa_{k+2}+qa_{k+1}+ra_k\\ a_{k+2}\\ a_{k+1}\end{bmatrix}=\begin{bmatrix} a_{k+3}\\ a_{k+2}\\ a_{k+1}\end{bmatrix}.$$

The result is proved by using mathematical induction. □

Corollary 2.8 *Let* $a_n=pa_{n-1}+qa_{n-2}+ra_{n-3}$ *with initial conditions* a_0, a_1, *and* a_2, *and let* $b_n=pb_{n-1}+qb_{n-2}+rb_{n-3}$ *with initial conditions* $a_0=a_1=0$, $a_2=b\neq 0$. *Then we have*

$$ba_n=b_ma_{n-m+2}+(qb_{m-1}+rb_{m-2})a_{n-m+1}+rb_{m-1}a_{n-m}. \tag{2.13}$$

Proof Let ϕ be defined by (2.11). Then from (2.2)

$$\begin{aligned}\begin{bmatrix} a_{n+2}\\ a_{n+1}\\ a_n\end{bmatrix}&=\phi^n\begin{bmatrix} a_2\\ a_1\\ a_0\end{bmatrix}=\phi^m\phi^{n-m}\begin{bmatrix} a_2\\ a_1\\ a_0\end{bmatrix}\\ &=\frac{1}{b}\begin{bmatrix} b_{m+2} & qb_{m+1}+rb_m & rb_{m+1}\\ b_{m+1} & qb_m+rb_{m-1} & rb_m\\ b_m & qb_{m-1}+rb_{m-2} & rb_{m-1}\end{bmatrix}\begin{bmatrix} a_{n-m+2}\\ a_{n-m+1}\\ a_{n-m}\end{bmatrix},\end{aligned}$$

which implies (2.13). □

Corollary 2.9 *Let sequences $\{a_n\}$ and $\{b_n\}$ be defined as Corollary 2.8 with $p = 0$ and $q = r = b = 1$. Then from (2.4),*

$$a_n = P_m a_{n-m+2} + P_{m+1} a_{n-m+1} + P_{m-1} a_{n-m}, \tag{2.14}$$

where P_n are Padovan numbers.

If $a_0 = a_1 = 0$ and $a_2 = 1$ and $a_0 = 3$, $a_1 = 0$, and $a_2 = 2$, then we have

$$P_n = P_m P_{n-m+2} + P_{m+1} P_{n-m+1} + P_{m-1} P_{n-m},$$
$$\hat{P}_n = P_m \hat{P}_{n-m+2} + P_{m+1} \hat{P}_{n-m+1} + P_{m-1} \hat{P}_{n-m},$$

that is, [3, Proposition 2.2]. Here $\hat{P}_n$ are Perrin numbers.

Similarly we may establish a polynomial sequence analogy of Theorem 2.7.

Theorem 2.10 *Let $a_n(x) = px^2 a_{n-1}(x) + qxa_{n-2}(x) + ra_{n-3}(x)$, with initial conditions $a_0(x)$, $a_1(x)$, and $a_2(x)$, and let*

$$\phi(x) = \begin{bmatrix} px^2 & qx & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{2.15}$$

Then we have the following matrix representation of $\{a_n(x)\}$:

$$\begin{bmatrix} a_{n+2}(x) \\ a_{n+1}(x) \\ a_n(x) \end{bmatrix} = \phi^n(x) \begin{bmatrix} a_2(x) \\ a_1(x) \\ a_0(x) \end{bmatrix}. \tag{2.16}$$

Similar to Corollary 2.8, there holds

Corollary 2.11 *Let $a_n(x) = px^2 a_{n-1} + qxa_{n-2}(x) + ra_{n-3}(x)$ with initial conditions $a_0(x)$, $a_1(x)$, and $a_2(x)$, and let $b_n(x) = px^2 b_{n-1}(x) + qxb_{n-2}(x) + rb_{n-3}(x)$ with initial conditions $b_0(x) = b_1(x) = 0$, $b_2(x) = b \neq 0$. Then we have*

$$ba_n(x) = b_m(x) a_{n-m+2}(x) + (qxb_{m-1}(x) + rb_{m-2}(x)) a_{n-m+1}(x) + rb_{m-1}(x) a_{n-m}(x). \tag{2.17}$$

Particularly, if $p = q = r = 1$, $a_0(x) = b_0(x) = a_1(x) = b_1(x) = 0$ and $a_2(x) = b_2(x) = 1$, then

$$T_n(x) = T_m(x) T_{n-m+2}(x) + (xT_{m-1}(x) + T_{m-2}(x)) T_{n-m+1}(x) + T_{m-1}(x) T_{n-m}(x). \tag{2.18}$$

Theorem 2.12 *Let $\{a_n\}$ and ϕ^n $(n \geq 1)$ be defined as in Theorem 2.1, and let $r := |\phi|$ be the determinant of ϕ. Then*

$$\begin{vmatrix} a_n & a_{n+1} & a_{n+2} \\ a_{n-1} & a_n & a_{n+1} \\ a_{n-2} & a_{n-1} & a_n \end{vmatrix} = r^{n-2} a \tag{2.19}$$

for all $n \geq 2$.

Proof From the definitions shown above, we have

$$r^n = |\phi^n| = \frac{1}{a} \begin{vmatrix} a_{n+2} & qa_{n+1} + ra_n & ra_{n+1} \\ a_{n+1} & qa_n + ra_{n-1} & ra_n \\ a_n & qa_{n-1} + ra_{n-2} & ra_{n-1} \end{vmatrix}$$

$$= \frac{1}{a}\begin{vmatrix} a_{n+2} & ra_n & ra_{n+1} \\ a_{n+1} & ra_{n-1} & ra_n \\ a_n & ra_{n-2} & ra_{n-1} \end{vmatrix} = \frac{r^2}{a}\begin{vmatrix} a_n & a_{n+1} & a_{n+2} \\ a_{n-1} & a_n & a_{n+1} \\ a_{n-2} & a_{n-1} & a_n \end{vmatrix},$$

which implies (2.19). □

Corollary 2.13 *Let $\{P_n\}$ be the Padovan sequence. Then*

$$\begin{vmatrix} P_n & P_{n+1} & P_{n+2} \\ P_{n-1} & P_n & P_{n+1} \\ P_{n-2} & P_{n-1} & P_n \end{vmatrix} = 1$$

for all $n \geq 2$.

Denote by $\{T_n(x)\}$ the Tribonacci polynomial sequence. Then

$$\begin{vmatrix} T_n(x) & T_{n+1}(x) & T_{n+2}(x) \\ T_{n-1}(x) & T_n(x) & T_{n+1}(x) \\ T_{n-2}(x) & T_{n-1}(x) & T_n(x) \end{vmatrix} = 1$$

for $n \geq 2$. Particularly, for $x = 1$ we notice that $T_n(1) = T_n$, the Tribonacci numbers, satisfy

$$\begin{vmatrix} T_n & T_{n+1} & T_{n+2} \\ T_{n-1} & T_n & T_{n+1} \\ T_{n-2} & T_{n-1} & T_n \end{vmatrix} = 1$$

for $n \geq 2$.

Proof By noting $r = |\phi| = 1$ and $a = 1$, we immediately obtain the corollary from Theorem 2.12. □

Remark 2.14 Similar results can be derived from the Perrin polynomial sequence $\{Q_n(x)\}$: $Q_n(x) = x^2Q_{n-2}(x) + Q_{n-3}(x)$ ($n > 3$) with initial conditions $Q_1(x) = 0$, $Q_2(x) = 2$, and $Q_3(x) = 3x$, and the cordonnier polynomial sequence $\{P_n(x)\}$: $P_n(x) = x^2P_{n-2}(x) + P_{n-3}(x)$ ($n > 3$) with initial conditions $P_1(x) = 1$, $P_2(x) = x$, and $P_3(x) = x^2$, studied in [2]. $\{P_n(1)\}$ is the cordonnier number sequence (see Remark 2.3).

3. Applications of the matrices

The characteristic polynomial of the recursive relation (1.1) with $p = 0$, $q = r = 1$ is $p(x) = x^3 - x - 1$, which can be written as

$$p(x) = \det(xI - \phi),$$

where

$$\phi = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{3.1}$$

From the Cayley-Hamilton Theorem, ϕ satisfies $p(\phi) = 0$, i.e.,

$$\phi^3 - \phi - I = 0. \tag{3.2}$$

Hence, we have

Proposition 3.1 *Let ϕ be defined as (3.1). Then*

$$I = \phi^3 - \phi = \phi^5 - \phi^4, \tag{3.3}$$

or equivalently

$$\phi^n = \phi^{n+3} - \phi^{n+1} = \phi^{n+5} - \phi^{n+4}. \tag{3.4}$$

Furthermore, (3.4) implies

$$\left(\phi^{n+5} - \phi^{n+4}\right) - \left(\phi^{n+3} - \phi^{n+2}\right) = \phi^{n+2} - \phi^{n+1}. \tag{3.5}$$

Particularly, for $n = 3$ we have

$$\phi^8 - \phi^7 - \phi^6 + \phi^5 = \phi^5 - \phi^4 = I. \tag{3.6}$$

Proof From (3.2), we have the first equation of (3.3) and

$$I = \phi^3 - \phi = (\phi + I)(\phi^2 - \phi) = \phi^3(\phi^2 - \phi),$$

which implies the second equation of (3.3). Equations (3.4) follow. From the second equation of (3.4), there holds

$$(\phi^{n+5} - \phi^{n+4}) - (\phi^{n+3} - \phi^{n+2}) = (\phi^{n+3} - \phi^{n+1}) - (\phi^{n+3} - \phi^{n+2}),$$

which yields (3.5). The special case follows. □

The second equation of (3.4) gives immediately the well known identity of Padovan numbers

$$P_{n+5} = P_{n+4} + P_n. \tag{3.7}$$

More results of the Padovan number identities can be derived from the first equation and the second equation of (3.3), which are presented in the following two propositions, respectively.

Proposition 3.2 *Let ϕ be defined by (3.1). Then for the Padovan sequence $\{P_n\}$ we have*

$$P_m = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} P_{n+2k+m}, \tag{3.8}$$

$$P_{(\ell+3)n+m} = \sum_{k=0}^{n} \binom{n}{k} P_{\ell n+k+m}, \tag{3.9}$$

$$P_{2n+m+3} - P_{m+1} = \sum_{k=0}^{n} P_{2k+m}, \tag{3.10}$$

$$P_{3n+m+2} - P_{m-1} = \sum_{k=0}^{n} P_{3k+m}. \tag{3.11}$$

Particularly,

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} P_{n+2k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} P_{n+2k+1} = 0,$$

$$P_{(\ell+3)n} = \sum_{k=0}^{n} \binom{n}{k} P_{\ell n+k}, \quad P_{(\ell+3)n+1} = \sum_{k=0}^{n} \binom{n}{k} P_{\ell n+k+1},$$

$$P_{2n+3}=\sum_{k=0}^{n}P_{2k},\ P_{2n+4}=1+\sum_{k=0}^{n}P_{2k+1},\ P_{3n+3}=\sum_{k=0}^{n}P_{3k+1}.$$

Proof From the first equation of (3.3),

$$\phi^m=\phi^m I^n=\phi^{n+m}(\phi^2-I)^n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\phi^{n+2k+m},$$

which implies (3.8) by comparing the entries at the position of the third row and the first column of the matrices on the left-hand and right-hand sides of the above equation. Following the first equation of (3.3), we may have $\phi^{\ell+3}=\phi^{\ell}(\phi+I)$. Thus

$$\phi^{(\ell+3)n}=\phi^{\ell n}(\phi+I)^n=\sum_{k=0}^{n}\phi^{\ell n+k},$$

which implies (3.9). Since the first equation of (3.3) gives $(\phi^2-I)^{-1}=\phi$, we have

$$\phi(\phi^{2n+2}-I)=\frac{\phi^{2n+2}-I}{\phi^2-I}=\sum_{k=0}^{n}\phi^{2k}.$$

Therefore,

$$\phi^{2n+m+3}-\phi^{m+1}=\sum_{k=0}^{n}\phi^{2k+m},$$

which leads to (3.10). Similarly, (3.3) also gives $\phi^{-1}=(\phi^3-I)^{-1}$. Thus

$$\phi^{-1}(\phi^{3(n+1)}-\phi)=\frac{\phi^{3(n+1)}-I}{\phi^3-I}=\sum_{k=0}^{n}\phi^{3k},$$

which is equivalent to (3.11). The special cases of (3.8)–(3.11) for $m=0$ and/or $m=1$ immediately follow. This completes the proof of the theorem. □

Similar to Proposition 3.2, we may prove the following results from the second equation of (3.3).

Proposition 3.3 *Let ϕ be defined by (3.1). Then for the Padovan sequence $\{P_n\}$. Then*

$$P_m=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P_{4n+k+m}, \tag{3.12}$$

$$P_{(\ell+5)n+m}=\sum_{k=0}^{n}\binom{n}{k}P_{\ell n+4k+m}, \tag{3.13}$$

$$P_{5n+m+1}-P_{m-4}=\sum_{k=0}^{n}P_{5k+m}, \tag{3.14}$$

$$P_{n+m+5}-P_{m+4}=\sum_{k=0}^{n}P_{k+m}. \tag{3.15}$$

Particularly,

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P_{4n+k}=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}P_{4n+k+1}=0,$$

$$P_{(\ell+5)n} = \sum_{k=0}^{n} \binom{n}{k} P_{\ell n+4k}, \quad P_{(\ell+5)n+1} = \sum_{k=0}^{n} \binom{n}{k} P_{\ell n+4k+1},$$

$$P_{5n+5} = \sum_{k=0}^{n} P_{5k+4}, \quad P_{5n+6} = \sum_{k=0}^{n} P_{5k+5},$$

$$P_{5n+7} = 1 + \sum_{k=0}^{n} P_{2k+6}, \quad P_{n+5} = 1 + \sum_{k=0}^{n} P_k.$$

We now extend all of the identities shown in Propositions 3.2 and 3.3 to other recursive sequences including the Perrin number sequence.

Proposition 3.4 *Let ϕ be defined as (3.1), and let sequence $\{a_n\}$ be defined by $a_n = a_{n-2}+a_{n-3}$ with arbitrary initial conditions a_0, a_1, and a_2. Then*

$$\phi^n \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = (\phi^{n+3} - \phi^{n+1}) \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix} = (\phi^{n+5} - \phi^{n+4}) \begin{bmatrix} a_2 \\ a_1 \\ a_0 \end{bmatrix}, \tag{3.16}$$

or equivalently,

$$a_n = a_{n+3} - a_{n+1} = a_{n+5} - a_{n+4}. \tag{3.17}$$

Particularly, for Perrin numbers $\hat{P}_n$

$$\hat{P}_{n+5} = \hat{P}_{n+4} + \hat{P}_n. \tag{3.18}$$

Proof Multiplying all sides of (3.4) by $(a_2, a_1, a_0)^T$, we may obtain (3.14). Comparing the entries on both sides of (3.14) leads to (3.15). Taking $a_0 = 3$, $a_1 = 0$, and $a_2 = 2$, we have (3.16), a special case of (3.15). □

Similar to Proposition 3.2, we can establish

Proposition 3.5 *Let ϕ be defined by (3.1), and let sequence $\{a_n\}$ be defined by $a_n = a_{n-2}+a_{n-3}$ with arbitrary initial conditions a_0, a_1, and a_2. Then we have*

$$a_m = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_{n+2k+m}, \tag{3.19}$$

$$a_{(\ell+3)n+m} = \sum_{k=0}^{n} \binom{n}{k} a_{\ell n+k+m}, \tag{3.20}$$

$$a_{2n+m+3} - a_{m+1} = \sum_{k=0}^{n} a_{2k+m}, \tag{3.21}$$

$$a_{3n+m+2} - a_{m-1} = \sum_{k=0}^{n} a_{3k+m}. \tag{3.22}$$

Particularly,

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_{n+2k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_{n+2k+1} = 0,$$

$$a_{(\ell+3)n} = \sum_{k=0}^{n} \binom{n}{k} a_{\ell n+k}, \quad a_{(\ell+3)n+1} = \sum_{k=0}^{n} \binom{n}{k} a_{\ell n+k+1},$$

$$a_{2n+3} = \sum_{k=0}^{n} a_{2k}, \; a_{2n+4} = 1 + \sum_{k=0}^{n} a_{2k+1}, \; a_{3n+3} = \sum_{k=0}^{n} a_{3k+1}.$$

Furthermore, for the Perrin sequence $\{\hat{P}_n\}$ we have

$$\hat{P}_m = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \hat{P}_{n+2k+m}, \tag{3.23}$$

$$\hat{P}_{(\ell+3)n+m} = \sum_{k=0}^{n} \binom{n}{k} \hat{P}_{\ell n+k+m}, \tag{3.24}$$

$$\hat{P}_{2n+m+3} - \hat{P}_{m+1} = \sum_{k=0}^{n} \hat{P}_{2k+m}, \tag{3.25}$$

$$\hat{P}_{3n+m+2} - \hat{P}_{m-1} = \sum_{k=0}^{n} \hat{P}_{3k+m}. \tag{3.26}$$

Particularly,

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \hat{P}_{n+2k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \hat{P}_{n+2k+1} = 0,$$

$$\hat{P}_{(\ell+3)n} = \sum_{k=0}^{n} \binom{n}{k} \hat{P}_{\ell n+k}, \quad \hat{P}_{(\ell+3)n+1} = \sum_{k=0}^{n} \binom{n}{k} \hat{P}_{\ell n+k+1},$$

$$\hat{P}_{2n+3} = \sum_{k=0}^{n} \hat{P}_{2k}, \quad \hat{P}_{2n+4} = 1 + \sum_{k=0}^{n} \hat{P}_{2k+1},$$

$$\hat{P}_{3n+3} = \sum_{k=0}^{n} \hat{P}_{3k+1}.$$

Proposition 3.6 *Let ϕ be defined by (3.1), and let sequence $\{a_n\}$ be defined by $a_n = a_{n-2}+a_{n-3}$ with arbitrary initial conditions a_0, a_1, and a_2. Then we have*

$$a_m = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_{4n+k+m}, \tag{3.27}$$

$$a_{(\ell+5)n+m} = \sum_{k=0}^{n} \binom{n}{k} a_{\ell n+4k+m}, \tag{3.28}$$

$$a_{5n+m+1} - a_{m-4} = \sum_{k=0}^{n} a_{5k+m}, \tag{3.29}$$

$$a_{n+m+5} - a_{m+4} = \sum_{k=0}^{n} a_{k+m}. \tag{3.30}$$

Particularly,

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_{4n+k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} a_{4n+k+1} = 0,$$

$$a_{(\ell+5)n} = \sum_{k=0}^{n} \binom{n}{k} a_{\ell n+4k}, \quad a_{(\ell+5)n+1} = \sum_{k=0}^{n} \binom{n}{k} a_{\ell n+4k+1},$$

$$a_{5n+5} = \sum_{k=0}^{n} a_{5k+4}, \quad a_{5n+6} = \sum_{k=0}^{n} a_{5k+5},$$

$$a_{5n+7} = 1 + \sum_{k=0}^{n} a_{2k+6}, \quad a_{n+5} = 1 + \sum_{k=0}^{n} a_k.$$

Furthermore, for the Perrin sequence $\{\hat{P}_n\}$ we have

$$\hat{P}_m = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \hat{P}_{4n+k+m}, \tag{3.31}$$

$$\hat{P}_{(\ell+5)n+m} = \sum_{k=0}^{n} \binom{n}{k} \hat{P}_{\ell n+4k+m}, \tag{3.32}$$

$$\hat{P}_{5n+m+1} - \hat{P}_{m-4} = \sum_{k=0}^{n} \hat{P}_{5k+m}, \tag{3.33}$$

$$\hat{P}_{n+m+5} - \hat{P}_{m+4} = \sum_{k=0}^{n} \hat{P}_{k+m}. \tag{3.34}$$

Particularly,

$$\sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \hat{P}_{4n+k} = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \hat{P}_{4n+k+1} = 0,$$

$$\hat{P}_{(\ell+5)n} = \sum_{k=0}^{n} \binom{n}{k} \hat{P}_{\ell n+4k}, \quad \hat{P}_{(\ell+5)n+1} = \sum_{k=0}^{n} \binom{n}{k} \hat{P}_{\ell n+4k+1},$$

$$\hat{P}_{5n+5} = \sum_{k=0}^{n} \hat{P}_{5k+4}, \quad \hat{P}_{5n+6} = \sum_{k=0}^{n} \hat{P}_{5k+5},$$

$$\hat{P}_{5n+7} = 1 + \sum_{k=0}^{n} \hat{P}_{2k+6}, \quad \hat{P}_{n+5} = 1 + \sum_{k=0}^{n} \hat{P}_k.$$

4. More applications of the general recursive sequences of the third order

We now consider the recursive number sequences of order 3 defined by $a_{n+3} = pa_{n+2} + qa_{n+1} + ra_n$, $n \geq 0$, with arbitrary initial conditions a_0, a_1, and a_2. The characteristic polynomial of the recursive relation is $p(x) = x^3 - px^2 - qx - r$, which can be written as

$$p(x) = \det(xI - \phi),$$

where

$$\phi = \begin{bmatrix} p & q & r \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}. \tag{4.1}$$

From the Cayley-Hamilton Theorem, ϕ satisfies $p(\phi)=0$, i.e.,

$$\phi^3 - p\phi^2 - q\phi - rI = 0. \tag{4.2}$$

Hence, we have

Proposition 4.1 *Let ϕ be defined as (4.1). Then*

$$(\phi^3 - rI)^n = \phi^n(p\phi + qI)^n, \tag{4.3}$$

or equivalently

$$\sum_{k=0}^{n}(-r)^{n-k}\binom{n}{k}\phi^{3k+\ell} = \sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}\phi^{n+k+\ell} \tag{4.4}$$

for all integers $\ell \geq 0$. Furthermore, (4.4) implies

$$\sum_{k=0}^{n}(-r)^{n-k}\binom{n}{k}a_{3k+\ell} = \sum_{k=0}^{n}\binom{n}{k}p^k q^{n-k}a_{n+k+\ell}. \tag{4.5}$$

Particularly, for $p=q=r=1$, we have

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}T_{3k+\ell} = \sum_{k=0}^{n}\binom{n}{k}T_{n+k+\ell}, \tag{4.6}$$

where $\{T_n\}$ is the Tribonacci sequence, i.e., $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ $(n \geq 3)$ with initial conditions $T_0 = T_1 = 0$ and $T_2 = 1$.

Similarly, from (4.2) there is $\phi^2(\phi - pI) = q\phi + rI$. Hence, we have

Proposition 4.2 *Let ϕ be defined as (4.1). Then*

$$\phi^{2n}(\phi - pI)^n = (q\phi + rI)^n, \tag{4.7}$$

or equivalently

$$\sum_{k=0}^{n}(-p)^{n-k}\binom{n}{k}\phi^{2n+k+\ell} = \sum_{k=0}^{n}\binom{n}{k}q^k r^{n-k}\phi^{k+\ell} \tag{4.8}$$

for all integer $\ell \geq 0$. Furthermore, (4.4) implies

$$\sum_{k=0}^{n}(-p)^{n-k}\binom{n}{k}a_{2n+k+\ell} = \sum_{k=0}^{n}\binom{n}{k}q^k r^{n-k}a_{k+\ell}. \tag{4.9}$$

Particularly, for $p=q=r=1$, we have

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}T_{2n+k+\ell} = \sum_{k=0}^{n}\binom{n}{k}T_{k+\ell} \tag{4.10}$$

for Tribonacci sequence $\{T_n\}$.

Inspired by the properties of sequence $\{F_nF_{n+1}\}$, where $\{F_n\}$ is the Fibonacci sequence, shown in Barry [10], we now present a unified approach to connect recursive sequences of order two and certain recursive sequences of order three.

Proposition 4.3 *Let $\{a_n\}$ be the recursive number sequences of order 3 defined by $a_{n+3} =$*

$pa_{n+2} + qa_{n+1} + ra_n$, $n \geq 0$, *with arbitrary initial conditions* a_0, a_1, *and* a_2. *If* p, q, *and* r *satisfy*

$$q = -pr^{1/3} \tag{4.11}$$

then sequence $\{a\}$ *can be written as* $a_n = b_n b_{n+1}$ *for a recursive sequence* $\{b_n\}$ *of order* 2 *that satisfies* $b_{n+2} = ab_{n+1} + bb_n$ *with initial conditions* b_0 *and* b_1 *and the recursive coefficients*

$$a = (p + r^{1/3})^{1/2} \text{ and } b = -r^{1/3}. \tag{4.12}$$

Conversely, if $\{b_n\}$ *is a recursive sequence of order* 2 *satisfying* $b_{n+2} = ab_{n+1} + bb_n$ *with initial conditions* b_0 *and* b_1, *then sequence* $\{a_n = b_n b_{n+1}\}$ *is a recursive sequence of order* 3 *satisfying*

$$a_{n+3} = (a^2 + b)a_{n+2} + b(a^2 + b)a_{n+1} - b^3 a_n. \tag{4.13}$$

Proof We prove the converse case first. If $\{b_n\}$ is the recursive sequence of order 2 satisfying $b_{n+2} = ab_{n+1} + bb_n$ with initial conditions b_0 and b_1, then

$$\begin{aligned} a_{n+3} &= b_{n+3}b_{n+4} = b_{n+3}(ab_{n+3} + bb_{n+2}) \\ &= bb_{n+2}b_{n+3} + ab_{n+3}(ab_{n+2} + bb_{n+1}) \\ &= (a^2 + b)b_{n+2}b_{n+3} + abb_{n+1}(ab_{n+2} + bb_{n+1}) \\ &= (a^2 + b)b_{n+2}b_{n+3} + a^2 bb_{n+1}b_{n+2} + b^2 b_{n+1}(a_{n+2} - bb_n) \\ &= (a^2 + b)b_{n+2}b_{n+3} + b(a^2 + b)b_{n+1}b_{n+2} - b^3 b_n b_{n+1}, \end{aligned}$$

which implies (4.13). Let $\{a_n\}$ be the recursive number sequences of order 3 defined by $a_{n+3} = pa_{n+2} + qa_{n+1} + ra_n$, $n \geq 0$, with arbitrary initial conditions a_0, a_1, and a_2, where p, q, and r satisfy (4.11). By defining a and b as (4.12), we have $p = a^2 + b$, $q = b(a^2 + b)$, and $r = -b^3$, which implies that $\{a_n\}$ can be written as $a_n = b_n b_{n+1}$, where $\{b_n\}$ is the recursive sequence satisfying $b_{n+3} = b_{n+2} + b_{n+1} + b_n$. □

The result shown in Barry [2] can be considered as a special case of Proposition 4.3.

Corollary 4.4 *Let* $\{F_n\}$ *and* $\{L_n\}$ *be the Fibonacci and Lucas number sequences, respectively. Then* $\{F_n F_{n+1}\}$ *and* $\{L_n L_{n+1}\}$ *are the recursive sequences of order* 3 *satisfying*

$$F_{n+3}F_{n+4} = 2F_{n+2}F_{n+3} + 2F_{n+1}F_{n+2} - F_n F_{n+1}, \tag{4.14}$$

$$L_{n+3}L_{n+4} = 2L_{n+2}L_{n+3} + 2L_{n+1}L_{n+2} - L_n L_{n+1}, \tag{4.15}$$

respectively.

Proof Since $F_{n+2} = F_{n+1} + F_n$, we may use Proposition 4.3 with $a = b = 1$ to obtain that $\{F_n F_{n+1}\}$ is the recursive sequence with recursive coefficients $p = a^2 + b = 2$, $q = b(a^2 + b) = 2$, and $r = -b^3 = -1$. □

Let matrix ϕ be defined by (4.1). By using the Cayley-Hamilton Theorem, from the characteristic polynomial of the recursive relation shown in (4.13), we obtain $\phi^3 - (a^2 + b)\phi^2 - b(a^2 + b)\phi + b^3 I = 0$. Hence, there is

$$(\phi^3 + b^3 I)^n = (a^2 + b)^n \phi^n (\phi + b)^n,$$

or equivalently,

$$\sum_{k=0}^{n}\binom{n}{k}b^{3(n-k)}\phi^{3k}=(a^2+b)^n\sum_{k=0}^{n}b^{n-k}\binom{n}{k}\phi^{n+k}. \tag{4.16}$$

Therefore, we have

Proposition 4.5 *Let $\{b_n\}$ be the recursive sequence of order 2 satisfying $b_{n+2}=ab_{n+1}+bb_n$ with initial conditions b_0 and b_1. Then*

$$\sum_{k=0}^{n}\binom{n}{k}b^{3(n-k)}b_{3k+\ell}b_{3k+\ell+1}=(a^2+b)^n\sum_{k=0}^{n}b^{n-k}\binom{n}{k}b_{n+k+\ell}b_{n+k+\ell+1}. \tag{4.17}$$

Particularly, for the Fibonacci sequence $\{F_n\}$ we have

$$\sum_{k=0}^{n}\binom{n}{k}F_{3k+\ell}F_{3k+\ell+1}=2^n\sum_{k=0}^{n}\binom{n}{k}F_{n+k+\ell}F_{n+k+\ell+1} \tag{4.18}$$

and

$$F_{3k+\ell}F_{3k+\ell+1}=\sum_{k=0}^{n}\sum_{j=0}^{k}2^k(-1)^{n-k}\binom{n}{k}\binom{k}{j}F_{k+\ell+j}F_{k+\ell+j+1}. \tag{4.19}$$

Proof Multiplying ϕ^ℓ on the both sides of (4.16) and applying (2.12), one may obtain (4.17). For the Fibonacci sequence, (4.18) follows from (4.17) due to $a=b=1$. Since there exists an inverse relationship

$$f_n=\sum_{k=0}^{n}\binom{n}{k}g_k\Leftrightarrow g_n=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f_k$$

for all $n\geq 0$, we immediately obtain (4.19) by applying the above relationship to the both sides of (4.18). $\square$

There are two ways to treat a non-homogeneous recessive relationship: One way is to change it to a homogeneous recursive relationship of the same order, and another way is to change it to a homogeneous recursive relationship of one higher order. For instance, [11] finds the equivalence between the non-homogeneous recurrence relation of order 2, $a_n=pa_{n-1}+qa_{n-2}+k$, and the homogeneous recurrence relation of order 3, $a_n=(p+1)a_{n-1}+(q-p)a_{n-2}-qa_{n-3}$, where $k=a_2-pa_1-qa_0$. If $p+q\neq 1$, by denoting $\ell=k/(1-p-q)$ and $b_n=a_n+\ell$, then the non-homogeneous recursive relation of a_n can be changed to the homogeneous recursive relation of b_n as $b_n=pb_{n-1}+qb_{n-2}$.

The above results can be extended to the higher order recursive sequences. More precisely, a number sequence $\{a_n\}$ is called a sequence of order $r\geq 4$ if it satisfies the linear recurrence relation of order r

$$a_n=\sum_{j=1}^{r}p_ja_{n-j},\quad n\geq r, \tag{4.20}$$

for some constants p_j $(j=1,2,\ldots,r)$, and initial conditions a_j $(j=0,1,\ldots,r-1)$. In [4] the generating function $P_r(t)$ of the sequence $\{a_n\}$ is presented as

$$P_r(t)=\{a_0+\sum_{n=1}^{r-1}\Big(a_n-\sum_{j=1}^{n}p_ja_{n-j}\Big)t^n\}/\{1-\sum_{j=1}^{r}p_jt^j\}. \tag{4.21}$$

We define the impulse response sequence (IRS) satisfying (4.20) with initial conditions $a_0 = a_{r-2} = 0$ and $a_{r-1} = a$, $a \neq 0$. Then its generating function is

$$\tilde{P}_r(t) = \frac{t^{r-1}}{1 - \sum_{j=1}^{r} p_j t^j}. \tag{4.22}$$

Some other properties of IRS were given in [11]. Denote

$$\phi = \begin{bmatrix} p_1 & p_2 & \cdots & p_{r-1} & p_r \\ 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}. \tag{4.23}$$

Then we have the following matrix representation of $\{a_n\}$:

$$\phi^n = \frac{1}{a} \begin{bmatrix} a_{n+r-1} & p_2 a_{n+r-2} + \cdots + p_r a_n & \cdots & p_r a_{n+r-2} \\ a_{n+r-2} & p_2 a_{n+r-3} + \cdots + p_r a_{n-1} & \cdots & p_r a_{n+r-3} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & p_2 a_{n-1} + \cdots + p_r a_{n-r+1} & \cdots & p_r a_{n-1} \end{bmatrix}. \tag{4.24}$$

The matrix representation can be used to derive similar results about $\{a_n\}$ as those of recursive sequences of order 3 by using the same arguments shown before and their generating functions.

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