Fekete-Szegő Problems for Certain Classes of Meromorphic Functions Using q-Derivative Operator

Huo TANG1,∗, H. M. ZAYED2, A. O. MOSTAFA3, M. K. AOUF3
1. School of Mathematics and Statistics, Chifeng University, Inner Mongolia 024000, P. R. China;
2. Department of Mathematics, Faculty of Science, Menofia University, Shebin Elkom 32511, Egypt;
3. Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt

Abstract In this paper, we introduce two subclasses \(\Sigma^*_q(\varphi)\) and \(\Sigma^*_{q,\alpha}(\varphi)\) of meromorphic functions \(f(z)\) for which 
\[
\frac{qzD_qf(z)}{f(z)} \prec \varphi(z) \quad \text{and} \\
\frac{-(1 - \frac{q}{q-1})qzD_qf(z) + \alpha qzD_q[zD_qf(z)]}{(1 - \frac{q}{q-1})f(z) - \alpha zD_qf(z)} \prec \varphi(z), \quad \alpha \in \mathbb{C}\setminus(0,1], \quad 0 < q < 1,
\]
respectively. Sharp bounds for the Fekete-Szegő functional \(|a_1 - \mu a_2|\) of the above classes are obtained. Also, we consider some applications of the results obtained to functions defined by q-Bessel function.

Keywords analytic function; meromorphic function; Fekete-Szegő problem; q-derivative operator; q-Bessel function

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1. Introduction

The theory of \(q\)-analysis has important role in many areas of mathematics and physics, for example, in the areas of ordinary fractional calculus, optimal control problems, \(q\)-difference, \(q\)-integral equations and in \(q\)-transform analysis [1–4].

Let \(\Sigma\) denote the class of meromorphic functions of the form:
\[
f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} a_k z^k,
\]
which are analytic in the open punctured unit disc \(\mathbb{U}^* = \{ z : z \in \mathbb{C} \text{ and } 0 < |z| < 1 \} = \mathbb{U}\setminus\{0\} \).

A function \(f \in \Sigma\) is meromorphic starlike of order \(\alpha\), denoted by \(\Sigma^*(\alpha)\), if
\[
-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha, \quad 0 \leq \alpha < 1; \quad z \in \mathbb{U}.
\]

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* Corresponding author
E-mail address: thth2009@163.com (Huo TANG); hanaa_zayed42@yahoo.com (H. M. ZAYED);
adelaeg254@yahoo.com (A. O. MOSTAFA); mkaouf127@yahoo.com (M. K. AOUF)
The class $\Sigma^*(\alpha)$ was introduced and studied by Pommerenke [5] (see also Miller [6]).

Let $\varphi(z)$ be an analytic function with positive real part on $U$ satisfying $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps $U$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Let $\Sigma^*(\varphi)$ be the class of functions $f \in \Sigma$ for which

$$-\frac{z f'(z)}{f(z)} < \varphi(z), \quad z \in \mathbb{U}.$$  

The class $\Sigma^*(\varphi)$ was introduced and studied by Silverman et al. [7]. We note that, the class $\Sigma^*(\alpha)$ is the special case of the class $\Sigma^*(\varphi)$ when $\varphi(z) = \frac{1 + (1 - 2\alpha)z}{1 - z} (0 \leq \alpha < 1)$.

For a function $f(z) \in \Sigma$ given by (1.1) and $0 < q < 1$, the $q$-derivative of a function $f(z)$ is defined by [2]

$$D_q f(z) = \frac{f(qz) - f(z)}{(q - 1)z}, \quad z \in \mathbb{U}^*.$$  

(1.2)

From (1.2), we deduce that $D_q f(z)$ for a function $f(z)$ of the form (1.1) is given by

$$D_q f(z) = -\frac{1}{qz^2} + \sum_{k=0}^{\infty} [k]_q a_k z^{k-1}, \quad z \neq 0,$$

where

$$[k]_q = \frac{1 - q^k}{1 - q}.$$  

As $q \to 1^-$, $[k]_q \to k$, we have $\lim_{q \to 1^-} D_q f(z) = f'(z)$.

Making use of the $q$-derivative $D_q$, we introduce the subclasses $\Sigma^*_q(\varphi)$ and $\Sigma^*_{q,a}(\varphi)$ as follows:

**Definition 1.1** A function $f(z) \in \Sigma$ is said to be in the class $\Sigma^*_q(\varphi)$, if and only if

$$-\frac{q z D_q f(z)}{f(z)} < \varphi(z), \quad z \in \mathbb{U}.$$  

(1.3)

We note that:

(i) $\lim_{q \to 1^-} \Sigma^*_q(\varphi) = \Sigma^*(\varphi)$ (see [7] and [8, with $\alpha = 0$]);

(ii) $\lim_{q \to 1^-} \Sigma^*_q(\frac{1 + (1 - 2\alpha)z}{1 - z}) = \Sigma^*(\alpha)$ ($0 \leq \alpha < 1$) (see [5]);

(iii) $\lim_{q \to 1^-} \Sigma^*_q(1 + \frac{z}{q}) = F^*(1) = F^*$ (see [9, with $b = 1$]);

(iv) $\lim_{q \to 1^-} \Sigma^*_q(\frac{1 + (1 - 2\alpha)z}{1 + (1 - 2\alpha)z}) = \Sigma(\delta, \beta, \gamma)$ ($0 \leq \delta < 1, 0 < \beta \leq 1, \frac{1}{2} \leq \gamma \leq 1$) (see [10]);

(v) $\lim_{q \to 1^-} \Sigma^*_q(\frac{1 + Aq}{1 + Bq}) = K_1(A, B)$ ($0 \leq B < 1, -B < A < B$) (see [11]).

**Definition 1.2** For $\alpha \in \mathbb{C}\setminus[0,1]$, let $\Sigma^*_{q,a}(\varphi)$ be the subclass of $\Sigma$ consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion:

$$-\frac{(1 - \frac{\alpha}{q}) q z D_q f(z) + \alpha q z D_q[z D_q f(z)]}{(1 - \frac{\alpha}{q}) f(z) - \alpha z D_q f(z)} < \varphi(z).$$  

(1.4)

From (1.3) and (1.4), we note that

$$\Sigma^*_{q,0}(\varphi) = \Sigma^*_q(\varphi)$$

and $\lim_{q \to 1^-} \Sigma^*_{q,a}(\varphi) = \Sigma^*_q(\varphi)$ (see [7]).
2. Fekete-Szegő problems

To prove our results, we need the following lemmas.

**Lemma 2.1** ([12]) If \( p(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is a function with positive real part in \( U \) and \( \mu \) is a complex number, then

\[
|c_2 - \mu c_1^2| \leq 2 \max\{1; |2\mu - 1|\}.
\]

The result is sharp for the functions given by

\[
p(z) = \frac{1 + z^2}{1 - z^2} \quad \text{and} \quad p(z) = \frac{1 + z}{1 - z}.
\]

**Lemma 2.2** ([12]) If \( p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots \) is a function with positive real part in \( U \), then

\[
|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \text{if } \nu \leq 0, \\ 2, & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2, & \text{if } \nu \geq 1. \end{cases}
\]

When \( \nu < 0 \) or \( \nu > 1 \), the equality holds if and only if \( p_1(z) = \frac{1 + z^2}{1 - z^2} \) or one of its rotations. If \( 0 < \nu < 1 \), then the equality holds if and only if \( p_1(z) = \frac{1 + z^2}{1 - z^2} \) or one of its rotations. If \( \nu = 0 \), the equality holds if and only if

\[
p_1(z) = \left( \frac{1}{2} + \lambda \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} - \lambda \right) \frac{1 - z}{1 + z}, \quad 0 \leq \lambda \leq 1,
\]

or one of its rotations. If \( \nu = 1 \), the equality holds if and only if

\[
\frac{1}{p_1(z)} = \left( \frac{1}{2} + \lambda \right) \frac{1 + z}{1 - z} + \left( \frac{1}{2} - \lambda \right) \frac{1 - z}{1 + z}, \quad 0 \leq \lambda \leq 1,
\]

or one of its rotations. Also the above upper bound is sharp and it can be improved as follows when \( 0 < \nu < 1 \):

\[
|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2, \quad 0 < \nu \leq \frac{1}{2}
\]

and

\[
|c_2 - \nu c_1^2| + (1 - \nu) |c_1|^2 \leq 2, \quad \frac{1}{2} < \nu < 1.
\]

Unless otherwise mentioned, we assume throughout this paper that \( \alpha \in \mathbb{C}\setminus(0, 1] \) and \( 0 < q < 1 \).

**Theorem 2.3** Let \( \varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 \geq 0) \). If \( f(z) \) given by (1.1) belongs to the class \( \Sigma_{q}^*(\varphi) \) and \( \mu \) is a complex number, then

\[
|a_1 - \mu a_0^2| \leq \frac{|B_1|}{(1 + q)} \max\{1; \frac{B_2}{B_1} - [1 - \mu(1 + q)B_1] \}, \quad B_1 \neq 0, \quad \text{if} \quad B_1 \neq 0, \quad (2.1)
\]

\[
|a_1| \leq \frac{|B_2|}{(1 + q)}, \quad B_1 = 0. \quad (2.2)
\]

The result is sharp.
Fekete-Szegő problems for certain classes of meromorphic functions using \( q \)-derivative operator

Proof If \( f(z) \in \Sigma_q^*(\varphi) \), then there is a Schwarz function \( w(z) \) in \( \mathbb{U} \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( \mathbb{U} \) and such that

\[
-qzD_qf(z) = \varphi(w(z)). \tag{2.3}
\]

Define the function \( p_1(z) \) by

\[
p_1(z) = 1 + \frac{w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \cdots. \tag{2.4}
\]

Since \( w(z) \) is a Schwarz function, we see that \( \Re\{p_1(z)\} > 0 \) and \( p_1(0) = 1 \).

Define

\[
p(z) = -\frac{qzD_qf(z)}{f(z)} = 1 + b_1z + b_2z^2 + \cdots. \tag{2.5}
\]

In view of (2.3)–(2.5), we have

\[
p(z) = \varphi(p_1(z) - \frac{1}{p_1(z) + 1}). \tag{2.6}
\]

Since

\[
p_1(z) - \frac{1}{p_1(z) + 1} = \frac{1}{2}(c_1z + (c_2 - \frac{c_1^2}{2})z^2 + (c_3 + \frac{c_1^3}{4} - c_1c_2)z^3 + \cdots).
\]

Therefore, we have

\[
\varphi\left(\frac{p_1(z) - 1}{p_1(z) + 1}\right) = 1 + \frac{1}{2}B_1c_1z + \left[\frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2c_1^2\right]z^2 + \cdots. \tag{2.7}
\]

From (2.6) and (2.7), we obtain

\[
b_1 = \frac{1}{2}B_1c_1,
\]

and

\[
b_2 = \frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2c_1^2.
\]

Then, from (2.5) and (1.1), we see that \( b_1 = -a_0 \), and \( b_2 = a_0^2 - (q + 1)a_1 \), or, equivalently, we have

\[
a_0 = -\frac{B_1c_1}{2}, \tag{2.8}
\]

and

\[
a_1 = -\frac{B_1}{2(1 + q)}[c_2 - \frac{c_1^2}{2}(1 - \frac{B_2}{B_1} + B_1)]. \tag{2.9}
\]

Therefore

\[
a_1 - \mu a_0^2 = -\frac{B_1}{2(1 + q)}\{c_2 - \nu c_1^2\},
\]

where

\[
\nu = \frac{1}{2}[1 - \frac{B_2}{B_1} + B_1 - \mu B_1(1 + q)]. \tag{2.10}
\]

Now, the result (2.1) follows by an application of Lemma 2.1. Also, if \( B_1 = 0 \), then

\[
a_0 = 0, \quad a_1 = -\frac{B_2c_1^2}{4(1 + q)}.
\]

Since \( p(z) \) has positive real part, then \( |c_1| \leq 2 \) (see [13]). Hence

\[
|a_1| \leq \frac{|B_2|}{1 + q}.
\]
this proving (2.2). The result is sharp for the functions

\[
-qzD_\varphi f(z) \frac{f(z)}{\varphi(z)} = \varphi(z^2), \quad -qzD_\varphi f(z) \frac{f(z)}{\varphi(z)} = \varphi(z).
\]

This completes the proof of Theorem 2.3. □

**Remark 2.4** For \( q \to 1^- \) in Theorem 2.3, we obtain the result obtained by [7, Theorem 2.1].

Putting \( q \to 1^- \) and \( \varphi(z) = \frac{1 + z}{1 - z} \) in Theorem 2.3, we obtain the following corollary.

**Corollary 2.5** If \( f(z) \) given by (1.1) belongs to the class \( F^* \) and \( \mu \) is a complex number, then

\[
|a_1 - \mu a_0^2| \leq \max\{1, |1 - 2(1 - 2\mu)|\}.
\]

The result is sharp.

By using Lemma 2.2, we can obtain the following theorem.

**Theorem 2.6** Let \( \varphi(z) = 1 + B_1z + B_2z^2 + \cdots \) \((B_i > 0, i = 1, 2)\). If \( f(z) \) given by (1.1) belongs to the class \( \Sigma_\varphi(\varphi) \), then

\[
|a_1 - \mu a_0^2| \leq \begin{cases} 
\frac{B_2 - [1 - \mu(1 + q)]B_1^2}{1 + q}, & \text{if } \mu \leq \sigma_1, \\
\frac{B_1}{1 + q}, & \text{if } \sigma_1 \leq \mu \leq \sigma_2, \\
\frac{-B_2 + [1 - \mu(1 + q)]B_1^2}{1 + q}, & \text{if } \mu \geq \sigma_2,
\end{cases}
\]

where

\[
\sigma_1 = \frac{-B_1 - B_2 + B_1^2}{(1 + q)B_1^2}, \quad \sigma_2 = \frac{B_1 - B_2 + B_1^2}{(1 + q)B_1^2}.
\]

The result is sharp. Further, let \( \sigma_3 = \frac{-B_2 + B_1^2}{(1 + q)B_1^2} \).

(i) If \( \sigma_1 \leq \mu \leq \sigma_3 \), then

\[
|a_1 - \mu a_0^2| + \frac{((B_1 + B_2) + [\mu(1 + q)] - 1B_1^2)|a_0|^2}{(1 + q)B_1^2} \leq \frac{B_2}{1 + q}.
\]

(ii) If \( \sigma_3 \leq \mu \leq \sigma_2 \), then

\[
|a_1 - \mu a_0^2| + \frac{((B_1 - B_2) + [1 - \mu(1 + q)]B_1^2)|a_0|^2}{(1 + q)B_1^2} \leq \frac{B_2}{1 + q}.
\]

**Proof** First, let \( \mu \leq \sigma_1 \). Then

\[
|a_1 - \mu a_0^2| \leq \frac{B_2}{(1 + q)} \left\{ \frac{B_2}{B_1} - [1 - \mu(1 + q)]B_1 \right\} \\
\leq \frac{B_2 - [1 - \mu(1 + q)]B_1^2}{1 + q}.
\]

Let, now \( \sigma_1 \leq \mu \leq \sigma_2 \). Then, using the above calculations, we obtain

\[
|a_1 - \mu a_0^2| \leq \frac{B_2}{1 + q}.
\]

Finally, if \( \mu \geq \sigma_2 \), then

\[
|a_1 - \mu a_0^2| \leq \frac{B_2}{(1 + q)} \left\{ \frac{-B_2}{B_1} + [1 - \mu(1 + q)]B_1 \right\} \leq \frac{-B_2 + [1 - \mu(1 + q)]B_1^2}{1 + q}.
\]
To show that the bounds are sharp, we define the functions $K_{\varphi_n}$ \((n \geq 2)\) by
\[ -\frac{qzD_qK_{\varphi_n}(z)}{K_{\varphi_n}(z)} = \varphi(z^{n-1})(z^2K_{\varphi_n}(z)|_{z=0} = 0 = -z^2K'_{\varphi_n}(z)|_{z=0} = 1) \]
and the functions $F_\gamma$ and $G_\gamma$ \((0 \leq \gamma \leq 1)\) by
\[ -\frac{qzD_qF_\gamma(z)}{F_\gamma(z)} = \varphi(z \zeta + \gamma)(z^2F_\gamma(z)|_{z=0} = 0 = -z^2F'_\gamma(z)|_{z=0} = 1) \]
and
\[ -\frac{qzD_qG_\gamma(z)}{G_\gamma(z)} = \varphi(-z \zeta + \gamma)(z^2G_\gamma(z)|_{z=0} = 0 = -z^2G'_\gamma(z)|_{z=0} = 1). \]
Clearly, the functions $K_{\varphi_n}$, $F_\gamma$ and $G_\gamma \in \Sigma^*_q(\varphi)$. Also we write $K_{\varphi} = K_{\varphi_2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if $f(z)$ is $K_{\varphi}$ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if $f(z)$ is $K_{\varphi_3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f(z)$ is $F_\gamma$ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f(z)$ is $G_\gamma$ or one of its rotations. This completes the proof of Theorem 2.6. □

**Remark 2.7** (i) For $q \to 1^-$ in Theorem 2.6, we obtain the result obtained by [8, Theorem 5.1]:

(ii) Putting $q \to 1^-$ and $\varphi(z) = \frac{z + \bar{z}}{2}$ in Theorem 2.6, we obtain a new result for the class $\mathcal{F}^*.$

**Theorem 2.8** Let $\varphi(z) = 1 + B_1z + B_2z^2 + \cdots$ \((B_1 \geq 0)\). If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{q,\alpha}(\varphi)$ and $\mu$ is a complex number, then
\[
|a_1 - \mu a_0^2| \leq \left(\frac{q}{1+q}\right)\frac{B_1}{q - \alpha/(1+q)}\max\{1, \left|\frac{B_2}{B_1} - \left[1 - \frac{q(1+q)}{q - \alpha(1+q)}\right]B_1\right|\}, \ B_1 \neq 0, \quad \text{(2.14)}
\]
\[
|a_1| \leq \left(\frac{q}{1+q}\right)\frac{B_2}{q - \alpha/(1+q)}, \ B_1 = 0. \quad \text{(2.15)}
\]

The result is sharp.

**Proof** If $f(z) \in \Sigma^*_{q,\alpha}(\varphi)$, then there is a Schwarz function $w(z)$ in $U$ with $w(0) = 0$ and $|w(z)| < 1$ in $U$ and such that
\[
-(1 - \frac{q}{\alpha})qzD_qf(z) + \alpha_0zD_q[zD_qf(z)]
\]
\[
(1 - \frac{q}{\alpha})f(z) - \alpha zD_qf(z)
\]
\[
= \varphi(w(z)).
\]
Since $w(z)$ is a Schwarz function, we see that $\Re\{p_1(z)\} > 0$ and $p_1(0) = 1$.

Define
\[
p(z) = \frac{-(1 - \frac{q}{\alpha})qzD_qf(z) + \alpha_0zD_q[zD_qf(z)]}{(1 - \frac{q}{\alpha})f(z) - \alpha zD_qf(z)} = 1 + b_1z + b_2z^2 + \cdots. \quad \text{(2.16)}
\]
Then, from (2.6), (2.7), (2.16) and (1.1), we see that
\[
\frac{1}{2}B_1c_1 = -(1 - \frac{\alpha}{\alpha})a_0,
\]
and
\[
\frac{1}{2}B_1(c_2 - \frac{q}{\alpha}) + \frac{1}{4}B_2c_1^2 = (1 - \frac{\alpha}{\alpha})a_0^2 - (1 + q)(1 - \alpha - \frac{\alpha}{\alpha})a_1,
\]
or, equivalently, we have
\[ a_0 = \frac{qB_1c_1}{2(q-\alpha)}, \quad a_1 = -\frac{qB_1}{2(1+q)[q-\alpha(1+q)]} \left[c_2 - \frac{\alpha}{2} \left(1 - \frac{B_2}{B_1} + B_1 \right) \right]. \]
Therefore
\[ a_1 - \mu a_0^2 = -\frac{qB_1}{2(1+q)[q-\alpha(1+q)]} \left[c_2 - \nu c_2^2 \right], \]
where
\[ \nu = \frac{1}{2}[1 - \frac{B_2}{B_1} + B_1(1 - \mu \frac{q(1+q)[q-\alpha(1+q)]}{(q-\alpha)^2})]. \tag{2.17} \]
Now, the result (2.14) follows by an application of Lemma 2.1. Also, if \( B_1 = 0 \), then
\[ a_0 = 0, \quad a_1 = \frac{qB_2c_1^2}{4(1+q)[q-\alpha(1+q)]}. \]
Since \( p(z) \) has positive real part, then \( |c_1| \leq 2 \) (see [13]), hence
\[ |a_1| \leq \left(\frac{q}{1+q}\right)[\frac{B_2}{q-\alpha(1+q)}], \]
this proving (2.15). The result is sharp for the functions
\[ -(1 - \frac{\alpha}{q})qzD_q f(z) + \alpha qzD_q[zD_q f(z)] \]
\[ \frac{(1 - \frac{\alpha}{q})f(z) - \alpha zD_q f(z)}{(1 - \frac{\alpha}{q})f(z) - \alpha zD_q f(z)} = \varphi(z^2) \]
and
\[ -(1 - \frac{\alpha}{q})qzD_q f(z) + \alpha qzD_q[zD_q f(z)] \]
\[ \frac{(1 - \frac{\alpha}{q})f(z) - \alpha zD_q f(z)}{(1 - \frac{\alpha}{q})f(z) - \alpha zD_q f(z)} = \varphi(z). \]
This completes the proof of Theorem 2.8. □

**Remark 2.9**
(i) Putting \( \alpha = 0 \) in Theorem 2.8, we obtain the result of Theorem 2.3;
(ii) Putting \( q \to 1^- \) in Theorem 2.8, we obtain the result obtained by [7, Theorem 2.2].

Using arguments similar to those in the proof of Theorem 2.6, we obtain the following theorem.

**Theorem 2.10** Let \( \varphi(z) = 1 + B_1z + B_2z^2 + \cdots \quad (B_i > 0, i \in 1, 2, 0 < \alpha < \frac{q}{1+q}). \) If \( f(z) \) given by (1.1) belongs to the class \( \Sigma_{q,\alpha}^\phi(\varphi) \), then
\[
|a_1 - \mu a_0^2| \leq \begin{cases} 
\frac{q}{1+q}[q-\alpha(1+q)] \left\{B_2 - [1 - \mu \frac{q(1+q)[q-\alpha(1+q)]}{(q-\alpha)^2} B_1^2] \right\}, & \text{if } \mu \leq \sigma_4, \\
\frac{q}{1+q}[q-\alpha(1+q)] \left\{-B_2 + [1 - \mu \frac{q(1+q)[q-\alpha(1+q)]}{(q-\alpha)^2} B_1^2] \right\}, & \text{if } \sigma_4 \leq \mu \leq \sigma_5, \\
\frac{q}{1+q}[q-\alpha(1+q)] \left\{-B_2 + [1 - \mu \frac{q(1+q)[q-\alpha(1+q)]}{(q-\alpha)^2} B_1^2] \right\}, & \text{if } \mu \geq \sigma_5, 
\end{cases}
\]
where
\[
\sigma_4 = \frac{(q-\alpha)^2[-B_1 - B_2 + B_1^2]}{q(1+q)[q-\alpha(1+q)]B_1^2}, \quad \sigma_5 = \frac{(q-\alpha)^2[B_1 - B_2 + B_1^2]}{q(1+q)[q-\alpha(1+q)]B_1^2},
\]
The result is sharp. Further, let
\[
\sigma_6 = \frac{(q-\alpha)^2[-B_2 + B_1^2]}{q(1+q)[q-\alpha(1+q)]B_1^2}.
\]
(i) If \( \sigma_4 \leq \mu \leq \sigma_5 \), then
\[
|a_1 - \mu a_0|^2 + \frac{(q - \alpha)^2}{(1 + q)[q - \alpha(1 + q)]B_1^2} \times \\
\{(B_1 + B_2) + \left[ \mu \frac{q(1 + q)[q - \alpha(1 + q)]}{(q - \alpha)^2} - 1 \right]B_1^2 \} |a_0|^2 \leq \frac{qB_1}{(1 + q)[q - \alpha(1 + q)]}.
\]

(ii) If \( \sigma_5 \leq \mu \leq \sigma_6 \), then
\[
|a_1 - \mu a_0|^2 + \frac{(q - \alpha)^2}{(1 + q)[q - \alpha(1 + q)]B_1^2} \times \\
\{(B_1 - B_2) + \left[ 1 - \mu \frac{q(1 + q)[q - \alpha(1 + q)]}{(q - \alpha)^2} - 1 \right]B_1^2 \} |a_0|^2 \leq \frac{qB_1}{(1 + q)[q - \alpha(1 + q)]}.
\]

3. Applications to functions defined by \( q \)-Bessel function

We recall some definitions of \( q \)-calculus which will be used in our paper.

For any complex number \( \alpha \), the \( q \)-shifted factorials are defined by
\[
(\alpha; q)_0 = 1; \quad (\alpha; q)_n = \prod_{k=0}^{n-1} (1 - \alpha q^k), \quad n \in \mathbb{N} = \{1, 2, \ldots\}. \tag{3.1}
\]

If \( |q| < 1 \), the definition (3.1) remains meaningful for \( n = \infty \) as a convergent infinite product
\[
(\alpha; q)_\infty = \prod_{j=0}^{\infty} (1 - \alpha q^j).
\]

In terms of the analogue of the gamma function
\[
(q^\alpha; q)_\infty = \frac{\Gamma_q(\alpha + n)[1 - q]^n}{\Gamma_q(\alpha)}, \quad n > 0,
\]
where the \( q \)-gamma function is defined by
\[
\Gamma_q(x) = \frac{(q; q)_\infty(1 - q)^{1-x}}{(q^x; q)_\infty}, \quad 0 < q < 1.
\]

We note that \( \lim_{q \to 1^-} \frac{q^\alpha}{(1 - q)^n} = (\alpha)_n \), where
\[
(\alpha)_n = \begin{cases} 
1, & \text{if } n = 0, \\
\alpha(\alpha + 1)(\alpha + 2) \cdots (\alpha + n - 1), & \text{if } n \in \mathbb{N}.
\end{cases}
\]

Now, consider the \( q \)-analogue of Bessel function defined by [3]
\[
\mathcal{J}_v^{(1)}(z; q) = \frac{(q^v + 1; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k (z^2/2)^{2k+\nu}}{q^k q^{v+1} q^k} \mathcal{J}_v^{(1)}(z^{1/2}(1 - q); q), \quad 0 < q < 1.
\]

Also, let us define
\[
\mathcal{L}_v(z; q) = \frac{2^v(q; q)_\infty}{(q^v + 1; q)_\infty(1 - q)^v z^{v/2 + \nu + 1}} \mathcal{J}_v^{(1)}(z^{1/2}(1 - q); q)
\]
\[
= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1} (1 - q)^{2(k+1)}}{4(k+1)(q; q)_{k+1}(q^{v+1}; q)_{k+1}} z^k, \quad z \in \mathbb{U}.
\]
By using the Hadamard product (or convolution), we define the linear operator $L_{q,v} : \Sigma \rightarrow \Sigma$, as follows:

$$\left(L_{q,v}(z)\right) = \Phi_v(z) * f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(1-q)^{2(k+1)}}{q^{k+1}(q^{1+1};q)_{k+1}} a_k z^k.$$ 

The linear operator $L_{q,v}$ was introduced and studied by Mostafa et al.\cite{14}. Also, as $q \rightarrow 1^-$, the linear operator $L_{q,v}$ reduces to the operator $L_1$, introduced and studied by Aouf et al.\cite{15}.

For $0 < q < 1$ and $\alpha \in \mathbb{C} \setminus (0, 1]$, let $\Sigma^*_{q,v}(\varphi)$ and $\Sigma^*_{q,\alpha,v}(\varphi)$ be the subclasses of $\Sigma$ consisting of functions $f(z)$ of the form (1.1) and satisfy the analytic criterions, respectively:

$$-\frac{q z D_q(L_{q,v}f)(z)}{(L_{q,v}f)(z)} < \varphi(z), \quad z \in \mathbb{U},$$

$$-(1 - \frac{q}{2}) q z D_q(L_{q,v}f) + \alpha q z D_q[z D_q(L_{q,v}f)](1 - \frac{q}{2})f(z) - \alpha z D_q(L_{q,v}f)(z) < \varphi(z), \quad z \in \mathbb{U}.$$

Using similar arguments to those in the proof of the above theorems, we obtain the following theorems.

**Theorem 3.1** Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ $(B_1 \geq 0)$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{q,v}(\varphi)$ and $\mu$ is a complex number, then

$$|a_1 - \mu a_0^2| \leq \frac{4^2(1-q^{v+1})(1-q^{v+2})|B_1|}{(1-q)^2} \times \max\{1, \frac{B_2}{B_1} - [1 - \mu(1-q^{v+1})]B_1]\}, \quad B_1 \neq 0,$n

$$|a_1| \leq \frac{4^2(1-q^{v+1})(1-q^{v+2})|B_2|}{(1-q)^2}, \quad B_1 = 0.$$

The result is sharp.

**Theorem 3.2** Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ $(B_i > 0, i = 1, 2)$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{q,v}(\varphi)$, then

$$|a_1 - \mu a_0^2| \leq \begin{cases} \frac{4^2(1-q^{v+1})(1-q^{v+2})B_2 - [1 - \mu(1-q^{v+1})]B_1^2}{(1-q)^2}, & \text{if } \mu \leq \sigma_1^*, \\ \frac{4^2(1-q^{v+1})(1-q^{v+2})B_2}{(1-q)^2}, & \text{if } \sigma_1^* \leq \mu \leq \sigma_2^*, \\ \frac{4^2(1-q^{v+1})(1-q^{v+2})}{(1-q)^2} \{-B_2 + [1 - \mu(1-q^{v+1})]B_1^2\}, & \text{if } \mu \geq \sigma_2^*, \end{cases}$$

where

$$\sigma_1^* = \frac{-B_1 - B_2 + B_2^2(1-q^{v+2})}{(1-q^{v+1})B_1^2}, \quad \sigma_2^* = \frac{B_1 - B_2 + B_2^2(1-q^{v+2})}{(1-q^{v+1})B_1^2}.$$

The result is sharp. Further, let

$$\sigma_3^* = \frac{[-B_1 + B_2^2](1-q^{v+2})}{(1-q^{v+1})B_1^2}.$$
(i) If $\sigma_1^* \leq \mu \leq \sigma_3^*$, then
\[
|a_1 - \mu a_0^2| + \frac{(1-q^{v+2})}{(1-q^{v+1})B_1^2} \{(B_1 + B_2) + [\mu(1-q^{v+1}) - 1]B_1^2\}|a_0|^2 \\
\leq \frac{4^2(1-q^{v+1})(1-q^{v+2})B_1}{(1-q)^2}.
\]

(ii) If $\sigma_3^* \leq \mu \leq \sigma_2^*$, then
\[
|a_1 - \mu a_0^2| + \frac{(1-q^{v+2})}{(1-q^{v+1})B_1^2} \{(B_1 - B_2) + [1 - \mu(1-q^{v+1})]B_1^2\}|a_0|^2 \\
\leq \frac{4^2(1-q^{v+1})(1-q^{v+2})B_1}{(1-q)^2}.
\]

**Theorem 3.3** Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \ (B_1 \geq 0)$. If $f(z)$ given by (1.1) belongs to the class $\Sigma_{\mu,\alpha,v}(\varphi)$ and $\mu$ is a complex number, then
\[
|a_1 - \mu a_0^2| \leq \frac{4^2q(1-q^{v+1})(1-q^{v+2})B_1}{(1-q)^2|q - \alpha(1+q)|} \\
\times \max\{1, \frac{B_2}{B_1} - [1 - \mu q(1-q^{v+1})|q - \alpha(1+q)|]/(\alpha - q)^2(1-q^{v+2})|B_1|\}, \\
B_1 \neq 0,
\]
\[
|a_1| \leq \frac{4^2q(1-q^{v+1})(1-q^{v+2})B_1}{(1-q)^2|q - \alpha(1+q)|}, \quad B_1 = 0.
\]
The result is sharp.

**Theorem 3.4** Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots \ (B_i > 0, i = 1, 2, 0 < \alpha < \frac{q}{1+q})$. If $f(z)$ given by (1.1) belongs to the class $\Sigma_{\mu,\alpha,v}(\varphi)$, then
\[
|a_1 - \mu a_0^2| \leq \begin{cases} \\
\frac{4^2(1-q^{v+1})(1-q^{v+2})}{(1-q)^2|q - \alpha(1+q)|} \\
\{B_2 - [1 - \mu q(1-q^{v+1})]/(1-q)^2(1-q^{v+2})|B_1^2\} \} \}, \\
\frac{4^2(1-q^{v+1})(1-q^{v+2})B_1}{(1-q)^2|q - \alpha(1+q)|}, & \text{if } \sigma_4^\ast \leq \mu \leq \sigma_5^\ast, \\
\frac{4^2(1-q^{v+1})(1-q^{v+2})B_1}{(1-q)^2|q - \alpha(1+q)|}, & \text{if } \mu \geq \sigma_5^\ast,
\end{cases}
\]
where
\[
\sigma_4^\ast = \frac{(q - \alpha)^2(1-q^{v+2})[-B_1 - B_2 + B_1^2]}{qB_1^2(1-q^{v+1})|q - \alpha(1+q)|},
\]
\[
\sigma_5^\ast = \frac{(q - \alpha)^2(1-q^{v+2})B_1 - B_2 + B_1^2]}{qB_1^2(1-q^{v+1})|q - \alpha(1+q)|}.
\]
The result is sharp. Further, let
\[
\sigma_6^\ast = \frac{(q - \alpha)^2(1-q^{v+2})[-B_2 + B_1^2]}{qB_1^2(1-q^{v+1})|q - \alpha(1+q)|}.
\]
(i) If $\sigma_4^* \leq \mu \leq \sigma_6^*$, then
\[
|a_1 - \mu a_0^2| + \frac{(q - \alpha)^2(1 - q^{v+2})}{q(1 - q^{v+1})[q - \alpha(1 + q)]B_1^2} \times \\
\{(B_1 + B_2) + \left[\mu \frac{q(1 - q^{v+2})[q - \alpha(1 + q)]}{(1 - q^{v+1})(\alpha - q)^2} - 1\right]B_2^2\}|a_0|^2 \\
\leq 4^2(1 - q^{v+1})(1 - q^{v+2})B_1 \frac{(1 - q^2)[q - \alpha(1 + q)]}{(1 - q)(1 - q[v+1])}.
\]

(ii) If $\sigma_6^* \leq \mu \leq \sigma_5^*$, then
\[
|a_1 - \mu a_0^2| + \frac{(q - \alpha)^2(1 - q^{v+2})}{q(1 - q^{v+1})[q - \alpha(1 + q)]B_1^2} \times \\
\{(B_1 - B_2) + \left[1 - \mu \frac{q(1 - q^{v+2})[q - \alpha(1 + q)]}{(1 - q^{v+1})(\alpha - q)^2} - 1\right]B_2^2\}|a_0|^2 \\
\leq 4^2(1 - q^{v+1})(1 - q^{v+2})B_1 \frac{1 - q^2}{(1 - q)(1 - q[v+1])}.
\]

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