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Fekete-Szegö Problems for Certain Classes of Meromorphic Functions Using q-Derivative Operator

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Abstract In this paper, we introduce two subclasses $\Sigma_q^*(\varphi)$ and $\Sigma_{q,\alpha}^*(\varphi)$ of meromorphic functions $f(z)$ for which $\frac{qzD_qf(z)}{f(z)} \prec \varphi(z)$ and

$$\frac{-(1-\frac{\alpha}{q})qzD_qf(z)+\alpha qzD_q[zD_qf(z)]}{(1-\frac{\alpha}{q})f(z)-\alpha zD_qf(z)} \prec \varphi(z), \quad \alpha \in \mathbb{C}\backslash(0,1],\ 0<q<1,$$

respectively. Sharp bounds for the Fekete-Szegö functional $|a_1-\mu a_0^2|$ of the above classes are obtained. Also, we consider some applications of the results obtained to functions defined by q-Bessel function.

Keywords analytic function; meromorphic function; Fekete-Szegö problem; q-derivative operator; q-Bessel function

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1. Introduction

The theory of q-analysis has important role in many areas of mathematics and physics, for example, in the areas of ordinary fractional calculus, optimal control problems, q-difference, q-integral equations and in q-transform analysis [1–4].

Let Σ denote the class of meromorphic functions of the form:

$$f(z)=\frac{1}{z}+\sum_{k=0}^{\infty}a_kz^k, \tag{1.1}$$

which are analytic in the open punctured unit disc $\mathbb{U}^*=\{z: z\in\mathbb{C} \text{ and } 0<|z|<1\}=\mathbb{U}\backslash\{0\}$.

A function $f\in\Sigma$ is meromorphic starlike of order α, denoted by $\Sigma^*(\alpha)$, if

$$-\Re\{\frac{zf'(z)}{f(z)}\}>\alpha, \quad 0\le\alpha<1;\ z\in\mathbb{U}.$$

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The class $\Sigma^*(\alpha)$ was introduced and studied by Pommerenke [5] (see also Miller [6]).

Let $\varphi(z)$ be an analytic function with positive real part on $\mathbb{U}$ satisfying $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps $\mathbb{U}$ onto a region starlike with respect to 1 and symmetric with respect to the real axis.

Let $\Sigma^*(\varphi)$ be the class of functions $f \in \Sigma$ for which

$$-\frac{zf'(z)}{f(z)} \prec \varphi(z), \quad z \in \mathbb{U}.$$

The class $\Sigma^*(\varphi)$ was introduced and studied by Silverman et al. [7]. We note that, the class $\Sigma^*(\alpha)$ is the special case of the class $\Sigma^*(\varphi)$ when $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \le \alpha < 1)$.

For a function $f(z) \in \Sigma$ given by (1.1) and $0 < q < 1$, the q-derivative of a function $f(z)$ is defined by [2]

$$D_q f(z) = \frac{f(qz) - f(z)}{(q-1)z}, \quad z \in \mathbb{U}^*. \tag{1.2}$$

From (1.2), we deduce that $D_q f(z)$ for a function $f(z)$ of the form (1.1) is given by

$$D_q f(z) = -\frac{1}{qz^2} + \sum_{k=0}^{\infty} [k]_q a_k z^{k-1}, \quad z \ne 0,$$

where

$$[k]_q = \frac{1-q^k}{1-q}.$$

As $q \to 1^-$, $[k]_q \to k$, we have $\lim_{q\to 1^-} D_q f(z) = f'(z)$.

Making use of the q-derivative D_q, we introduce the subclasses $\Sigma_q^*(\varphi)$ and $\Sigma_{q,\alpha}^*(\varphi)$ as follows:

Definition 1.1 *A function $f(z) \in \Sigma$ is said to be in the class $\Sigma_q^*(\varphi)$, if and only if*

$$-\frac{qzD_q f(z)}{f(z)} \prec \varphi(z), \quad z \in \mathbb{U}. \tag{1.3}$$

We note that:

(i) $\lim_{q\to 1^-} \Sigma_q^*(\varphi) = \Sigma^*(\varphi)$ (see [7] and [8, with $\alpha = 0$]);

(ii) $\lim_{q\to 1^-} \Sigma_q^*(\frac{1+(1-2\alpha)z}{1-z}) = \Sigma^*(\alpha)$ $(0 \le \alpha < 1)$ (see [5]);

(iii) $\lim_{q\to 1^-} \Sigma_q^*(\frac{1+z}{1-z}) = \mathcal{F}^*(1) = \mathcal{F}^*$ (see [9, with $b = 1$]);

(iv) $\lim_{q\to 1^-} \Sigma_q^*(\frac{1+\beta(1-2\gamma\delta)z}{1+\beta(1-2\gamma)z}) = \Sigma(\delta, \beta, \gamma)$ $(0 \le \delta < 1,\ 0 < \beta \le 1, \frac{1}{2} \le \gamma \le 1)$ (see [10]);

(v) $\lim_{q\to 1^-} \Sigma_q^*(\frac{1+Az}{1+Bz}) = K_1(A, B)$ $(0 \le B < 1, -B < A < B)$ (see [11]).

Definition 1.2 *For $\alpha \in \mathbb{C}\backslash(0, 1]$, let $\Sigma_{q,\alpha}^*(\varphi)$ be the subclass of Σ consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion:*

$$\frac{-(1-\frac{\alpha}{q})qzD_q f(z) + \alpha qzD_q[zD_q f(z)]}{(1-\frac{\alpha}{q})f(z) - \alpha zD_q f(z)} \prec \varphi(z). \tag{1.4}$$

From (1.3) and (1.4), we note that

$$\Sigma_{q,0}^*(\varphi) = \Sigma_q^*(\varphi)$$

and $\lim_{q\to 1^-} \Sigma_{q,\alpha}^*(\varphi) = \Sigma_\alpha^*(\varphi)$ (see [7]).

2. Fekete-Szegö problems

To prove our results, we need the following lemmas.

Lemma 2.1 ([12]) *If $p(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in $\mathbb{U}$ and μ is a complex number, then*

$$|c_2 - \mu c_1^2| \leq 2\max\{1; |2\mu - 1|\}.$$

The result is sharp for the functions given by

$$p(z) = \frac{1+z^2}{1-z^2} \text{ and } p(z) = \frac{1+z}{1-z}.$$

Lemma 2.2 ([12]) *If $p_1(z) = 1 + c_1 z + c_2 z^2 + \cdots$ is a function with positive real part in $\mathbb{U}$, then*

$$|c_2 - \nu c_1^2| \leq \begin{cases} -4\nu + 2, & \text{if } \nu \leq 0, \\ 2, & \text{if } 0 \leq \nu \leq 1, \\ 4\nu - 2, & \text{if } \nu \geq 1. \end{cases}$$

When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p_1(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p_1(z) = \frac{1+z^2}{1-z^2}$ or one of its rotations. If $\nu = 0$, the equality holds if and only if

$$p_1(z) = (\frac{1}{2} + \frac{\lambda}{2})\frac{1+z}{1-z} + (\frac{1}{2} - \frac{\lambda}{2})\frac{1-z}{1+z}, \quad 0 \leq \lambda \leq 1,$$

or one of its rotations. If $\nu = 1$, the equality holds if and only if

$$\frac{1}{p_1(z)} = (\frac{1}{2} + \frac{\lambda}{2})\frac{1+z}{1-z} + (\frac{1}{2} - \frac{\lambda}{2})\frac{1-z}{1+z}, \quad 0 \leq \lambda \leq 1,$$

or one of its rotations. Also the above upper bound is sharp and it can be improved as follows when $0 < \nu < 1$:

$$|c_2 - \nu c_1^2| + \nu |c_1|^2 \leq 2, \quad 0 < \nu \leq \frac{1}{2}$$

and

$$|c_2 - \nu c_1^2| + (1-\nu)|c_1|^2 \leq 2, \quad \frac{1}{2} < \nu < 1.$$

Unless otherwise mentioned, we assume throughout this paper that $\alpha \in \mathbb{C}\backslash(0,1]$ and $0 < q < 1$.

Theorem 2.3 *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots (B_1 \geq 0)$. If $f(z)$ given by (1.1) belongs to the class $\Sigma_q^*(\varphi)$ and μ is a complex number, then*

$$|a_1 - \mu a_0^2| \leq \frac{|B_1|}{(1+q)} \max\{1, |\frac{B_2}{B_1} - [1 - \mu(1+q)]B_1|\}, \quad B_1 \neq 0, \tag{2.1}$$

$$|a_1| \leq \frac{|B_2|}{(1+q)}, \quad B_1 = 0. \tag{2.2}$$

The result is sharp.

Proof If $f(z) \in \Sigma_q^*(\varphi)$, then there is a Schwarz function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$ and such that

$$-\frac{qzD_qf(z)}{f(z)} = \varphi(w(z)). \tag{2.3}$$

Define the function $p_1(z)$ by

$$p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + \cdots. \tag{2.4}$$

Since $w(z)$ is a Schwarz function, we see that $\Re\{p_1(z)\} > 0$ and $p_1(0) = 1$.

Define

$$p(z) = -\frac{qzD_qf(z)}{f(z)} = 1 + b_1z + b_2z^2 + \cdots. \tag{2.5}$$

In view of (2.3)–(2.5), we have

$$p(z) = \varphi(\frac{p_1(z) - 1}{p_1(z) + 1}). \tag{2.6}$$

Since

$$\frac{p_1(z) - 1}{p_1(z) + 1} = \frac{1}{2}[c_1z + (c_2 - \frac{c_1^2}{2})z^2 + (c_3 + \frac{c_1^3}{4} - c_1c_2)z^3 + \cdots].$$

Therefore, we have

$$\varphi(\frac{p_1(z) - 1}{p_1(z) + 1}) = 1 + \frac{1}{2}B_1c_1z + [\frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2c_1^2]z^2 + \cdots. \tag{2.7}$$

From (2.6) and (2.7), we obtain

$$b_1 = \frac{1}{2}B_1c_1,$$

and

$$b_2 = \frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2c_1^2.$$

Then, from (2.5) and (1.1), we see that $b_1 = -a_0$, and $b_2 = a_0^2 - (q + 1)a_1$, or, equivalently, we have

$$a_0 = -\frac{B_1c_1}{2}, \tag{2.8}$$

and

$$a_1 = -\frac{B_1}{2(1 + q)}[c_2 - \frac{c_1^2}{2}(1 - \frac{B_2}{B_1} + B_1)]. \tag{2.9}$$

Therefore

$$a_1 - \mu a_0^2 = -\frac{B_1}{2(1 + q)}\{c_2 - \nu c_1^2\},$$

where

$$\nu = \frac{1}{2}[1 - \frac{B_2}{B_1} + B_1 - \mu B_1(1 + q)]. \tag{2.10}$$

Now, the result (2.1) follows by an application of Lemma 2.1. Also, if $B_1 = 0$, then

$$a_0 = 0, \quad a_1 = -\frac{B_2c_1^2}{4(1 + q)}.$$

Since $p(z)$ has positive real part, then $|c_1| \leq 2$ (see [13]). Hence

$$|a_1| \leq \frac{|B_2|}{1 + q},$$

this proving (2.2). The result is sharp for the functions

$$-\frac{qzD_qf(z)}{f(z)}=\varphi(z^2),\quad -\frac{qzD_qf(z)}{f(z)}=\varphi(z).$$

This completes the proof of Theorem 2.3. □

Remark 2.4 For $q\to 1^-$ in Theorem 2.3, we obtain the result obtained by [7, Theorem 2.1].

Putting $q\to 1^-$ and $\varphi(z)=\frac{1+z}{1-z}$ in Theorem 2.3, we obtain the following corollary.

Corollary 2.5 *If $f(z)$ given by (1.1) belongs to the class $\mathcal{F}^*$ and μ is a complex number, then*

$$|a_1-\mu a_0^2|\le \max\{1,|1-2(1-2\mu)|\}.$$

The result is sharp.

By using Lemma 2.2, we can obtain the following theorem.

Theorem 2.6 *Let $\varphi(z)=1+B_1z+B_2z^2+\cdots$ $(B_i>0,i=1,2)$. If $f(z)$ given by (1.1) belongs to the class $\Sigma_q^*(\varphi)$, then*

$$|a_1-\mu a_0^2|\le\begin{cases}\frac{B_2-[1-\mu(1+q)]B_1^2}{1+q}, & \text{if } \mu\le\sigma_1,\\ \frac{B_1}{1+q}, & \text{if } \sigma_1\le\mu\le\sigma_2,\\ \frac{-B_2+[1-\mu(1+q)]B_1^2}{1+q}, & \text{if } \mu\ge\sigma_2,\end{cases}\tag{2.11}$$

where

$$\sigma_1=\frac{-B_1-B_2+B_1^2}{(1+q)B_1^2},\quad \sigma_2=\frac{B_1-B_2+B_1^2}{(1+q)B_1^2}.$$

The result is sharp. Further, let $\sigma_3=\frac{-B_2+B_1^2}{(1+q)B_1^2}$.

(i) If $\sigma_1\le\mu\le\sigma_3$, then

$$|a_1-\mu a_0^2|+\frac{\{(B_1+B_2)+[\mu(q+1)-1]B_1^2\}|a_0|^2}{(1+q)B_1^2}\le\frac{B_1}{1+q}.\tag{2.12}$$

(ii) If $\sigma_3\le\mu\le\sigma_2$, then

$$|a_1-\mu a_0^2|+\frac{\{(B_1-B_2)+[1-\mu(q+1)]B_1^2\}|a_0|^2}{(1+q)B_1^2}\le\frac{B_1}{1+q}.\tag{2.13}$$

Proof First, let $\mu\le\sigma_1$. Then

$$\begin{aligned}|a_1-\mu a_0^2|&\le\frac{B_1}{(1+q)}\{\frac{B_2}{B_1}-[1-\mu(1+q)]B_1\}\\&\le\frac{B_2-[1-\mu(1+q)]B_1^2}{1+q}.\end{aligned}$$

Let, now $\sigma_1\le\mu\le\sigma_2$. Then, using the above calculations, we obtain

$$|a_1-\mu a_0^2|\le\frac{B_1}{1+q}.$$

Finally, if $\mu\ge\sigma_2$, then

$$|a_1-\mu a_0^2|\le\frac{B_1}{(1+q)}\{-\frac{B_2}{B_1}+[1-\mu(1+q)]B_1\}\le\frac{-B_2+[1-\mu(1+q)]B_1^2}{1+q}.$$

To show that the bounds are sharp, we define the functions $K_{\varphi n}$ $(n \geq 2)$ by

$$-\frac{qzD_qK_{\varphi n}(z)}{K_{\varphi n}(z)} = \varphi(z^{n-1})(z^2K_{\varphi n}(z)|_{z=0} = 0 = -z^2K'_{\varphi n}(z)|_{z=0} - 1)$$

and the functions F_γ and G_γ $(0 \leq \gamma \leq 1)$ by

$$-\frac{qzD_qF_\gamma(z)}{F_\gamma(z)} = \varphi(\frac{z(z+\gamma)}{1+\gamma z})(z^2F_\gamma(z)|_{z=0} = 0 = -z^2F'_\gamma(z)|_{z=0} - 1)$$

and

$$-\frac{qzD_qG_\gamma(z)}{G_\gamma(z)} = \varphi(-\frac{z(z+\gamma)}{1+\gamma z})(z^2G_\gamma(z)|_{z=0} = 0 = -z^2G'_\gamma(z)|_{z=0} - 1).$$

Clearly, the functions $K_{\varphi n}$, F_γ and $G_\gamma \in \Sigma_q^*(\varphi)$. Also we write $K_\varphi = K_{\varphi 2}$. If $\mu < \sigma_1$ or $\mu > \sigma_2$, then the equality holds if and only if $f(z)$ is K_φ or one of its rotations. When $\sigma_1 < \mu < \sigma_2$, then the equality holds if $f(z)$ is $K_{\varphi 3}$ or one of its rotations. If $\mu = \sigma_1$, then the equality holds if and only if $f(z)$ is F_γ or one of its rotations. If $\mu = \sigma_2$, then the equality holds if and only if $f(z)$ is G_γ or one of its rotations. This completes the proof of Theorem 2.6. □

Remark 2.7 (i) For $q \to 1^-$ in Theorem 2.6, we obtain the result obtained by [8, Theorem 5.1];

(ii) Putting $q \to 1^-$ and $\varphi(z) = \frac{1+z}{1-z}$ in Theorem 2.6, we obtain a new result for the class $\mathcal{F}^*$.

Theorem 2.8 *Let $\varphi(z) = 1 + B_1z + B_2z^2 + \cdots$ $(B_1 \geq 0)$. If $f(z)$ given by (1.1) belongs to the class $\Sigma_{q,\alpha}^*(\varphi)$ and μ is a complex number, then*

$$|a_1 - \mu a_0^2| \leq (\frac{q}{1+q})|\frac{B_1}{q-\alpha(1+q)}|\max\{1, |\frac{B_2}{B_1} - [1 - \mu\frac{q(1+q)[q-\alpha(1+q)]}{(q-\alpha)^2}]B_1|\}, \quad B_1 \neq 0, \tag{2.14}$$

$$|a_1| \leq (\frac{q}{1+q})|\frac{B_2}{q-\alpha(1+q)}|, \quad B_1 = 0. \tag{2.15}$$

The result is sharp.

Proof If $f(z) \in \Sigma_{q,\alpha}^*(\varphi)$, then there is a Schwarz function $w(z)$ in $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ in $\mathbb{U}$ and such that

$$\frac{-(1-\frac{\alpha}{q})qzD_qf(z) + \alpha qzD_q[zD_qf(z)]}{(1-\frac{\alpha}{q})f(z) - \alpha zD_qf(z)} = \varphi(w(z)).$$

Since $w(z)$ is a Schwarz function, we see that $\Re\{p_1(z)\} > 0$ and $p_1(0) = 1$.

Define

$$p(z) = \frac{-(1-\frac{\alpha}{q})qzD_qf(z) + \alpha qzD_q[zD_qf(z)]}{(1-\frac{\alpha}{q})f(z) - \alpha zD_qf(z)} = 1 + b_1z + b_2z^2 + \cdots. \tag{2.16}$$

Then, from (2.6), (2.7), (2.16) and (1.1), we see that

$$\frac{1}{2}B_1c_1 = -(1-\frac{\alpha}{q})a_0,$$

and

$$\frac{1}{2}B_1(c_2 - \frac{c_1^2}{2}) + \frac{1}{4}B_2c_1^2 = (1-\frac{\alpha}{q})^2a_0^2 - (1+q)(1-\alpha-\frac{\alpha}{q})a_1,$$

or, equivalently, we have

$$a_0 = -\frac{qB_1c_1}{2(q-\alpha)}, \quad a_1 = -\frac{qB_1}{2(1+q)[q-\alpha(1+q)]}[c_2 - \frac{c_1^2}{2}(1 - \frac{B_2}{B_1} + B_1)].$$

Therefore

$$a_1 - \mu a_0^2 = -\frac{qB_1}{2(1+q)[q-\alpha(1+q)]}\{c_2 - \upsilon c_1^2\},$$

where

$$\nu = \frac{1}{2}[1 - \frac{B_2}{B_1} + B_1(1 - \mu\frac{q(1+q)[q-\alpha(1+q)]}{(q-\alpha)^2})]. \tag{2.17}$$

Now, the result (2.14) follows by an application of Lemma 2.1. Also, if $B_1 = 0$, then

$$a_0 = 0, \quad a_1 = \frac{qB_2c_1^2}{4(1+q)[q-\alpha(1+q)]}.$$

Since $p(z)$ has positive real part, then $|c_1| \le 2$ (see [13]), hence

$$|a_1| \le (\frac{q}{1+q})|\frac{B_2}{q-\alpha(1+q)}|,$$

this proving (2.15). The result is sharp for the functions

$$\frac{-(1-\frac{\alpha}{q})qzD_qf(z) + \alpha qzD_q[zD_qf(z)]}{(1-\frac{\alpha}{q})f(z) - \alpha zD_qf(z)} = \varphi(z^2)$$

and

$$\frac{-(1-\frac{\alpha}{q})qzD_qf(z) + \alpha qzD_q[zD_qf(z)]}{(1-\frac{\alpha}{q})f(z) - \alpha zD_qf(z)} = \varphi(z).$$

This completes the proof of Theorem 2.8. □

Remark 2.9 (i) Putting $\alpha = 0$ in Theorem 2.8, we obtain the result of Theorem 2.3;

(ii) Putting $q \to 1^-$ in Theorem 2.8, we obtain the result obtained by [7, Theorem 2.2].

Using arguments similar to those in the proof of Theorem 2.6, we obtain the following theorem.

Theorem 2.10 *Let $\varphi(z) = 1 + B_1z + B_2z^2 + \cdots$ $(B_i > 0, i \in 1, 2, 0 < \alpha < \frac{q}{1+q})$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{q,\alpha}(\varphi)$, then*

$$|a_1 - \mu a_0^2| \le \begin{cases} \frac{q}{(1+q)[q-\alpha(1+q)]}\{B_2 - [1 - \mu\frac{q(1+q)[q-\alpha(1+q)]}{(q-\alpha)^2}]B_1^2\}, & \text{if } \mu \le \sigma_4, \\ \frac{qB_1}{(1+q)[q-\alpha(1+q)]}, & \text{if } \sigma_4 \le \mu \le \sigma_5, \\ \frac{q}{(1+q)[q-\alpha(1+q)]}\{-B_2 + [1 - \mu\frac{q(1+q)[q-\alpha(1+q)]}{(q-\alpha)^2}]B_1^2\}, & \text{if } \mu \ge \sigma_5, \end{cases}$$

where

$$\sigma_4 = \frac{(q-\alpha)^2[-B_1 - B_2 + B_1^2]}{q(1+q)[q-\alpha(1+q)]B_1^2}, \quad \sigma_5 = \frac{(q-\alpha)^2[B_1 - B_2 + B_1^2]}{q(1+q)[q-\alpha(1+q)]B_1^2}.$$

The result is sharp. Further, let

$$\sigma_6 = \frac{(q-\alpha)^2[-B_2 + B_1^2]}{q(1+q)[q-\alpha(1+q)]B_1^2}.$$

(i) *If* $\sigma_4 \leq \mu \leq \sigma_6$, *then*

$$|a_1 - \mu a_0^2| + \frac{(q-\alpha)^2}{q(1+q)[q-\alpha(1+q)]B_1^2} \times$$
$$\{(B_1 + B_2) + [\mu \frac{q(1+q)[q-\alpha(1+q)]}{(q-\alpha)^2} - 1]B_1^2\}|a_0|^2 \leq \frac{qB_1}{(1+q)[q-\alpha(1+q)]}.$$

(ii) *If* $\sigma_6 \leq \mu \leq \sigma_5$, *then*

$$|a_1 - \mu a_0^2| + \frac{(q-\alpha)^2}{q(1+q)[q-\alpha(1+q)]B_1^2} \times$$
$$\{(B_1 - B_2) + [1 - \mu \frac{q(1+q)[q-\alpha(1+q)]}{(q-\alpha)^2}]B_1^2\}|a_0|^2 \leq \frac{qB_1}{(1+q)[q-\alpha(1+q)]}.$$

3. Applications to functions defined by q-Bessel function

We recall some definitions of q-calculus which will be used in our paper.

For any complex number α, the q-shifted factorials are defined by

$$(\alpha; q)_0 = 1; \ (\alpha; q)_n = \prod_{k=0}^{n-1} (1 - \alpha q^k), \quad n \in \mathbb{N} = \{1, 2, \ldots\}. \tag{3.1}$$

If $|q| < 1$, the definition (3.1) remains meaningful for $n = \infty$ as a convergent infinite product

$$(\alpha; q)_\infty = \prod_{j=0}^{\infty} (1 - \alpha q^j).$$

In terms of the analogue of the gamma function

$$(q^\alpha; q)_n = \frac{\Gamma_q(\alpha + n)(1-q)^n}{\Gamma_q(\alpha)}, \quad n > 0,$$

where the q-gamma function is defined by

$$\Gamma_q(x) = \frac{(q; q)_\infty (1-q)^{1-x}}{(q^x; q)_\infty}, \quad 0 < q < 1.$$

We note that $\lim_{q \to 1^-} \frac{(q^\alpha; q)_n}{(1-q)^n} = (\alpha)_n$, where

$$(\alpha)_n = \begin{cases} 1, & \text{if } n = 0, \\ \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+n-1), & \text{if } n \in \mathbb{N}. \end{cases}$$

Now, consider the q-analoge of Bessel function defined by [3]

$$\mathcal{J}_v^{(1)}(z; q) = \frac{(q^{v+1}; q)_\infty}{(q; q)_\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{(q; q)_k (q^{v+1}; q)_k} (\frac{z}{2})^{2k+\nu}, \quad 0 < q < 1.$$

Also, let us define

$$\mathcal{L}_v(z; q) = \frac{2^v (q; q)_\infty}{(q^{v+1}; q)_\infty (1-q)^v z^{v/2+1}} \mathcal{J}_v^{(1)}(z^{1/2}(1-q); q)$$
$$= \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(1-q)^{2(k+1)}}{4^{(k+1)}(q; q)_{k+1}(q^{v+1}; q)_{k+1}} z^k, \quad z \in \mathbb{U}.$$

By using the Hadamard product (or convolution), we define the linear operator $\mathcal{L}_{q,\upsilon} : \Sigma \to \Sigma$, as follows:

$$(\mathcal{L}_{q,\upsilon} f)(z) = \mathcal{L}_{\upsilon}(z;q) * f(z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(1-q)^{2(k+1)}}{4^{(k+1)}(q;q)_{k+1}(q^{\upsilon+1};q)_{k+1}} a_k z^k.$$

The linear operator $\mathcal{L}_{q,\upsilon}$ was introduced and studied by Mostafa et al. [14]. Also, as $q \to 1^-$, the linear operator $\mathcal{L}_{q,\upsilon}$ reduces to the operator $\mathcal{L}_{\upsilon}$ introduced and studied by Aouf et al. [15].

For $0 < q < 1$ and $\alpha \in \mathbb{C} \backslash (0, 1]$, let $\Sigma^*_{q,\upsilon}(\varphi)$ and $\Sigma^*_{q,\alpha,\upsilon}(\varphi)$ be the subclasses of Σ consisting of functions $f(z)$ of the form (1.1) and satisfy the analytic criterions, respectively:

$$-\frac{qzD_q(\mathcal{L}_{q,\upsilon} f)(z)}{(\mathcal{L}_{q,\upsilon} f)(z)} \prec \varphi(z), \quad z \in \mathbb{U},$$

$$\frac{-(1-\frac{\alpha}{q})qzD_q(\mathcal{L}_{q,\upsilon} f) + \alpha qzD_q[zD_q(\mathcal{L}_{q,\upsilon} f)]}{(1-\frac{\alpha}{q})f(z) - \alpha zD_q(\mathcal{L}_{q,\upsilon} f)(z)} \prec \varphi(z), \quad z \in \mathbb{U}.$$

Using similar arguments to those in the proof of the above theorems, we obtain the following theorems.

Theorem 3.1 *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ $(B_1 \geq 0)$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{q,\upsilon}(\varphi)$ and μ is a complex number, then*

$$\begin{aligned} |a_1 - \mu a_0^2| \leq & \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})|B_1|}{(1-q)^2} \times \\ & \max\{1, |\frac{B_2}{B_1} - [1 - \mu\frac{(1-q^{\upsilon+1})}{(1-q^{\upsilon+2})}]B_1|\}, \quad B_1 \neq 0, \end{aligned}$$

$$|a_1| \leq \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})|B_2|}{(1-q)^2}, \quad B_1 = 0.$$

The result is sharp.

Theorem 3.2 *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ $(B_i > 0, i = 1, 2)$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{q,\upsilon}(\varphi)$, then*

$$|a_1 - \mu a_0^2| \leq \begin{cases} \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})}{(1-q)^2}\{B_2 - [1 - \mu(\frac{1-q^{\upsilon+1}}{1-q^{\upsilon+2}})]B_1^2\}, & \text{if } \mu \leq \sigma_1^*, \\ \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})B_1}{(1-q)^2}, & \text{if } \sigma_1^* \leq \mu \leq \sigma_2^*, \\ \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})}{(1-q)^2}\{-B_2 + [1 - \mu(\frac{1-q^{\upsilon+1}}{1-q^{\upsilon+2}})]B_1^2\}, & \text{if } \mu \geq \sigma_2^*, \end{cases}$$

where

$$\sigma_1^* = \frac{[-B_1 - B_2 + B_1^2](1-q^{\upsilon+2})}{(1-q^{\upsilon+1})B_1^2}, \quad \sigma_2^* = \frac{[B_1 - B_2 + B_1^2](1-q^{\upsilon+2})}{(1-q^{\upsilon+1})B_1^2}.$$

The result is sharp. Further, let

$$\sigma_3^* = \frac{[-B_2 + B_1^2](1-q^{\upsilon+2})}{(1-q^{\upsilon+1})B_1^2}.$$

(i) If $\sigma_1^* \le \mu \le \sigma_3^*$, then

$$|a_1 - \mu a_0^2| + \frac{(1-q^{\upsilon+2})}{(1-q^{\upsilon+1})B_1^2}\{(B_1+B_2)+[\mu(\frac{1-q^{\upsilon+1}}{1-q^{\upsilon+2}})-1]B_1^2\}|a_0|^2 \le \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})B_1}{(1-q)^2}.$$

(ii) If $\sigma_3^* \le \mu \le \sigma_2^*$, then

$$|a_1 - \mu a_0^2| + \frac{(1-q^{\upsilon+2})}{(1-q^{\upsilon+1})B_1^2}\{(B_1-B_2)+[1-\mu(\frac{1-q^{\upsilon+1}}{1-q^{\upsilon+2}})]B_1^2\}|a_0|^2 \le \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})B_1}{(1-q)^2}.$$

Theorem 3.3 *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ $(B_1 \ge 0)$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{q,\alpha,\upsilon}(\varphi)$ and μ is a complex number, then*

$$|a_1 - \mu a_0^2| \le \frac{4^2 q(1-q^{\upsilon+1})(1-q^{\upsilon+2})|B_1|}{(1-q)^2|q-\alpha(1+q)|}\times \max\{1, |\frac{B_2}{B_1} - [1-\mu\frac{q(1-q^{\upsilon+1})[q-\alpha(1+q)]}{(\alpha-q)^2(1-q^{\upsilon+2})}]B_1|\}, \quad B_1 \ne 0,$$

$$|a_1| \le \frac{4^2 q(1-q^{\upsilon+1})(1-q^{\upsilon+2})|B_1|}{(1-q)^2|q-\alpha(1+q)|}, \quad B_1 = 0.$$

The result is sharp.

Theorem 3.4 *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \cdots$ $(B_i > 0, i = 1, 2, 0 < \alpha < \frac{q}{1+q})$. If $f(z)$ given by (1.1) belongs to the class $\Sigma^*_{q,\alpha,\upsilon}(\varphi)$, then*

$$|a_1 - \mu a_0^2| \le \begin{cases} \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})}{(1-q)^2[q-\alpha(1+q)]}\times & \\ \{B_2 - [1-\mu\frac{[\alpha+(1-\alpha)q](1-2\alpha)(1-q^{\upsilon+1})}{q(1-\alpha)^2(1-q^{\upsilon+2})}]B_1^2\}, & \text{if } \mu \le \sigma_4^*, \\ \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})B_1}{(1-q)^2[q-\alpha(1+q)]}, & \text{if } \sigma_4^* \le \mu \le \sigma_5^*, \\ \frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})}{(1-q)^2[q-\alpha(1+q)]}\times & \\ \{-B_2 + [1-\mu\frac{q(1-q^{\upsilon+1})[q-\alpha(1+q)]}{(q-\alpha)^2(1-q^{\upsilon+2})}]B_1^2\}, & \text{if } \mu \ge \sigma_5^*, \end{cases}$$

where

$$\sigma_4^* = \frac{(q-\alpha)^2(1-q^{\upsilon+2})[-B_1-B_2+B_1^2]}{qB_1^2(1-q^{\upsilon+1})[q-\alpha(1+q)]},$$

$$\sigma_5^* = \frac{(q-\alpha)^2(1-q^{\upsilon+2})[B_1-B_2+B_1^2]}{qB_1^2(1-q^{\upsilon+1})[q-\alpha(1+q)]}.$$

The result is sharp. Further, let

$$\sigma_6^* = \frac{(q-\alpha)^2(1-q^{\upsilon+2})[-B_2+B_1^2]}{qB_1^2(1-q^{\upsilon+1})[q-\alpha(1+q)]}.$$

(i) If $\sigma_4^* \le \mu \le \sigma_6^*$*, then*

$$|a_1-\mu a_0^2|+\frac{(q-\alpha)^2(1-q^{\upsilon+2})}{q(1-q^{\upsilon+1})[q-\alpha(1+q)]B_1^2}\times$$
$$\{(B_1+B_2)+[\mu\frac{q(1-q^{\upsilon+2})[q-\alpha(1+q)]}{(1-q^{\upsilon+1})(\alpha-q)^2}-1]B_1^2\}|a_0|^2$$
$$\le\frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})B_1}{(1-q)^2[q-\alpha(1+q)]}.$$

(ii) If $\sigma_6^* \le \mu \le \sigma_5^*$*, then*

$$|a_1-\mu a_0^2|+\frac{(q-\alpha)^2(1-q^{\upsilon+2})}{q(1-q^{\upsilon+1})[q-\alpha(1+q)]B_1^2}\times$$
$$\{(B_1-B_2)+[1-\mu\frac{q(1-q^{\upsilon+2})[q-\alpha(1+q)]}{(1-q^{\upsilon+1})(\alpha-q)^2}]B_1^2\}|a_0|^2$$
$$\le\frac{4^2(1-q^{\upsilon+1})(1-q^{\upsilon+2})B_1}{(1-q)^2[q-\alpha(1+q)]}.$$

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