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G-Kernel-Open Sets, G-Kernel-Neighborhoods and G-Kernel-Derived Sets

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Abstract Based on G-hulls and G-kernels under the meaning of G-methods on sets, we introduce the concepts of G-hull-closed sets, G-kernel-open sets, G-kernel-neighborhoods and G-kernel-derived sets, discuss some related properties. In particular, we define pointwise G-methods, prove the consistency of G-closed sets and G-hull-closed sets, G-open sets and G-kernel-open sets, G-neighborhoods and G-kernel-neighborhoods, G-derived sets and G-kernel-derived sets under this method, and enrich some results about G-closed sets, G-open sets, G-interiors, G-neighborhoods and G-derived sets in sets. At the same time, we put forward some problems for further research.

Keywords topological spaces; G-methods; pointwise G-methods; G-hull-closed sets; G-kernel-open sets; G-kernel-neighborhoods; G-kernel-derived sets

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1. Introduction

Sequential convergence is an important research object in topology and analysis. On one hand, convergence is closely related to continuity, compactness and other related properties. On the other hand, it has played a fundamental role in mathematics and its applications. As a generalization of convergence, Zygmund [1] put forward the thought of statistical convergence. Fast [2] and Steinhaus [3] introduced the concepts of statistical convergence in real number and complex number space independently. Di Maio and Kočinac [4] defined statistical convergence in topological spaces. Tang and Lin [5] discussed the statistically sequential spaces and statistically Fréchet-Urysohn spaces in topological spaces. Recently, Renukadevi and Prakash [6] introduced the concept of statistically sequentially quotient mappings. Besides the ordinary and statistical convergence, there exists a wide variety of convergence, for example, A-convergence of the matrix method in summability theory, almost convergence in functional analysis, Cesàro convergence [7] in real analysis and so on. Based on several kinds of convergence properties of real analysis, Connor and Grosse-Erdmann [8] introduced G-methods and G-convergences defined on a linear subspace of the vector space of all real sequences. Since then, Çakallı [9] defined the concepts of G-accumulation points, G-derived sets and G-boundaries on Hausdorff topological groups satisfying the first axiom of countability. At the same time, he also discussed G-continuity by

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means of G-closures and G-closed sets. Mucuk and Şahan [10] introduced the notions of G-open sets and G-neighborhoods of first-countable topological groups, studied the operations of G-closed sets and G-open sets, and gave a further investigation of G-continuity in topological groups.

Recently, Lin and Liu [11] introduced the concepts of G-method and G-convergence in topological spaces, gave the definitions of G-neighborhoods and a series of related contents, and studied the properties of G-continuity of mappings and so on. As we all know, open sets, closed sets, neighborhoods and derived sets are important collections to describe topology of spaces, define convergence and depict continuity in topological spaces. Therefore, Liu [12,13] discussed some properties of G-neighborhoods, G-continuity at a point, G-derived sets and G-boundaries of a set. These properties are educed by G-closures and G-interiors of a set. For another, we can use G-methods to generate G-hulls and G-kernels. In this paper, we introduce the concepts of G-hull-closed sets, G-kernel-open sets, G-kernel-neighborhoods and G-kernel-derived sets, discuss some relative properties. Especially, we pay attention to relationships among these collections under certain conditions. For example, we define pointwise G-methods, prove the consistency of G-closed sets and G-hull-closed sets, G-open sets and G-kernel-open sets, G-neighborhoods and G-kernel-neighborhoods, G-derived sets and G-kernel-derived sets under this method, enrich some results about G-closed sets, G-open sets, G-interiors, G-neighborhoods and G-derived sets in sets.

1. Basic concepts and lemmas

Let X be a set and $s(X)$ denote the set of all X-valued sequences, i.e., $\boldsymbol{x} \in s(X)$ if and only if $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}}$ is a sequence with each $x_n \in X$. If X is a topological space, $c(X)$ denotes the set of all X-valued convergent sequences. By a method on X, we mean an additive function $G : c_G(X) \to X$ defined on a subset $c_G(X)$ of $s(X)$ into X (see [11]). A sequence $\boldsymbol{x}$ on X is said to be G-convergent to $l \in X$ if $\boldsymbol{x} \in c_G(X)$ and $G(\boldsymbol{x}) = l$ (see [11]). A method $G : c_G(X) \to X$ is called regular if $c(X) \subset c_G(X)$ and $G(\boldsymbol{x}) = \lim \boldsymbol{x}$ for each $\boldsymbol{x} \in c(X)$ (see [11]). A method $G : c_G(X) \to X$ is called subsequential if whenever $\boldsymbol{x} \in c_G(X)$ is G-convergent to $l \in X$, then there exists a subsequence $\boldsymbol{x}' \in c(X)$ of $\boldsymbol{x}$ with $\lim \boldsymbol{x}' = l$ (see [11]). The G-method with the name "convergence" is only a function relation. It is not related with the topology on a space X. Based on the regular or subsequential methods, we can establish close ties between G-convergence and convergent sequences on X.

We recall definitions and properties on some special subsets of G-methods in sets.

Definition 1.1 ([11]) *Let G be a method on a set X. For each $A \subset X$,*

(1) A is called a G-closed set of X if, whenever $\boldsymbol{x} \in s(A) \cap c_G(X)$, then $G(\boldsymbol{x}) \in A$.

(2) The G-closure of A is defined as the intersection of all G-closed sets containing A, and the G-closure of A is denoted by $\mathrm{cl}_G(A)$ or $\overline{A}^G$.

(3) The G-hull of A is defined as the set $\{G(\boldsymbol{x}) : \boldsymbol{x} \in s(A) \cap c_G(X)\}$, and the G-hull of A is denoted by $\mathrm{hu}_G(A)$ or $[A]_G$.

Obviously, $A \subset \overline{A}^G$, if $A \subset B \subset X$, then $[A]_G \subset [B]_G$.

Lemma 1.2 ([11]) *Let G be a method on a set X. If $A \subset X$, then A is a G-closed set if and only if $\overline{A}^G \subset A$ (i.e., $\overline{A}^G = A$), if and only if $[A]_G \subset A$.*

Thus, $[A]_G \subset \overline{A}^G$ (see [11]).

Definition 1.3 ([11]) *Let G be a method on a set X. For each $A \subset X$,*

(1) A is called a G-open set if $X \setminus A$ is G-closed in X.

(2) The G-interior of A is defined as the union of all G-open sets contained in A, and the G-interior of A is denoted by $\mathrm{int}_G(A)$ or $A^{\circ G}$.

(3) The G-kernel of A is defined as the set

$$\{l \in X: \text{ there is no } \boldsymbol{x} \in s(X \setminus A) \cap c_G(X) \text{ with } l = G(\boldsymbol{x})\},$$

and the G-kernel of A is denoted by $\ker_G(A)$ or $(A)_G$.

Obviously, $A^{\circ G} \subset A$, if $A \subset B \subset X$, then $(A)_G \subset (B)_G$.

Lemma 1.4 ([11]) *Let G be a method on a set X. If $A \subset X$, then A is a G-open set if and only if $A \subset A^{\circ G}$ (i.e., $A^{\circ G} = A$), if and only if $A \subset (A)_G$.*

Thus, $A^{\circ G} \subset (A)_G$ (see [11]).

Similar to the relationship between the closures and the interiors in topological spaces, there is a dual relation as follows.

Lemma 1.5 ([11]) *Let G be a method on a set X and $A \subset X$. Then*

(1) $A^{\circ G} = X \setminus \overline{X \setminus A}^G$.

(2) $(A)_G = X \setminus [X \setminus A]_G$.

Readers may refer to [14] for some terminology unstated here.

2. G-kernel-open sets

Lemma 1.4 shows that a subset A of a set X is a G-open set if and only if $A = A^{\circ G}$. A another "interior" $(A)_G$ of A, if it is equal to A, then what kind of situation would it be? In order to investigate this problem, we introduce the definition of G-kernel-open sets.

Definition 2.1 *Let G be a method on a set X. For each $A \subset X$,*

(1) A is called a G-kernel-open set if $A = (A)_G$.

(2) A is called a G-hull-closed set if $A = [A]_G$.

The following lemma is obvious by Lemma 1.5.

Lemma 2.2 *Let G be a method on a set X. If $A \subset X$, then A is a G-kernel-open set if and only if $X \setminus A$ is a G-hull-closed set.*

The following are some relations between G-kernel-open sets and G-hull-closed sets.

Theorem 2.3 *Let G be a method on a set X. If $A \subset X$, then the following are equivalent.*

(1) A is a G-kernel-open (resp., G-hull-closed) set.

(2) A is a G-open (resp., G-closed) set and $(A)_G \subset A$ (resp., $A \subset [A]_G$).

(3) A is a G-open(resp., G-closed) set and $(A)_G \subset A^{\circ G}$ (resp., $\overline{A}^G \subset [A]_G$).

Proof We only prove the situation of open sets by Lemmas 1.5 and 2.2.

(1) $\Rightarrow$ (2). Suppose that A is a G-kernel-open set in X, that is $(A)_G = A$. By Lemma 1.4, we know that A is a G-open set of X and $(A)_G \subset A$.

(2) $\Rightarrow$ (3). Suppose that A is a G-open set in X and $(A)_G \subset A$. By Lemma 1.4, $A = A^{\circ G}$, thus $(A)_G \subset A^{\circ G}$.

(3) $\Rightarrow$ (1). Suppose that A is a G-open set in X and $(A)_G \subset A^{\circ G}$. By Lemma 1.4, $A \subset (A)_G$ and $A = A^{\circ G}$, then $A = (A)_G$, thus A is a G-kernel-open set of X. $\square$

Since a G-open (resp., G-closed) set is not necessarily a G-kernel open (resp., G-hull closed) set, Example 5.3(1) shows that some conditions in Theorem 2.3 cannot be omitted.

Corollary 2.4 *Let G be a method on a set X. The following are equivalent.*

(1) For each $A \subset X$, $A^{\circ} = (A)_G$.

(2) For each $A \subset X$, $A^{\circ} = A^{\circ G} = (A)_G$.

(3) For each $A \subset X$, $(A)_G \subset A^{\circ}$ and A is an open set if and only if A is a G-open (or G-kernel open) set.

(4) For each $A \subset X$, $(A)_G \subset A^{\circ}$. If A is an open set, then A is a G-open (or G-kernel open) set.

Proof It follows from [11, Corollary 3.10] that (1) $\Leftrightarrow$ (2), and (2) $\Rightarrow$ (3) $\Rightarrow$ (4) is obvious. We need only prove (4) $\Rightarrow$ (1).

For each $A \subset X$, A° is always an open set, so A° is a G-open set by condition (4) and Theorem 2.3. Then we have $A^{\circ} \subset (A^{\circ})_G \subset (A^{\circ})^{\circ} = A^{\circ}$ by Lemma 1.4 and condition (4), therefore $(A)_G \subset A^{\circ} = (A^{\circ})_G \subset (A)_G$. Hence $A^{\circ} = (A)_G$. $\square$

Obviously, Corollary 2.4 has a dual representation about "closed set".

Example 5.2 shows that even if G is the ordinary convergence method on the topological space X (it is a regular subsequential method), we cannot guarantee: For each $A \subset X$, $(A)_G = A^{\circ G}$. Since $A^{\circ G} \subset (A)_G$ is always true, then $(A)_G = A^{\circ G}$ if and only if $(A)_G \subset A^{\circ G}$. There is an interesting question to seek a simple sufficient condition for $(A)_G \subset A^{\circ G}$.

As we all know, the union of any family of open sets on a set X is an open set. We can prove that the union of any family of G-open sets is a G-open set [11, Proposition 3.2]. Naturally, we have the following question.

Question 2.5 Let G be a method on a set X. Is the union of any family of G-kernel-open sets a G-kernel-open set?

Next, we give a partial answer to this question. In order to describe convenience, we call the method G on a set X a pointwise method, if for each $x \in X$, $x \in [\{x\}]_G$. Namely the constant sequence $\boldsymbol{x} = \{x, x, x, \ldots\}$ is G-convergent to x. Obviously, a regular method on a topological space X is a pointwise method; and the method which makes every singleton be a G-hull-closed set is also a pointwise method.

Theorem 2.6 *Let G be a pointwise method on a set X. Then, for $A \subset X$, we have*

(1) $(A)_G \subset A \subset [A]_G$;

(2) A is a G-open (resp., G-closed) set if and only if A is a G-kernel-open (resp., G-hull-closed) set.

Proof Since G is a pointwise method, then $A = \bigcup_{x\in A}\{x\} \subset \bigcup_{x\in A}[\{x\}]_G \subset [A]_G$. Thus $X \setminus A \subset [X \setminus A]_G$. By Lemma 1.5, $(A)_G \subset A$ for each $A \subset X$. So (1) is obtained. Then A is G-open (resp., G-closed) set if and only if A is a G-kernel-open (resp., G-hull-closed) set by (1) and Theorem 2.3. □

By Theorem 2.6, the results on G-open sets are also applicable to G-kernel-open sets. In particular, we have the following corollary.

Corollary 2.7 *If G is a pointwise method on a set X, then the union of any family of G-kernel-open sets is a G-kernel-open set.*

Proof Suppose $\{U_s\}_{s\in S}$ is a family of G-kernel-open sets of X. Since each $U_s \subset \bigcup_{s\in S} U_s$, then $U_s = (U_s)_G \subset (\bigcup_{s\in S} U_s)_G$, thus $\bigcup_{s\in S} U_s \subset (\bigcup_{s\in S} U_s)_G \subset \bigcup_{s\in S} U_s$ in which the last inclusion relation is derived from Theorem 2.6(1). Hence $\bigcup_{s\in S} U_s = (\bigcup_{s\in S} U_s)_G$ is a G-kernel-open set of X. □

It is a natural question: whether the intersection of two G-kernel-open sets is always a G-kernel-open set? Example 5.1(1) shows that the answer is negative even in the regular method.

Question 2.8 Let G be a method on a set X. If each G-open set is G-kernel-open set of X, is the G a pointwise method?

3. G-kernel-neighborhoods

The function of neighborhood systems is equivalent to the role of the family of open sets on topological spaces. Some properties on G-open sets and G-neighborhoods have been discussed in [12,13]. For completeness, corresponding to the G-kernel-open sets, we introduce the concept of G-kernel-neighborhoods.

Definition 3.1 *Let G be a method on a set X. For each $A \subset X$,*

(1) A is called a G-neighborhood of a point $x \in X$ if there exists a G-open set U with $x \in U \subset A$ (see [11]).

(2) A is called a G-kernel-neighborhood of a point $x \in X$ if there exists a G-kernel-open set U with $x \in U \subset A$.

The family of all G-neighborhoods of a point $x \in X$ are expressed as $\mathcal{U}^G(x)$ (see [12]). In this paper, the family of all G-kernel-neighborhoods of a point $x \in X$ are denoted as $\mathcal{U}_G(x)$.

The following are some properties on G-kernel-neighborhoods.

Theorem 3.2 *Let G be a pointwise method on a set X, $U \subset X$ and $x \in X$.*

(1) If U is a G-kernel-neighborhood of x, then U is a G-neighborhood of x, that is $\mathcal{U}_G(x) \subset$

$\mathcal{U}^G(x)$.

(2) If U is a G-kernel-open set, then U is a G-kernel-neighborhood of each point $x \in U$.

Proof (1) Suppose U is a G-kernel-neighborhood of x, then there exists a G-kernel-open set A with $x \in A \subset U$ by Definition 3.1. By Theorem 2.3, we know that A is a G-open set containing x, thus U is a G-neighborhood of x.

(2) By Definition 3.1, (2) is easy to prove. □

Example 5.3(1) shows that the converse of Theorem 3.2(1) is not always true. We have the following question.

Question 3.3 Let G be a method on a set X. If $U \subset X$ is a G-kernel-neighborhood of each point $x \in U$, is U a G-kernel-open set?

By Corollary 2.4, Theorem 3.2 and Corollary 2.7 we have the following corollary.

Corollary 3.4 *Let G be a pointwise method on a set X. If $U \subset X$, then*

(1) U is a G-kernel-neighborhood of a point $x \in X$ if and only if U is a G-neighborhood of x;

(2) U is a G-kernel-open set if and only if U is a G-kernel-neighborhood of each point $x \in U$.

The following theorem concentrates on some basic properties of $\mathcal{U}_G(x)$.

Theorem 3.5 *Let G be a method on a set X. For each $x \in X$,*

(1) $X \in \mathcal{U}_G(x)$;

(2) If $U \in \mathcal{U}_G(x)$, then $x \in U$;

(3) If $U \in \mathcal{U}_G(x)$ and $V \supset U$, then $V \in \mathcal{U}_G(x)$;

(4) If $U \in \mathcal{U}_G(x)$, then there exists a subset V such that for $x \in V \subset U$ and any $x' \in V$, $V \in \mathcal{U}_G(x')$.

Proof (1)–(3) is obvious. We only prove (4).

Suppose $U \in \mathcal{U}_G(x)$, by Definition 3.1, then there exists a G-kernel-open set V with $x \in V \subset U$, thus V is a G-kernel-neighborhood of any point $x \in V$, that is $V \in \mathcal{U}_G(x')$ for any $x' \in V$. □

Naturally, there is a question as follows: if $U, V \in \mathcal{U}_G(x)$, then $U \bigcap V \in \mathcal{U}_G(x)$? Example 5.1(2) shows the answer is negative.

Theorem 3.6 *Let G be a method on a set X, $A \subset X$. If $x \in [A]_G$, then for any G-kernel-neighborhood U of x, $U \bigcap A \neq \emptyset$.*

Proof Suppose there is a G-kernel-neighborhood U of x with $U \bigcap A = \emptyset$. Without loss of generality, we may assume that U is a G-kernel-open set, then $X \setminus U$ is a G-hull-closed set and $A \subset X \setminus U$, thus $[A]_G \subset [X \setminus U]_G \subset X \setminus U$, hence $x \notin [A]_G$. □

Example 5.3(2) shows that the converse of Theorem 3.6 is not always true.

Corollary 3.7 *Let G be a method on a set X. If U is a G-kernel-neighborhood of a point $x \in X$, then $x \notin [X \setminus U]_G$.*

Proof Suppose U is a G-kernel-neighborhood of a point $x \in X$. If $x \in [X \setminus U]_G$, then $U \bigcap (X \setminus U) \neq \emptyset$ by Theorem 3.5. It is a contradiction. □

Example 5.3(3) shows that the converse of Corollary 3.7 is not always true.

4. G-kernel-derived sets

First, we recall the notions of G-accumulation points and G-derived sets, then introduce the definitions of G-kernel-accumulation points and G-kernel-derived sets.

Definition 4.1 ([13]) *Let G be a method on a set X. For each $A \subset X$,*

(1) $x \in X$ is called a G-accumulation point of A if for any G-neighborhood U of x, $U \bigcap (A \setminus \{x\}) \neq \emptyset$.

(2) The G-derived set of A is defined as the set which consists of all G-accumulation points of A, and the G-derived set of A is denoted by A^{dG}.

Definition 4.2 *Let G be a method on a set X. For each $A \subset X$,*

(1) $x \in X$ is called a G-kernel-accumulation point of A if for any G-kernel-neighborhood U of x, $U \bigcap (A \setminus \{x\}) \neq \emptyset$.

(2) The G-kernel-derived set of A is defined as the set which consists of all G-kernel-accumulation points of A, and the G-kernel-derived set of A is denoted by $[A]_{dG}$.

Obviously, $x \in [A]_{dG}$ if and only if there is a point different from x in any G-kernel-neighborhood of x. By Corollary 3.4, if G is a pointwise method on a set X, then $A^{dG} = [A]_{dG}$ for any $A \subset X$.

The following theorem concentrates on some operation properties for G-kernel-derived sets.

Theorem 4.3 *Let G be a method on a set X. If $A, B \subset X$, then*

(1) $(\emptyset)_{dG} = \emptyset$;

(2) $A \subset B \Rightarrow [A]_{dG} \subset [B]_{dG}$;

(3) $[A]_{dG} \bigcup [B]_{dG} \subset [A \bigcup B]_{dG}$.

Proof (1) is obvious. We will prove (2) and (3).

(2) Suppose $A \subset B$. For any $x \in [A]_{dG}$, by Definition 4.2, we know that for any G-kernel-neighborhood U of x, $U \bigcap (A \setminus \{x\}) \neq \emptyset$, hence $U \bigcap (B \setminus \{x\}) \neq \emptyset$, so $x \in [B]_{dG}$, thus $[A]_{dG} \subset [B]_{dG}$.

(3) Since $A \subset A \bigcup B$ and $B \subset A \bigcup B$, then $[A]_{dG} \subset [A \bigcup B]_{dG}$ and $[B]_{dG} \subset [A \bigcup B]_{dG}$ by (2), thus $[A]_{dG} \bigcup [B]_{dG} \subset [A \bigcup B]_{dG}$. □

Naturally, we have the following question.

Question 4.4 Let G be a method on a set X. Does $[[A]_{dG}]_{dG} \subset A \bigcup [A]_{dG}$ hold for any $A \subset X$?

There are many similar properties between G-derived sets and derived sets [13]. For example, for each $A \subset X$, (1) $x \in A^{dG} \Leftrightarrow x \in \overline{A \setminus \{x\}}^G$; (2) $\overline{A}^G = A \bigcup A^{dG}$; (3) $(A^{dG})^{dG} \subset A \bigcup A^{dG}$. But for G-kernel-derived sets, the situation is a bit complicated. Next, we prove some properties of G-kernel-derived sets.

Theorem 4.5 *Let G be a method on a set X. If $A \subset X$, then $A^{dG} \subset [A]_{dG}$.*

Proof Let $x \in A^{dG}$. For any G-kernel-neighborhood U of x, by Theorem 3.2, U is a G-neighborhood of x, thus $U \bigcap (A \setminus \{x\}) \neq \emptyset$. By Definition 4.2, $x \in [A]_{dG}$, i.e., $A^{dG} \subset [A]_{dG}$. $\square$

Example 5.3(4) shows that the converse of Theorem 4.5 is not always true.

Theorem 4.6 *Let G be a method on a set X. For each $A \subset X$, if $x \in [A \setminus \{x\}]_G$, then $x \in [A]_{dG}$.*

Proof If $x \notin [A]_{dG}$, then there exists a G-kernel-neighborhood U of x with $U \bigcap (A \setminus \{x\}) = \emptyset$. Without loss of generality, we may assume that U is a G-kernel-open set, since $A \setminus \{x\} \subset X \setminus U$, then $[A \setminus \{x\}]_G \subset [X \setminus U]_G$. By Lemma 1.5, $[X \setminus U]_G = X \setminus (U)_G = X \setminus U$. And because $x \in U$, then $x \notin [A \setminus \{x\}]_G$. $\square$

Example 5.3(5) shows that the converse of Theorem 4.6 is not always true.

Theorem 4.7 *Let G be a method on a set X. If $A \subset X$, then $[A]_G \subset A \bigcup [A]_{dG}$.*

Proof Let $x \in [A]_G \setminus A$. By Theorem 3.6, for any G-kernel-neighborhood U of x, $U \bigcap (A \setminus \{x\}) = U \bigcap A \neq \emptyset$, therefore, $x \in [A]_{dG}$. Thus $[A]_G \setminus A \subset [A]_{dG}$, i.e., $[A]_G \subset A \bigcup [A]_{dG}$. $\square$

The converse of Theorem 4.7 is not always true. In fact, there is a method G on X and $A \subset X$ such that $A \not\subset [A]_G$, $[A]_{dG} \not\subset [A]_G$, see Example 5.3(6).

5. Some examples

In this section, some examples are given to illustrate some relations mentioned in the previous sections, and the G-continuity is discussed.

Example 5.1 Let X be the set of all real numbers endowed with the usual topology. Put

$$c_G(X) = \{\{x_n\}_{n \in \mathbb{N}} \in s(X) : \{x_n + x_{n+1}\}_{n \in \mathbb{N}} \in c(X)\}.$$

Define $G : c_G(X) \to X$ by $G(\boldsymbol{x}) = \lim_{n \to \infty} \frac{x_n + x_{n+1}}{2}$, $\forall \boldsymbol{x} = \{x_n\}_{n \in \mathbb{N}} \in c_G(X)$. Then G is a regular method on X.

(1) The intersection of two G-kernel-open sets is not always a G-kernel-open set.

Obviously, $[\{0\}]_G = \{0\}$, $[\{1\}]_G = \{1\}$ and $[\{0,1\}]_G = \{0, 1/2, 1\}$. Then $\{0\}$ and $\{1\}$ are G-hull-closed sets in X, but $\{0,1\}$ is not G-hull-closed. Let $U = X \setminus \{0\}$ and $V = X \setminus \{1\}$. By Lemma 2.2, U and V are G-kernel-open sets in X, but $U \bigcap V = X \setminus \{0,1\}$ is not a G-kernel-open set.

(2) The intersection of two G-kernel-neighborhoods of a point is not always a G-kernel-neighborhood.

Since U and V are the G-kernel-open sets in (1), then $U, V \in \mathcal{U}_G(\frac{1}{2})$. If $U \bigcap V \in \mathcal{U}_G(\frac{1}{2})$, then there exists a G-kernel-open set W with $1/2 \in W \subset U \bigcap V$, thus $\{0,1\} \subset X \setminus W$, hence $1/2 \in [\{0,1\}]_G \subset [X \setminus W]_G = X \setminus W$, it is a contradiction. So $U \bigcap V$ is not a G-kernel-neighborhood of $1/2$. □

Example 5.2 ([11, Example 2.13(3)]) There is the ordinal convergence method G on a topological space X such that $(A)_G \neq A^{\circ G}$ for some $A \subset X$.

Let $X = \{0\} \cup \bigcup_{i\in\mathbb{N}} X_i$, where $X_i = \{1/i\} \cup \{1/i + 1/k : k \in \mathbb{N}, k \geq i^2\}$ for each $i \in \mathbb{N}$; suppose that X is endowed with the following topology.

(1) Each point of the form $1/i + 1/j$ is isolated.

(2) Each neighborhood of each point of the form $1/i$ contains a set of the form $\{1/i\} \cup \{1/i + 1/k : k \geq j\}$, where $j \geq i^2$.

(3) Each neighborhood of the point 0 contains a set obtained from X by removing a finite number of X_i's and a finite number of points of the form $1/i + 1/j$ in all the remaining X_i's.

The topological space X is called Arens' space and is denoted by S_2 (see [14, Example 1.6.19]). Let G be the ordinary convergence method on the topological space X. Let $A = X \setminus \{1/i + 1/k : i, k \in \mathbb{N}, k \geq i^2\}$. Then $(A)_G = X \setminus [X \setminus A]_G = \{0\} \neq \emptyset = X \setminus \overline{X \setminus A}^G = A^{\circ G}$. □

Example 5.3 *Let X be the set of all integers. Put $c_G(X) = s(X)$, and $G : c_G(X) \to X$ is defined by $G(\boldsymbol{x}) = 0$ for each $\boldsymbol{x} = \{x_n\}_{n\in\mathbb{N}} \in c_G(X)$. Obviously, G is not a pointwise method on X.*

For each $A \subset X$, it is obvious that $[A]_G = \{0\}$ if and only if $A \neq \emptyset$. Then, the G-hull-closed sets in X are only $\{0\}$ and $\emptyset$, and the G-kernel-open sets in X are only $X \setminus \{0\}$ and X. Thus, $\mathcal{U}_G(0) = \{X\}$; $\mathcal{U}_G(x) = \{X \setminus \{0\}, X\}$, $x \in X \setminus \{0\}$.

Suppose $A \subset X$. Then

$$[A]_{dG} = \begin{cases} \emptyset, & A \text{ is } \emptyset \text{ or singleton,} \\ X \setminus \{a\}, & A = \{0, a\}, \forall\, a \in X \setminus \{0\}, \\ X, & \text{others.} \end{cases}$$

In fact: if A is $\emptyset$ or singleton, by Definition 4.2, $[A]_{dG} = \emptyset$.

Let $A = \{a, 0\}$, $\forall\, a \in X \setminus \{0\}$. Then $A \setminus \{a\} = \{0\}$. Let $U = X \setminus \{0\}$. Then U is a G-kernel-neighborhood of a and $U \bigcap (A \setminus \{a\}) = U \bigcap \{0\} = \emptyset$, thus $a \notin [A]_{dG}$. $0 \in [A]_{dG}$ since the G-kernel-neighborhood of 0 is only X. For any $b \in X \setminus A$, $A \setminus \{b\} = A$. For any G-kernel-neighborhood U of b, $a \in U \bigcap A$ and $b \in [A]_{dG}$. In summary, $[A]_{dG} = X \setminus \{a\}$.

If there are at least two elements in $A \setminus \{0\}$, by Definition 4.2, $[A]_{dG} = X$.

(1) The condition "$(A)_G \subset A$" or "$(A)_G \subset A^{\circ G}$" in Theorem 2.3 cannot be omitted.

Let A be the set of all positive integers. Then $(A)_G = X \setminus \{0\}$. A is a G-open set since $0 \notin A$ (see [12, Example 3.2]), thus $A = A^{\circ G}$. But A is not a G-kernel-open set in X.

(2) The converse of Theorem 3.6 is not always true.

Let $B = \{b, c\} \subset X \setminus \{0\}$. Then $b \in U \bigcap B \neq \emptyset$ for any G-kernel-neighborhood U of b, but $[B]_G = \{0\}$, $b \notin \{0\}$.

(3) The converse of Corollary 3.7 is not always true.

Let $A=\{a\}$, $\forall\, b\in X\setminus\{a,0\}$. Then $[X\setminus A]_G=\{0\}\not\ni b$, but A is not G-kernel-neighborhood of b.

(4) The converse of Theorem 4.5 is not always true.

Let $B=\{b,c\}\subset X\setminus\{0\}$. Then $[B]_{dG}=X$ and $B^{dG}=\{0\}$, thus $[B]_{dG}\not\subset B^{dG}$.

(5) The converse of Theorem 4.6 is not always true.

Let $B=\{b,c\}\subset X\setminus\{0\}$. Then $b\in X=[B]_{dG}$. But $b\notin[B\setminus\{b\}]_G=[\{c\}]_G=\{0\}$.

(6) The converse of Theorem 4.7 is not always true.

Let $B=\{b,c\}\subset X\setminus\{0\}$. Then $b\in X=[B]_{dG}$ and $[B]_G=\{0\}$, thus $B\not\subset[B]_G$ and $[B]_{dG}\not\subset[B]_G$. □

The final example of this section discusses G-continuity. Let G_1, G_2 be methods on sets X and Y, respectively. A mapping $f:X\to Y$ is called (G_1,G_2)-continuous [11] if $f(\boldsymbol{x})\in c_{G_2}(Y)$ and $G_2(f(\boldsymbol{x}))=f(G_1(\boldsymbol{x}))$ for each $\boldsymbol{x}\in c_{G_1}(X)$.

Example 5.4 There are topological spaces X and Y, a mapping $f:X\to Y$ and methods G_1, G_2 on topological spaces X and Y, respectively, satisfying the following conditions:

(1) $f^{-1}(W)$ is a G_1-kernel-open set of X for each G_2-kernel-open set W of Y;

(2) f is not a (G_1,G_2)-continuous mapping.

Let X be the set of all integers endowed with the discrete topology. Put $c_{G_1}(X)=\{\{x_n\}_{n\in\mathbb{N}}\in s(X)$: there exists $m\in\mathbb{N}$ such that $\{x_n-x_{n-1}\}_{n>m}$ is a constant sequence $\}$. $G_1:c_{G_1}(X)\to X$ is defined by $G_1(\boldsymbol{x})=\lim_{n\to\infty}(x_{n+1}-x_n)$, $\forall\boldsymbol{x}=\{x_n\}_{n\in\mathbb{N}}\in c_{G_1}(X)$. Then G_1 is a method of X. Let $Y=\{0,1\}$ be a subspace of X endowed with the submethod $G_1|_Y$ of G_1 denoted as $G_2=G_1|_Y$. $f:X\to Y$ is defined by $f(x)=0$ if and only if $x=2k$, $\forall\, k\in\mathbb{Z}$. f is a continuous mapping because X is discrete space.

Obviously, F is a G_2-hull-closed set in Y if and only if F is $\emptyset$ or $\{0\}$. It is clear that $f^{-1}(\emptyset)=\emptyset$, and $f^{-1}(\{0\})=\{2k:k\in\mathbb{Z}\}$ is G_1-hull-closed set in X. Thus, U is a G_2-kernel-open set in Y if and only if U is Y or $\{1\}$, in addition, $f^{-1}(Y)=X$, and $f^{-1}(\{1\})=\{2k+1:k\in\mathbb{Z}\}$ is a G_1-kernel-open set in X.

Let $\boldsymbol{x}=\{n\}_{n\in\mathbb{N}}$. Then $\boldsymbol{x}\in c_{G_1}(X)$ and $G_1(\boldsymbol{x})=1$, but $f(\boldsymbol{x})=\{0,1,0,1,\ldots\}\notin c_{G_2}(Y)$, thus f is not a (G_1,G_2)-continuous mapping. □

Question 5.5 Suppose G_1, G_2 are methods on sets X and Y, respectively. If $f:X\to Y$ is a (G_1,G_2)-continuous mapping, is $f^{-1}(W)$ a G_1-kernel-open set of X for each G_2-kernel-open set W of Y?

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References

[1] A. ZYGMUND. *Trigonometric Series*. Cambridge Univ. Press, New York, 1959.
[2] H. FAST. *Sur la convergence statistique*. Colloq. Math., 1951, **2**(3-4): 241–244.
[3] H. STEINHAUS. *Sur la convergence ordinaire et la convergence asymptotique*. Colloq. Math., 1951, **2**(1): 73–74.

[4] G. DI MAIO, L. D. R. KOČINAC. *Statistical convergence in topology*. Topology Appl., 2008, **156**(1): 28–45.
[5] Zhongbao TANG, Fucai LIN. *Statistical versions of sequential and Fréchet-Urysohn spaces*. Adv. Math. (China), 2015, **44**(6): 945–954.
[6] V. RENUKADEVI, B. PRAKASH. *Some characterizations of spaces with weak form of cs-networks*. J. Math. Res. Appl., 2016, **36**(3): 369–378.
[7] J. CONNOR. *The statistical and strong p-Cesáro convergence of sequences*. Analysis, 1988, **8**(1-2): 47–64.
[8] J. CONNOR, K. GROSSE-ERDMANN. *Sequential definitions of continuity for real functions*. Rocky Mountain J. Math., 2003, **33**(1): 93–122.
[9] H. ÇAKALLı. *On G-continuity*. Comput. Math. Appl., 2011, **61**(2): 313–318.
[10] O. MUCUK, T. ŞAHAN. *On G-sequential continuity*. Filomat, 2012, **28**(6): 1181–1189.
[11] Shou LIN, Li LIU. *G-methods, G-sequential spaces and G-continuity in topological spaces*. Topology Appl., 2016, **212**: 29–48.
[12] Li LIU. *G-neighborhoods in topological spaces*. J. Minnan Normal Univ. Nat. Sci., 2016, **29**(3): 1–6.
[13] Li LIU. *G-derived sets and G-boundary*. J. Yangzhou Univ. Nat. Sci., 2017, **20**(1): 18–22.
[14] R. ENGELKING. *General Topology (revised and completed edition)*. Heldermann Verlag, Berlin, 1989.