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Maps Completely Preserving Jordan 1-$*$-Zero-Product on Factor Von Neumann Algebras

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Abstract Let H, K be infinite dimensional complex Hilbert spaces, and $\mathcal{A}$, $\mathcal{B}$ be factor von Neumann algebras on H and K, respectively. It is shown that every surjective map completely preserving Jordan 1-$*$-zero-product from $\mathcal{A}$ to $\mathcal{B}$ is a nonzero scalar multiple of either a linear $*$-isomorphism or a conjugate linear $*$-isomorphism.

Keywords factor von Neumann algebras; Jordan 1-$*$-zero-product; complete preserver problems

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1. Introduction

In recent years, many mathematicians devoted themselves to study the new products. Let $\mathcal{R}$ be a $*$-ring. For $A, B \in \mathcal{R}$, denote by $A \bullet B = AB + BA^*$ and $[A, B]_* = AB - BA^*$, which are two different kinds of new products. The former is called Jordan 1-$*$-product and the latter is called skew Lie product. They are found playing a more and more important role in some research topics. Jordan 1-$*$-product was extensively explored in the problem of characterizing ring [1]. A map Φ between $*$-rings $\mathcal{A}$ and $\mathcal{B}$ preserves Jordan 1-$*$-product if $\Phi(AB + BA^*) = \Phi(A)\Phi(B) + \Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{A}$; A map Φ between $*$-algebra $\mathcal{A}$ and $\mathcal{B}$ preserves Jordan η-$*$-product if $\Phi(AB + \eta BA^*) = \Phi(A)\Phi(B) + \eta\Phi(B)\Phi(A)^*$ for all $A, B \in \mathcal{A}$. In [2], Li, Lu and Fang have proved that if $\mathcal{A}$ and $\mathcal{B}$ are two factor von Neumann algebras, the nonlinear bijection $\Phi : \mathcal{A} \to \mathcal{B}$ preserves Jordan 1-$*$-product if and only if Φ is a $*$-ring isomorphism. In particular, if the von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ are type I factors, then Φ is a unitary isomorphism or a conjugate unitary isomorphism. Later, Dai and Lu investigated the bijective map Φ preserving Jordan η-$*$-product between two von Neumann algebras, one of which has no central abelian projections in [3]. It is shown that Φ is a linear $*$-isomorphism if η is not real and Φ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism if η is real. In [4], Li and Lu further proved that the map $\Phi(I)\Phi$ is a sum of a linear $*$-isomorphism and a conjugate linear $*$-isomorphism, when the nonlinear bijection Φ preserves Jordan triple 1-$*$-product on von Neumann algebras.

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A map $\Phi : \mathcal{A} \to \mathcal{B}$ preserves Jordan 1-$*$-zero-product in both directions if for every pair $A, B \in \mathcal{A}$, we have

$$AB + BA^* = 0 \Leftrightarrow \Phi(\mathrm{A})\Phi(\mathrm{B}) + \Phi(\mathrm{B})\Phi(\mathrm{A})^* = 0.$$

Whereas, depicting the maps that preserve Jordan 1-$*$-zero-product in both directions is not easy. In this paper, it is our goal to characterize general surjective maps between factor von Neumann algebras that completely preserve Jordan 1-$*$-zero-product in both directions. (Theorem 2.1)

As is well-known, complete preserver problems can be more precise in reflecting the homomorphic maps between operator algebras. So we propose to discuss the above questions under the frame of complete preserver problems. Let $\mathcal{B}(H)$ denote the operator algebra of all bounded linear operators from Hilbert space H into H. Let $\mathcal{S} \subseteq \mathcal{B}(H)$ and $\mathcal{T} \subseteq \mathcal{B}(K)$ be linear subspaces, and $\Phi : \mathcal{S} \to \mathcal{T}$ be a map. For each $n \in \mathbb{N}$, define a map $\Phi_n : \mathcal{S} \otimes M_n(\mathbb{F}) \to \mathcal{T} \otimes M_n(\mathbb{F})$ by

$$\Phi_n((s_{ij})_{n\times n}) = (\Phi(s_{ij}))_{n\times n}.$$

Then Φ is said to be n-Jordan 1-$*$-zero-product preserving if Φ_n preserves Jordan 1-$*$-zero-product; Φ is said to be completely Jordan 1-$*$-zero-product preserving if Φ is n-Jordan 1-$*$-zero-product preserving for every positive integer n. In this respect, some work has been done. At the earliest, Stinespring introduced the concept of completely positive in [5] and described the completely positive functions on C^*-Algebras. Later, Choi discussed the completely positive linear maps on complex matrices in [6]. In [7], Hadwin and Larson introduced the notion of completely rank-nonincreasing linear maps on $\mathcal{B}(H)$. After a year, they generalized it to $\mathcal{B}(\mathcal{X})$ in [8]. General surjective maps between standard operator algebras that completely preserve invertibility and spectrum, spectral functions, idempotents and square zero operators, commutativity and Jordan zero products have been characterized [9–12]. In this paper, we will discuss the maps completely preserving the Jordan 1-$*$-zero-product on factor von Neumann algebras.

Let $\mathbb{R}, \mathbb{C}$, respectively denote the real field and complex field. A von Neumann algebra $\mathcal{A}$ is a weakly closed, self-adjoint algebra of operators on Hilbert spaces H containing the identity operator I. The set $\mathcal{Z}_{\mathcal{A}} = \{S \in \mathcal{A} \mid ST = TS \text{ for all } T \in \mathcal{A}\}$ is called the center of $\mathcal{A}$. $\mathcal{A}$ is a factor von Neumann algebra means that its center only contains the scalar operators ($\mathcal{Z}_{\mathcal{A}} = \mathbb{C}I$). It comes to light that the factor von Neumann algebra $\mathcal{A}$ is prime, in the sense that $A\mathcal{A}B = 0$ for $A, B \in \mathcal{A}$ implies either $A = 0$ or $B = 0$.

2. Maps completely preserving Jordan 1-$*$-zero-product

Theorem 2.1 *Let H, K be infinite dimensional complex Hilbert spaces, and $\mathcal{A}, \mathcal{B}$ be factor von Neumann algebras on H and K, respectively. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a surjective map. Then the following statements are equivalent:*

(1) Φ is completely Jordan 1-$*$-zero-product preserving in both directions;

(2) Φ is 2-Jordan 1-$*$-zero-product preserving in both directions;

(3) Φ is a nonzero real scalar multiple of either a linear $*$-isomorphism or a conjugate linear $*$-isomorphism.

Proof Clearly, (3)⇒(1)⇒(2). We only need prove (2)⇒(3). We divide the proof of (2)⇒(3) into seven steps. Assume that Φ is 2-Jordan 1-$*$-zero-product preserving in both directions.

Claim 1 $\Phi(0) = 0, \Phi(I) = cI$, for some nonzero real scalar c.

For any $T \in \mathcal{A}$, we have

$$\begin{pmatrix} 0 & 0 \\ T & T \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ T & T \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Applying Φ_2 to the above equation, we get

$$\begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(T) & \Phi(T) \end{pmatrix}\begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix} + \begin{pmatrix} \Phi(0) & \Phi(0) \\ \Phi(0) & \Phi(0) \end{pmatrix}\begin{pmatrix} \Phi(0)^* & \Phi(T)^* \\ \Phi(0)^* & \Phi(T)^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So,

$$\Phi(0)^2 = \Phi(T)\Phi(0). \tag{2.1}$$

Due to the surjectivity of Φ, there exists some $T_0 \in \mathcal{A}$ such that $\Phi(T_0) = iI$. Taking $T = T_0$ in Eq. (2.1) yields that $\Phi(0)^2 = i\Phi(0)$, there still exists $T_1 \in \mathcal{A}$ such that $\Phi(T_1) = 0$. Taking $\mathrm{T} = \mathrm{T}_1$ again in Eq. (2.1) yields that $\Phi(0)^2 = 0$. Then $i\Phi(0) = 0$, this implies that $\Phi(0) = 0$.

Next, let us prove that $\Phi(I) = cI$, for some nonzero real scalar c.

For any $T \in \mathcal{A}$,

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix} + \begin{pmatrix} 0 & T \\ -T & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Applying Φ_2 to the above equation, we get

$$\begin{pmatrix} 0 & \Phi(I) \\ \Phi(I) & 0 \end{pmatrix}\begin{pmatrix} 0 & \Phi(T) \\ \Phi(-T) & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Phi(T) \\ \Phi(-T) & 0 \end{pmatrix}\begin{pmatrix} 0 & \Phi(I)^* \\ \Phi(I)^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So,

$$\Phi(I)\Phi(-T) + \Phi(T)\Phi(I)^* = 0. \tag{2.2}$$

On the other hand,

$$\begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}\begin{pmatrix} T & -T \\ -T & T \end{pmatrix} + \begin{pmatrix} T & -T \\ -T & T \end{pmatrix}\begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Then,

$$\begin{pmatrix} \Phi(I) & \Phi(I) \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \Phi(T) & \Phi(-T) \\ \Phi(-T) & \Phi(T) \end{pmatrix} + \begin{pmatrix} \Phi(T) & \Phi(-T) \\ \Phi(-T) & \Phi(T) \end{pmatrix}\begin{pmatrix} \Phi(I)^* & 0 \\ \Phi(I)^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So,

$$\Phi(I)\Phi(-T) + \Phi(I)\Phi(T) = 0. \tag{2.3}$$

Both Eqs. (2.2) and (2.3) yield that $\Phi(T)\Phi(I)^* = \Phi(I)\Phi(T)$. Due to the surjectivity of Φ, there exists some $T_1 \in \mathcal{A}$ such that $\Phi(T_1) = I$. Taking $T = T_1$ in the previous equation implies that $\Phi(I)^* = \Phi(I)$. Thus $\Phi(T)\Phi(I) = \Phi(I)\Phi(T) \Rightarrow \Phi(I) \in \mathcal{Z}_{\mathcal{A}}$. Because $\mathcal{A}$ is a factor, we have

$\Phi(I) = cI$ for some nonzero scalar $c \in \mathbb{C}$. And since $\Phi(I)^* = \Phi(I)$ that is $(cI)^* = cI$, we get c is a nonzero real scalar. This implies that there is a nonzero real scalar c such that $\Phi(I) = cI$.

If necessary, replace Φ by $c^{-1}\Phi$, which is still 2-Jordan 1-$*$-zero-product preserving in both directions. We can assume that $\Phi(I) = I$ in the sequel. $\square$

Claim 2 Φ is injective and $\Phi(-T) = -\Phi(T)$, for all $T \in \mathcal{A}$.

For any $T, S \in \mathcal{A}$, assume that $\Phi(T) = \Phi(S)$, we have

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix} + \begin{pmatrix} 0 & -T \\ T & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \tag{2.4}$$

Applying Φ_2 to the above equation, we get

$$\begin{pmatrix} 0 & \Phi(I) \\ \Phi(I) & 0 \end{pmatrix}\begin{pmatrix} 0 & \Phi(-T) \\ \Phi(T) & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Phi(-T) \\ \Phi(T) & 0 \end{pmatrix}\begin{pmatrix} 0 & \Phi(I)^* \\ \Phi(I)^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Replacing $\Phi(T)$ with $\Phi(S)$ in the above equation, we have

$$\begin{pmatrix} 0 & \Phi(I) \\ \Phi(I) & 0 \end{pmatrix}\begin{pmatrix} 0 & \Phi(-T) \\ \Phi(S) & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Phi(-T) \\ \Phi(S) & 0 \end{pmatrix}\begin{pmatrix} 0 & \Phi(I)^* \\ \Phi(I)^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

which implies

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & -T \\ S & 0 \end{pmatrix} + \begin{pmatrix} 0 & -T \\ S & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

To compare with the Eq. (2.4), it follows that $T = S$. Then Φ is an injection and hence a bijection from $\mathcal{A}$ onto $\mathcal{B}$.

Next according to Eq. (2.4) and $\Phi(I) = I$, we have

$$\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}\begin{pmatrix} 0 & \Phi(-T) \\ \Phi(T) & 0 \end{pmatrix} + \begin{pmatrix} 0 & \Phi(-T) \\ \Phi(T) & 0 \end{pmatrix}\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

it follows that

$$\Phi(T) + \Phi(-T) = 0.$$

That is,

$$\Phi(-T) = -\Phi(T), \quad \forall T \in \mathcal{A}. \ \square$$

Claim 3 $\Phi(T^*) = \Phi(T)^*$ *for all* $T \in \mathcal{A}$.

Note that, for any $T \in \mathcal{A}$, since

$$\begin{pmatrix} T & -I \\ -I & -T^* \end{pmatrix}\begin{pmatrix} I & T \\ T^* & -I \end{pmatrix} + \begin{pmatrix} I & T \\ T^* & -I \end{pmatrix}\begin{pmatrix} T & -I \\ -I & -T^* \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Applying Φ_2 to the above equation, we get

$$\begin{pmatrix} \Phi(T) & -I \\ -I & -\Phi(T^*) \end{pmatrix}\begin{pmatrix} I & \Phi(T) \\ \Phi(T^*) & -I \end{pmatrix} + \begin{pmatrix} I & \Phi(T) \\ \Phi(T^*) & -I \end{pmatrix}\begin{pmatrix} \Phi(T)^* & -I \\ -I & -\Phi(T^*)^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

It implies that

$$\Phi(T^*) - \Phi(T)^* = 0.$$

That is, $\Phi(T^*) = \Phi(T)^*$. $\square$

Claim 4 $\Phi(T+S) = \Phi(T) + \Phi(S)$ *for all* $T, S \in \mathcal{A}$.

For any $\mathrm{T}, \mathrm{S} \in \mathcal{A}$, since

$$\begin{pmatrix} I & -I \\ I & -I \end{pmatrix} \begin{pmatrix} S & S-T \\ T+S & S \end{pmatrix} + \begin{pmatrix} S & S-T \\ T+S & S \end{pmatrix} \begin{pmatrix} I & -I \\ I & -I \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

we see that

$$\begin{pmatrix} I & -I \\ I & -I \end{pmatrix} \begin{pmatrix} \Phi(S) & \Phi(S-T) \\ \Phi(T+S) & \Phi(S) \end{pmatrix} + \begin{pmatrix} \Phi(S) & \Phi(S-T) \\ \Phi(T+S) & \Phi(S) \end{pmatrix} \begin{pmatrix} I & I \\ -I & -I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we get

$$\Phi(T+S) + \Phi(S-T) = 2\Phi(S). \tag{2.5}$$

On the other hand,

$$\begin{pmatrix} I & -I \\ I & -I \end{pmatrix} \begin{pmatrix} T & T-S \\ T+S & T \end{pmatrix} + \begin{pmatrix} T & T-S \\ T+S & T \end{pmatrix} \begin{pmatrix} I & -I \\ I & -I \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

We see that

$$\begin{pmatrix} I & -I \\ I & -I \end{pmatrix} \begin{pmatrix} \Phi(T) & \Phi(T-S) \\ \Phi(T+S) & \Phi(T) \end{pmatrix} + \begin{pmatrix} \Phi(T) & \Phi(T-S) \\ \Phi(T+S) & \Phi(T) \end{pmatrix} \begin{pmatrix} I & I \\ -I & -I \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

So we get

$$\Phi(T+S) + \Phi(T-S) = 2\Phi(T). \tag{2.6}$$

Both Eqs. (2.5), (2.6) and Claim 2 yield that $\Phi(T+S) = \Phi(T) + \Phi(S)$. $\square$

Claim 5 $\Phi(TS) = \Phi(T)\Phi(S)$ for all $T, S \in \mathcal{A}$.

For any $T, S \in \mathcal{A}$, we have

$$\begin{pmatrix} I & -T \\ 0 & 0 \end{pmatrix} \begin{pmatrix} TST^* & TS \\ ST^* & S \end{pmatrix} + \begin{pmatrix} TST^* & TS \\ ST^* & S \end{pmatrix} \begin{pmatrix} I & -T \\ 0 & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Applying Φ_2 to the above equation, we get

$$\begin{pmatrix} I & -\Phi(T) \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \Phi(TST^*) & \Phi(TS) \\ \Phi(ST^*) & \Phi(S) \end{pmatrix} + \begin{pmatrix} \Phi(TST^*) & \Phi(TS) \\ \Phi(ST^*) & \Phi(S) \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Phi(T)^* & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, $\Phi(TS) - \Phi(T)\Phi(S) = 0$. That is, $\Phi(TS) = \Phi(T)\Phi(S)$. $\square$

Claim 6 $\Phi(iI) \in \mathcal{Z}(\mathcal{B})$.

For any $T \in \mathcal{A}$, we have

$$\begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix} \begin{pmatrix} T & iI \\ iI & -T \end{pmatrix} + \begin{pmatrix} T & iI \\ iI & -T \end{pmatrix} \begin{pmatrix} 0 & -iI \\ iI & 0 \end{pmatrix}^* = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Applying Φ_2 to the above equation, we can get $\Phi(iI)\Phi(T) + \Phi(-T)\Phi(-iI)^* = 0$. According to Claims 2 and 3, it implies that $\Phi(iI)\Phi(T) - \Phi(T)\Phi((-iI)^*) = 0 \Rightarrow \Phi(iI)\Phi(T) = \Phi(T)\Phi(iI) \Rightarrow \Phi(iI) \in \mathcal{Z}(\mathcal{B})$. $\square$

Claim 7 Φ preserves linear or conjugate linear and the statement (3) in the theorem holds true.

Since

$$\begin{pmatrix} I & iI \\ 0 & -I \end{pmatrix}\begin{pmatrix} I & -iI \\ iI & 0 \end{pmatrix}+\begin{pmatrix} I & -iI \\ iI & 0 \end{pmatrix}\begin{pmatrix} I & iI \\ 0 & -I \end{pmatrix}^*=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Applying Φ_2 to the above equation, we have that $2I+\Phi(iI)^2+\Phi(-iI)\Phi(iI)^*=0 \Rightarrow 2I+\Phi(iI)^2+\Phi(-iI)^2=0$. That is, $2I+2\Phi(iI)^2=0$. Therefore, $\Phi(iI)^2=-I$. According to claim 6, let $\Phi(iI)=\mu I$. Then, we have $(\mu I)^2=-I \Rightarrow \mu=\pm i$. So, $\Phi(iI)=iI$ or $\Phi(iI)=-iI$.

Furthermore, using the additivity of Φ, we get $\Phi(rI)=rI$, for every rational number r.

If α is a real number, then there are rational sequences $\{q_n\}$ and $\{p_n\}$ such that $\{q_n\}\leq \alpha \leq \{p_n\}$ for all n and $\lim_{n\to\infty} q_n=\lim_{n\to\infty} p_n=\alpha$. Since we have shown that Φ preserves star operation and is multiplicative, Φ preserves the order of self-adjoint elements. We get $q_nI\leq \alpha I\leq p_nI$, hence $\Phi(q_nI)\leq\Phi(\alpha I)\leq\Phi(p_nI)\Rightarrow q_nI\leq\Phi(\alpha I)\leq p_nI$ ($\{q_n\}$ and $\{p_n\}$ are rational sequences $\Rightarrow \Phi(q_nI)=q_nI$ and $\Phi(p_nI)=p_nI$). Taking the limit, we have $\Phi(\alpha I)=\alpha I$. So, $\Phi(\alpha A)=\Phi((\alpha I)A)=\Phi((\alpha I))\Phi(A)=\alpha\Phi(A)$.

If α is a complex number, let $\alpha=a+bi$, then $\Phi(\alpha I)=\Phi((a+bi)I)=\Phi(aI)+\Phi(b(iI))$. According to Claim 7, we have $\Phi(\alpha I)=(a+bi)I=\alpha I$ or $\Phi(\alpha I)=(a-bi)I=\bar{\alpha}I$. Therefore, $\Phi(\alpha A)=\alpha\Phi(A)$ or $\Phi(\alpha A)=\bar{\alpha}\Phi(A)$.

As desired, this completes the proof of (2) $\Rightarrow$ (3). $\square$

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