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The Crossing Numbers of $K_5 + P_n$

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Abstract A join graph denoted by $G + H$, is illustrated by connecting each vertex of graph G to each vertex of graph H. In this paper, we prove the crossing number of join product of $K_5 + P_n$ is $Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 4$ for $n \geq 2$.

Keywords crossing number; join product; complete graph; path

MR(2010) Subject Classification 05C10; 05C38

1. Introduction

Let G be a simple graph, whose vertex set and edge set are denoted by $V(G)$ and $E(G)$, respectively. A drawing of G is a representation of G in the plane such that its vertices are represented by distinct points and its edges by simple continuous arcs connecting the corresponding point pairs. For simplicity, we assume that in a drawing (i) no edge crosses itself, (ii) no two edges cross more than once, and (iii) no two edges are incident with the same vertex cross.

The crossing number, $cr(G)$ is the smallest number of edge crossings in any drawing of G. It is easy to see that a drawing with minimum number of crossings (an optimal drawing) is always a good drawing. Let ϕ be a drawing of graph G. We denote the number of crossings in ϕ by $cr_\phi(G)$. For definitions not explained in this paper, readers are referred to [1]. By definition and notation about crossing numbers, it is easy to get the following properties:

Property 1.1 *Let D be a good drawing of G, and A, B, C be mutually edge-disjoint subgraphs of G. Then*

(1) $cr_D(A \cup B, C) = cr_D(A, C) + cr_D(B, C)$;

(2) $cr_D(A \cup B) = cr_D(A) + cr_D(A, B) + cr_D(B)$.

Property 1.2 *(1) Let H be a subgraphs of G. Then $cr(H) \leq cr(G)$;*

(2) If H is isomorphic to G. Then $cr(H) = cr(G)$;

(3) Let H be the subdivision of G. Then $cr(H) = cr(G)$.

In fact, computing the crossing number of a graph is NP-complete problem, and the exact values are known only for very restricted classes of graphs. For example, these include the complete bipartite graph $K_{m,n}$ (see [2,3]) and the complete tripartite graph $K_{m,n,s}$ (see [4]) and

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so on. As a very important result of $K_{m,n}$, Kleitman [3] proved that:

$$cr(K_{m,n}) = Z(m,n) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{m-1}{2} \rfloor \lfloor \frac{n}{2} \rfloor \lfloor \frac{n-1}{2} \rfloor, \quad m \le 6, \ m \le n.$$

The join product of G and H, denoted by $G+H$, is illustrated by connecting each vertex of graph G to each vertex of graph H. In 2007, Klešč [5] obtained the crossing numbers of join of P_n+P_n, P_n+C_n and C_n+C_n. And in [6] the crossing numbers of $G+P_n$ and $G+C_n$ are also known for the special graph G of order six. Wang [7] proved the crossing numbers of S_m+P_n ($m=3,4$) and S_m+C_n ($m=3,4$). The up to date results of crossing numbers of G of order five with P_n are given in [8].

Let P_n be the path with n vertices and $n-1$ edges. In this paper, using the result of crossing number of $cr(K_{5,n}) = Z(5,n)$ by Kleitman, and together with the result of $cr(K_5 + nK_1) = Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$ by [9], we prove the crossing number of join product K_5+P_n is $Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 4$ for $n \ge 2$.

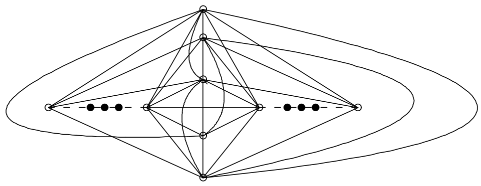

Figure 1 A good drawing of K_5+P_n

2. Some lemmas

In the graph of K_5+P_n, denote $V(K_5+P_n) = \{x_1,x_2,x_3,x_4,x_5\} \cup \{t_1,t_2,\dots,t_n\}$. Let for $i=1,2,\dots,n$, T^i denote the subgraph of $K_{5,n}$ which consists of the five edges incident with the vertex t_i. One can easily see that

$$K_5+P_n = K_5 \cup (\bigcup_{i=1}^{n} T^i) \cup P_n. \tag{2.1}$$

Lemma 2.1 ([1]) *Jordan Curve Theorem: Every Jordan curve divides the plane into an "interior" region bounded by the curve and an "exterior" region containing all of the nearby and far away exterior points, so that every continuous path connecting a point of one region to a point of the other intersects with that loop somewhere.*

Lemma 2.2 ([9]) *For $n \ge 1$, we have $cr(K_5+nK_1) = Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 1$.*

Lemma 2.3 *Let $K_5+P_2 = K_5 \cup T^1 \cup P_n$ and $K_5+P_3 = K_5 \cup T^1 \cup T^2 \cup P_n$. Then, we have $cr(K_5+P_2)=9$ and $cr(K_5+P_3)=15$.*

Proof Since K_5+P_2 is isomorphic to K_7 and K_5+P_3 is isomorphic to K_8-e, and in [2] and [10], $cr(K_7)=9$ and $cr(K_8-e)=15$. So by Property 1.2, we have $cr(K_5+P_2)=9$ and $cr(K_5+P_3)=15$. $\Box$

Lemma 2.4 *Let D be a good drawing of the graph $K_5 \cup T^1 \cup T^2$. If $cr_D(T^1, T^2) = 0$, then $cr_D(K_5, T^1 \cup T^2) \geq 5$.*

Proof Since $T^1 \cup T^2$ is isomorphic to $K_{2,5}$ and $cr_D(T^1 \cup T^2) = 0$, the subgraph $T^1 \cup T^2$ induced by D is isomorphic to Figure 2(a). Obviously, there are two vertices on the boundary of each region, so no matter whether the edges of C_5 belong to K_5 cross each other, by Lemma 2.1, the edges of $(K_5 - C_5)$ cross the edges of $T^1 \cup T^2$ at least five times, hence $cr_D(K_5, T^1 \cup T^2) \geq 5$ and this completes the proof. □

Lemma 2.5 ([11]) *Let ϕ and φ be the good drawings of graph $K_{m,n}$. Then there always holds $cr_\phi(K_{m,n}) \equiv cr_\varphi(K_{m,n}) \pmod 2$, where both m and n are odd.*

Lemma 2.6 *Let D be a good drawing of $K_5 + P_4$, in which for all t_i $(1 \leq i \leq 4)$, $cr_D(K_5, T^i) \geq 3$, and for two different $i, j \in \{1, 2, 3, 4\}$, $cr_D(T^i, T^j) \geq 1$. Then $cr_D(K_5 + P_4) \geq 22$.*

Proof Since $cr_D(K_5, T^i) \geq 3$, $cr_D(K_5, \bigcup_{i=1}^4 T^i) \geq 3 \cdot 4 = 12$. As $\bigcup_{i=1}^4 T^i$ is isomorphic to $K_{4,5}$, we have $cr_D(\bigcup_{i=1}^4 T^i) \geq cr(K_{4,5}) = 8$. Moreover, as $cr_D(K_5) \geq 1$, according to Properties 1.1 and 1.2, we have $cr_D(K_5 + P_4) \geq cr_D(K_5 \cup (\bigcup_{i=1}^4 T^i) \cup P_4) = cr_D(\bigcup_{i=1}^4 T^i) + cr_D(K_5, \bigcup_{i=1}^4 T^i) + cr_D(K_5) + cr_D(K_5 \cup (\bigcup_{i=1}^4 T^i), P_4) \geq 8 + 3 \cdot 4 + 1 = 21$.

(a) (b)

Figure 2 Two drawings of $T^1 \cup T^2$

To complete the proof of lemma, only proving "$\geq$" of the last formula cannot get "$=$". If "$=$" holds, then we have $cr_D(\bigcup_{i=1}^4 T^i) = 8$, $cr_D(K_5, T^i) = 3$ $(1 \leq i \leq 4)$, $cr_D(K_5) = 1$ and $cr_D(K_5 \cup (\bigcup_{i=1}^4 T^i), P_4) = 0$ are all satisfied. As $(\bigcup_{i=1}^4 T^i) = K_{3,5} \cup T^i$, and according to Lemma 2.5, we have $cr_D(K_{3,5}) = 4$ or $cr_D(K_{3,5}) \geq 6$. The following are divided into two cases.

Case 1 $cr_D(K_{3,5}) = 4$. Since $cr_D(T^i, T^j) \geq 1$, there exist T^i and T^j, such that $cr_D(T^i, T^j) = 1$ (otherwise $cr_D(\bigcup_{i=1}^4 T^i) \geq 2C_4^2 = 12 > 8$). Without loss of generality, let $cr_D(T^1, T^2) = 1$. Then the only drawing of $T^1 \cup T^2$ is shown in Figure 2(b). Obviously, t_3, t_4 are only placed in the regions which are marked with 1 and 2 (otherwise $cr_D(T^1 \cup T^2, T^3) \geq 5$, thus $cr_D(T^1 \cup T^2 \cup T^3) \geq 6$, turn to the following Case 2). When t_3 is placed in the region 1 or 2, T^3 must satisfy that $cr_D(T^1, T^3) \geq 1$ and $cr_D(T^2, T^3) \geq 2$, or $cr_D(T^1, T^3) \geq 2$ and $cr_D(T^2, T^3) \geq 1$. Then adding the edges of T^4 (t_4 also is only placed in the region 1 or 2), by Lemma 2.1, we always have $cr_D(T^1 \cup T^2 \cup T^3, T^4) \geq 5$. So $cr_D(\bigcup_{i=1}^4 T^i) \geq 9 > 8$.

Case 2 $cr_D(K_{3,5}) \geq 6$. Since $cr_D(T^i, T^j) \geq 1$, we have $cr_D(\bigcup_{i=1}^3 T^i, T^4) \geq 3$. So $cr_D(\bigcup_{i=1}^4 T^i) \geq 6 + 3 = 9 > 8$.

Thus $cr_D(K_5+P_4)\geq 22$. This completes the proof. $\square$

3. The main theorem and proof

Theorem 3.1 *For $n\geq 2$, we have $cr(K_5+P_n)=Z(5,n)+2n+\lfloor\frac{n}{2}\rfloor+4$.*

Proof The drawing in Figure 1 shows that $cr(K_5+P_n)\leq Z(5,n)+2n+\lfloor\frac{n}{2}\rfloor+4$ and that the theorem is true if the equality holds. We prove the reverse inequality by induction on n. By Lemma 2.3, the theorem is true for $n=2$ and $n=3$. Suppose now that for $n\geq 4$, $cr(K_5+P_{n-2})=Z(5,n-2)+2(n-2)+\lfloor\frac{n-2}{2}\rfloor+4$, and consider such an optimal drawing D of K_5+P_n that

$$cr_D(K_5+P_n)\leq Z(5,n)+2n+\lfloor\frac{n}{2}\rfloor+3. \tag{3.1}$$

Claim 1 The path P_n crosses at most two times.

Since $K_5\cup(\bigcup_{i=1}^n T^i)$ is isomorphic to K_5+nK_1, by Lemma 2.2 and Properties 1.1 and 1.2, we have $cr_D(K_5\cup(\bigcup_{i=1}^n T^i))\geq Z(5,n)+2n+\lfloor\frac{n}{2}\rfloor+1$. Moreover, using equality (2.1) and Properties 1.1 and 1.2, we have

$$\begin{aligned}cr_D(K_5+P_n)&=cr_D(K_5\cup(\bigcup_{i=1}^n T^i))+cr_D(K_5\cup(\bigcup_{i=1}^n T^i),P_n)+cr_D(P_n)\\&\geq Z(5,n)+2n+\lfloor\frac{n}{2}\rfloor+1+cr_D(K_5\cup(\bigcup_{i=1}^n T^i),P_n)+cr_D(P_n).\end{aligned}$$

This together with the assumption (3.1), implies that $cr_D(K_5\cup(\bigcup_{i=1}^n T^i),P_n)+cr_D(P_n)\leq 2$. Hence the path P_n crosses at most two times.

Claim 2 For every $1\leq i<j\leq n$, there holds $cr_D(T^i,T^j)\geq 1$.

Assume T^n and T^{n-1}, $cr_D(T^{n-1},T^n)=0$. Using Lemma 2.5, $cr_D(K_5,T^{n-1}\cup T^n)\geq 5$. By $cr(K_{3,5})=4$, and therefore $cr_D(T^i,T^{n-1}\cup T^n)\geq 4$ for $i=1,2,\ldots,n-2$. This, together with Eq. (2.1), leads to

$$\begin{aligned}cr_D(K_5+P_n)\geq&cr_D(K_5+P_{n-2})+\sum_{i=1}^{n-2}cr(T^i,T^{n-1}\cup T^n)+\\&cr_D(K_5,T^{n-1}\cup T^n)+cr_D(T^{n-1}\cup T^n,P_n)\\\geq&Z(5,n-2)+2(n-2)+\lfloor\frac{n-2}{2}\rfloor+4+4(n-2)+5\\>&Z(5,n)+2n+\lfloor\frac{n}{2}\rfloor+3.\end{aligned}$$

This contradicts the assumption (3.1). Hence $cr_D(T^i,T^j)\geq 1$.

Next we will divide 4 cases to discuss.

Case 1 There exists a vertex t_i $(1\leq i\leq n)$ such that $cr_D(K_5,T^i)=0$.

Without loss of generality, let $cr_D(K_5,T^n)=0$. Then we consider the subdrawing of $K_5\cup T^n$ induced by D. As $cr_D(K_5,T^n)=0$, there is a disk such that the vertices of K_5 are all placed on the boundary of disk. Assume the vertex t_n placed in the external of the disk, and the edges of

K_5 are all placed in the inner. As K_5 is a complete graph, the subdrawing of $K_5 \cup T^n$ is shown in Figure 3(a).

Now consider the subgraphs of $K_5 \cup T^n \cup T^i$. By Claim 1, $cr_D(K_5 \cup (\bigcup_{i=1}^n T^i), P_n) + cr_D(P_n) \leq 2$, this implies that the vertices $t_1, t_2, \ldots, t_{n-1}$ just placed in the regions which are marked with 1, 2 and 3 (otherwise by Lemma 2.1, $cr_D(K_5 \cup (\bigcup_{i=1}^n T^i), P_n) \geq 3$, this contradicts the Claim 1).

When t_i is placed in the region 1, we have $cr_D(T^i, K_5 \cup T^n) \geq 4$, if and only if $cr_D(T^i, T^n) = 4$ and the equality $cr_D(T^i, K_5) = 0$ holds. When t_i is placed in the region 2 and 3, we have $cr_D(T^i, K_5 \cup T^n) \geq 6$.

Subcase 1.1 If there does not exist the vertex t_i which satisfies $cr_D(T^i, K_5 \cup T^n) = 4$ in region 1, then all of t_i are placed in the regions 1, 2 and 3, and in this case we have $cr_D(T^i, K_5 \cup T^n) \geq 5$. By Eq. (2.1), we have

$$\begin{aligned}
cr_D(K_5 + P_n) \geq & cr_D(\bigcup_{i=1}^{n-1} T^i) + cr_D(K_5 \cup T^n, \bigcup_{i=1}^{n-1} T^i) + cr_D(K_5 \cup T^n) \\
\geq & Z(5, n-1) + 5(n-1) + 4 \\
> & Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 3.
\end{aligned}$$

This contradicts the assumption (3.1).

Subcase 1.2 If there exists a vertex t_i which satisfies $cr_D(T^i, K_5 \cup T^n) = 4$ in region 1. Let x be the number of vertices t_i which satisfy $cr_D(T^i, K_5 \cup T^n) = 4$ (then $cr_D(T^i, T^n) = 4$). So there are $(n-1-x)$ vertices t_i placed in the regions marked 1, 2 and 3, we have $cr_D(T^i, K_5 \cup T^n) \geq 5$ (then by claim 2, we have $cr_D(T^i, T^n) \geq 1$). So using Eq. (2.1), we have

$$\begin{aligned}
cr_D(K_5 + P_n) \geq & cr_D(K_5 + P_{n-1}) + cr_D((\bigcup_{i=1}^{n-1} T^i), T^n) \\
\geq & Z(5, n-1) + 2(n-1) + \lfloor \frac{n-1}{2} \rfloor + 4 + 4x + (n-1-x) \\
\geq & Z(5, n-1) + \lfloor \frac{n-1}{2} \rfloor + 3n + 3x + 1 \quad n \geq 4.
\end{aligned}$$

This together with assumption (3.1), implies that $x \leq \frac{n}{3}$. And as $(\bigcup_{i=1}^{n-1} T^i)$ is isomorphic to $K_{5,n-1}$, so using Eq. (2.1) and together with $x \leq \frac{n}{3}$, we have

$$\begin{aligned}
cr_D(K_5 + P_n) \geq & cr_D(\bigcup_{i=1}^{n-1} T^i) + cr_D(K_5 \cup T^n, \bigcup_{i=1}^{n-1} T^i) + cr_D(K_5 \cup T^n) \\
\geq & Z(5, n-1) + 4x + 5(n-1-x) + 5 \\
> & Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 3.
\end{aligned}$$

This contradicts the assumption (3.1).

Case 2 For every $1 \leq i < j \leq n$, there holds $cr_D(K_5, T^i) \geq 1$, and there exists a vertex t_i $(1 \leq i \leq n)$, such that $cr_D(K_5, T^i) = 1$.

Without loss of generality, assume $cr_D(K_5, T^n) = 1$. Now consider the good drawings of

$K_5 \cup T^n$. Next we will explain there is only a good drawing of $K_5 \cup T^n$.

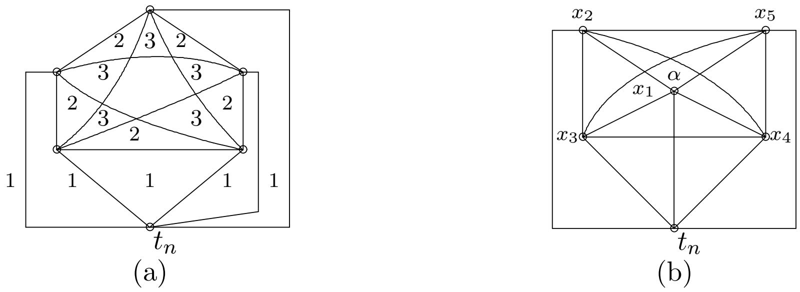

Figure 3 Two drawings of $K_5 \cup T^n$

First assume the edge t_nx_1 crosses with the edge x_3x_4. We can suppose the vertices of t_n, x_1, x_3, x_4 are placed on the plane R^2 as Figure 3(b), and the other 3 vertices of K_5 are placed around the vertices t_n, x_1, x_3, x_4. But as there is no cross on the other edges incident with vertex t_n, so t_n and x_1, x_2, x_3, x_4, x_5 will be connected as shown in Figure 3(b). And there is no cross on the edge x_2x_3, otherwise the edge x_2x_3 crosses the edge t_nx_1 at least once, thus $cr_D(K_5, T^n) \geq 2$. This contradicts the Case 2. The similar discussion can be made with the edges x_2x_5 and x_4x_5. So the rest edges x_2x_4, x_3x_5 can also be connected as shown in Figure 3(b).

In Figure 3(b), according to Claim 1, the vertices $t_1, t_2, \dots, t_{n-1}$ will be placed in the regions except α (otherwise by Lemma 2.1, $cr_D(K_5 \cup (\bigcup_{i=1}^{n} T^i), P_n) \geq 3$, leading to a contradiction). Then $cr_D(T^i, K_5 \cup T^n) \geq 5$. As $(\bigcup_{i=1}^{n-1} T^i)$ is isomorphic to $K_{5,n-1}$, together with $cr_D(K_5 \cup T^n) = 4$, proceeding with the similar calculating to Case 1.1, we get $cr_D(K_5 + P_n) \geq cr_D(\bigcup_{i=1}^{n-1} T^i) + cr_D(K_5 \cup T^n, \bigcup_{i=1}^{n-1} T^i) + cr_D(K_5 \cup T^n) \geq Z(5, n-1) + 5(n-1) + 4 > Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 3$. This contradicts the assumption (3.1).

Case 3 For every $1 \leq i < j \leq n$, there holds $cr_D(K_5, T^i) \geq 2$, and there exists a vertex t_i, such that $cr_D(K_5, T^i) = 2$.

Without loss of generality, let $cr_D(K_5, T^n) = 2$. Next we divide the subdrawings of $K_5 \cup T^n$ into 3 cases:

(i) Two edges of T^n cross with one of the edge of K_5. As the other edges of K_5 do not cross T^n, together with the structure of K_5. Then, there is only a good drawing of $K_5 \cup T^n$, see Figure 4(a).

(ii) One edge of T^n crosses with two edges of K_5. If one edge of T^n crosses two adjacent edges of K_5, the drawing of $K_5 \cup T^n$ is shown in Figure 4(b). If one edge of T^n crosses with two unadjacent edges e_1 and e_2 of K_5, where e_1 and e_2 cross each other, the drawing of $K_5 \cup T^n$ is shown in Figure 4(c); If one edge of T^n crosses with two unadjacent edges e_1 and e_2 of K_5, where e_1 and e_2 do not cross each other, the drawing of $K_5 \cup T^n$ is not a good drawing.

(iii) Two edges of T^n cross with two edges of K_5. Then there is only a good drawing of $K_5 \cup T^n$, see Figure 4(d).

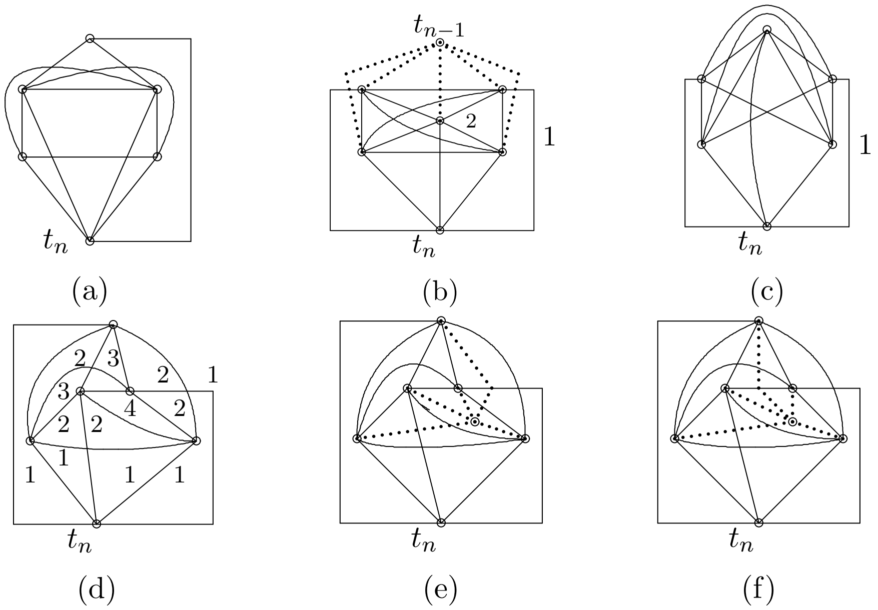

Figure 4 Some drawings of $K_5 \cup T^n$

Subcase 3.1 The subdrawing of $K_5 \cup T^n$ is isomorphic to Figure 4(a). No matter which region the vertex t_i is placed in, by Claim 2, $cr_D(T^i, T^j) \geq 1$, there always holds $cr_D(T^i, K_5 \cup T^n) \geq 5$. As $(\bigcup_{i=1}^{n-1} T^i)$ is isomorphic to $K_{5,n-1}$, $cr_D(K_5 \cup T^n) = 5$. Carrying out the similar calculating to Case 1(1), we can obtain that $cr_D(K_5 + P_n) \geq Z(5, n-1) + 5(n-1) + 5 > Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 3$. This contradicts the assumption (3.1).

Subcase 3.2 The subdrawing of $K_5 \cup T^n$ is isomorphic to Figure 4(b). If t_i is placed in the region 1, we have $cr_D(T^i, K_5 \cup T^n) \geq 4$ (if and only if $cr_D(T^i, K_5) = 2$ and $cr_D(T^i, T^n) = 2$ the equality holds). On the other regions, by Claim 2 and together with $cr_D(K_5, T^i) \geq 2$, it is easy to obtain that $cr_D(T^i, K_5 \cup T^n) \geq 5$.

Subcase 3.2.1 If there exist the vertices t_i placed in the region 1 which satisfy $cr_D(T^i, K_5 \cup T^n) = 4$. Without loss of generality, let $cr_D(T^{n-1}, K_5 \cup T^n) = 4$. Thus the drawing of $K_5 \cup T^n \cup T^{n-1}$ is shown in Figure 4(b). When $t_j (1 \leq j \leq n-2)$ is placed in the region 2, we have $cr_D(T^i, K_5 \cup T^n \cup T^{n-1}) \geq 6$. When t_j is placed in the other regions, there always holds $cr_D(T^i, K_5 \cup T^n \cup T^{n-1}) \geq 8$. Let x be the number of vertices t_j, which satisfy that $cr_D(T^i, K_5 \cup T^n \cup T^{n-1}) \geq 6$. By Eq. (2.1), we have

$$\begin{aligned}
cr_D(K_5 + P_n) \geq & cr_D(\bigcup_{i=1}^{n-2} T^i) + cr_D(K_5 \cup T^n \cup T^{n-1}, \bigcup_{i=1}^{n-2} T^i) + \\
& cr_D(K_5 \cup T^n \cup T^{n-1}) + cr_D(K_5 \cup (\bigcup_{i=1}^{n} T^i), P_n) \\
\geq & Z(5, n-2) + 6x + 8(n-2-x) + 9 + 2 \\
\geq & Z(5, n-2) + 8n - 2x - 5
\end{aligned}$$

This together with the assumeption (3.1) results in $x \geq \frac{3n}{4}$. As $(\bigcup_{i=1}^{n-1} T^i)$ is isomorphic to $K_{5,n-1}$, by Eq. (2.1) and $x \geq \frac{3n}{4}$, we get

$$\begin{aligned} cr_D(K_5+P_n) \geq & cr_D(\bigcup_{i=1}^{n-1} T^i) + cr_D(K_5 \cup T^n, \bigcup_{i=1}^{n-1} T^i) + cr_D(K_5 \cup T^n) \\ \geq & Z(5,n-1) + 5x + 4(n-1-x) + 5 \\ > & Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 3 \end{aligned}$$

This contradicts the assumption (3.1).

Subcase 3.2.2 There does not exist the vertex t_i placed in the region 1 which satisfies $cr_D(T^i, K_5 \cup T^n) = 4$. Then for all vertices t_i, there always holds $cr_D(T^i, K_5 \cup T^n) \geq 5$. As $(\bigcup_{i=1}^{n-1} T^i)$ is isomorphic to $K_{5,n-1}$, $cr_D(K_5 \cup T^n) = 4$. Carrying out the similar calculating to case 1(1), we have $cr_D(K_5+P_n) \geq Z(5,n-1) + 5(n-1) + 4 > Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 3$. This contradicts the assumption (3.1).

Subcase 3.3 The subdrawing of $K_5 \cup T^n$ is isomorphic to Figure 4(c). If t_i is placed in the region 1, we have $cr_D(T^i, K_5 \cup T^n) \geq 4$ (if and only if $cr_D(T^i, K_5) = 2$ and $cr_D(T^i, T^n) = 2$ the equality holds). In the other regions, there always holds $cr_D(T^i, K_5 \cup T^n) \geq 5$. So using the similar method to Figure 4(b), we can get the contradiction with the assumption (3.1).

Subcase 3.4 The subdrawing of $K_5 \cup T^n$ is isomorphic to Figure 4(d). By Claim 1, the vertices t_i are only placed in the regions marked with 1-4 (otherwise $cr_D(K_5 \cup (\bigcup_{i=1}^{n} T^i), P_n) \geq 3$).

(i) If t_i is placed in the region 1, there holds $cr_D(T^i, K_5 \cup T^n) \geq 6$.

(ii) If t_i is placed in the region 2, there holds $cr_D(T^i, K_5 \cup T^n) \geq 5$.

(iii) If t_i is placed in the region 3, there holds $cr_D(T^i, K_5 \cup T^n) \geq 4$, if and only if $cr_D(T^i, K_5) = 1$ and the equality $cr_D(T^i, T^n) = 3$ holds. Together with $cr_D(T^i, K_5) \geq 2$, we always have $cr_D(T^i, K_5 \cup T^n) \geq 5$.

(iv) If t_i is placed in the region 4, there holds $cr_D(T^i, K_5 \cup T^n) \geq 4$, if and only if $cr_D(T^i, K_5) = 2$ and $cr_D(T^i, T^n) = 2$ or $cr_D(T^i, K_5) = 3$ and the equality $cr_D(T^i, T^n) = 1$ holds.

Subcase 3.4.1 If there exists a vertex t_i which is placed in the region 2, then the other vertices t_j can only be placed in the region 1 and 2 (otherwise $cr_D(K_5 \cup (\bigcup_{i=1}^{n} T^i), P_n) \geq 3$). So for all t_i, there always holds $cr_D(T^i, K_5 \cup T^n) \geq 5$. As $(\bigcup_{i=1}^{n-1} T^i)$ is isomorphic to $K_{5,n-1}$, together with $cr_D(K_5 \cup T^n) = 3$ and $cr_D(K_5, P_n) \geq 1$, carrying out the similar calculating to Subcase 1.1, we have $cr_D(K_5+P_n) \geq Z(5,n-1) + 5(n-1) + 3 + 1 > Z(5,n) + 2n + \lfloor \frac{n}{2} \rfloor + 3$. This contradicts the assumption (3.1).

Subcase 3.4.2 If there exists a vertex t_i which is placed in the region 3, then the other vertices t_j can only placed in the region 1 and 3 (otherwise $cr_D(K_5 \cup (\bigcup_{i=1}^{n} T^i), P_n) \geq 3$). So for all t_i, there always holds $cr_D(T^i, K_5 \cup T^n) \geq 5$. As $(\bigcup_{i=1}^{n-1} T^i)$ is isomorphic to $K_{5,n-1}$, yet $cr_D(K_5 \cup T^n) = 3$, $cr_D(K_5, P_n) \geq 2$, so carrying out the similar calculating to Subcase 3.4.1, we can get the contradiction .

Subcase 3.4.3 If there exists a vertex t_i which is placed in the region 4, and there does not exist the vertex t_i which satisfies $cr_D(T^i, K_5 \cup T^n) = 4$, then we have $cr_D(T^i, K_5 \cup T^n) \geq 5$. And the other vertices t_j can only be placed in the region 1 and 4 (otherwise $cr_D(K_5 \cup (\bigcup_{i=1}^{n} T^i), P_n) \geq 3$). So for all t_i, there always holds $cr_D(T^i, K_5 \cup T^n) \geq 5$. As $(\bigcup_{i=1}^{n-1} T^i)$ is isomorphic to $K_{5,n-1}$, and also $cr_D(K_5 \cup T^n) = 3$, $cr_D(K_5, P_n) \geq 2$, so carrying out the similar calculating to Subcase 3.4.1, we can get the contradiction.

Subcase 3.4.4 If there exists a vertex t_i which is placed in the region 4 which satisfies $cr_D(T^i, K_5 \cup T^n) = 4$, then we have two cases $cr_D(T^i, K_5) = 2$, $cr_D(T^i, T^n) = 2$ and $cr_D(T^i, K_5) = 3$, $cr_D(T^i, T^n) = 1$. Assume the vertex t_{n-1} satisfies $cr_D(T^{n-1}, K_5 \cup T^n) = 4$. Then, the sub-drawings of $K_5 \cup T^n \cup T^{n-1}$ is shown in Figures 4(e) and (f).

As the other vertices can only be placed in the region 1 and 4, so for all t_i, there always holds $cr_D(T^i, K_5 \cup T^n \cup T^{n-1}) \geq 7$. As $(\bigcup_{i=1}^{n-2} T^i)$ is isomorphic to $K_{5,n-2}$, and also $cr_D(K_5 \cup T^n \cup T^{n-1}) = 7$, $cr_D(K_5, P_n) \geq 2$, so we get

$$\begin{aligned} cr_D(K_5 + P_n) \geq & cr_D(\bigcup_{i=1}^{n-2} T^i) + cr_D(K_5 \cup T^n \cup T^{n-1}, \bigcup_{i=1}^{n-2} T^i) + \\ & cr_D(K_5 \cup T^n \cup T^{n-1}) + cr_D(K_5, P_n) \\ \geq & Z(5, n-2) + 7(n-2) + 7 + 2 \\ > & Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 3 \end{aligned}$$

This contradicts the assumption (3.1).

Subcase 3.4.5 If the vertices t_i are all placed in the region 1, we have $cr_D(T^i, K_5 \cup T^n) \geq 6$. By Eq. (2.1), we have

$$\begin{aligned} cr_D(K_5 + P_n) \geq & cr_D(\bigcup_{i=1}^{n-1} T^i) + cr_D(K_5 \cup T^n, \bigcup_{i=1}^{n-1} T^i) + cr_D(K_5 \cup T^n) \\ \geq & Z(5, n-1) + 6(n-1) + 3 \\ > & Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 3. \end{aligned}$$

This contradicts the assumption (3.1).

Case 4 For every $1 \leq i < j \leq n$, such that $cr_D(K_5, T^i) \geq 3$.

By Lemmas 2.3 and 2.6, the theorem is true for $n = 2, 3$ and 4. For $n \geq 5$, together with Eq. (2.1) and $cr_D(K_5) \geq 1$, we have

$$\begin{aligned} cr_D(K_5 + P_n) \geq & cr_D(\bigcup_{i=1}^{n} T^i) + cr_D(K_5, \bigcup_{i=1}^{n} T^i) + cr_D(K_5) \\ \geq & Z(5, n) + 3n + 1 \\ > & Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 3, \quad n \geq 5 \end{aligned}$$

This contradicts the assumption (3.1). Now the theorem is completed.

Finally, we give a conjecture about the crossing number of join product of $K_5 + C_n$.

Conjecture 3.2 For $n \geq 3$, we have $cr(K_5 + C_n) = Z(5, n) + 2n + \lfloor \frac{n}{2} \rfloor + 7$.

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