Journal of Mathematical Research with Applications
Jul., 2018, Vol. 38, No. 4, pp. 341–350
DOI:10.3770/j.issn:2095-2651.2018.04.002
Http://jmre.dlut.edu.cn

The Measurement Topology and the Density Topology of Posets

Xuxin MAO[1,*], **Luoshan XU**[2]

1. *College of Science, Nanjing University of Aeronautics and Astronautics, Jiangsu* 210016*, P. R. China;*
2. *Department of Mathematics, Yangzhou University, Jiangsu* 225002*, P. R. China*

Abstract In this paper, the concepts of the essential topology and the density topology of dcpos are generalized to the setting of general posets. Basic properties of the essential topology and relations with other intrinsic topologies are explored. Comparisons between the density topology and the measurement topology are made. Via the essential topology, the density topology and the measurement topology, we obtain properties and characterizations of bases of continuous posets. We also provide some new conditions for a continuous poset to be an algebraic poset.

Keywords essential topology; density topology; measurement topology; continuous poset; basis

MR(2010) Subject Classification 06A06; 06A11; 06B35; 54A10

1. Introduction

The notion of continuous lattices as a model for the semantics of programming languages was introduced by Scott in [1]. Later, a more general notion of continuous directed complete partially ordered sets (i.e., continuous dcpos or domains) was introduced and extensively studied [2, 3]. Since some naturally arisen posets are important but fail to be directed complete, there are more and more occasions to study posets which miss suprema of directed sets [4–9]. Lawson [3] gave a remarkable characterization that a dcpo is continuous iff its Scott topology is completely distributive. By the technique of embedded bases and sobrification via the Scott topology, Xu [6] successfully embedded continuous posets into continuous domains and proved that a poset is continuous iff its Scott topology is completely distributive.

Martin [10] introduced a new intrinsic topology called μ topology on continuous dcpos and proved that the μ topology can be induced by measurements with certain conditions. Later, Xu [7] introduced the concept of the measurement topology of posets, a generalization of the μ topology, and studied properties of the measurement topology. In order to provide a topological

Received September 12, 2017; Accepted April 27, 2018
Supported by the National Natural Science Foundation of China (Grant Nos. 11671008; 11101212) and the University Science Research Project of Jiangsu Province (Grant No. 15KJD110006).
* Corresponding author
E-mail address: xuxinmao@yahoo.com (Xuxin MAO); luoshanxu@hotmail.com (Luoshan XU)

interpretation of bases of continuous dcpos, Rusu and Ciobanu [11] introduced the concepts of the essential topology and the density topology of dcpos and proved that bases are just dense sets in the density topology of continuous dcpos. In this paper, we manage to generalize the concepts of the essential topology and the density topology of dcpos to the setting of general posets. We investigate properties of the essential topology and relations with other intrinsic topologies. We make comparisons between the density topology and the measurement topology. Via the essential topology, the density topology and the measurement topology, we obtain several properties and characterizations of bases of continuous posets. We also provide new conditions for a continuous poset to be an algebraic poset.

2. Preliminaries

We quickly recall some basic notions and results [2, 5, 6].

Let $(L, \leqslant)$ be a poset. A principal ideal (resp., principal filter) is a set of the form $\downarrow x = \{y \in L | y \leqslant x\}$ (resp., $\uparrow x = \{y \in L | x \leqslant y\}$). For $A \subseteq L$, we write $\downarrow A = \{y \in L | \exists\, x \in A,\, y \leqslant x\}$, $\uparrow A = \{y \in L | \exists\, x \in A,\, x \leqslant y\}$. A subset A is a lower set (resp., an upper set) if $A = \downarrow A$ (resp., $A = \uparrow A$). We say that z is a lower bound (resp., an upper bound) of A if $A \subseteq \uparrow z$ (resp., $A \subseteq \downarrow z$). The supremum of A is denoted by $\bigvee A$ or $\sup A$. The infimum of A is denoted by $\bigwedge A$ or $\inf A$. A subset M of L is called order convex if $x, z \in M$ and $z \leqslant y \leqslant x$ implies $y \in M$. A nonempty subset D of L is directed if every finite subset of D has an upper bound in D. A poset L is a directed complete partially ordered set (dcpo, for short) if every directed subset of L has a supremum. A complete lattice is a poset in which every subset has a supremum.

In a poset L, we say that x approximates y, written $x \ll y$ if whenever D is a directed set that has a supremum $\sup D \geqslant y$, then $x \leqslant d$ for some $d \in D$. We say that x is compact if x approximates itself, i.e., $x \ll x$. The set of all compact elements is denoted by $K(L)$. For $x \in L$, we write $\Downarrow x = \{z \in L | z \ll x\}$ and $\Uparrow x = \{z \in L | x \ll z\}$. The poset L is said to be continuous (resp., algebraic) if every element is the directed supremum of all (resp., compact) elements that approximate it, i.e., for all $x \in L$, the set $\Downarrow x$ (resp., $\downarrow x \cap K(L)$) is directed and $x = \bigvee \Downarrow x$ (resp., $x = \bigvee(\downarrow x \cap K(L))$). A continuous poset (resp., an algebraic poset) which is also a dcpo is called a continuous domain (resp., an algebraic domain).

Proposition 2.1 ([2, 8]) *If L is a continuous poset, then the approximating relation $\ll$ has the interpolation property: $x \ll z \Longrightarrow \exists\, y \in L$ such that $x \ll y \ll z$.*

Definition 2.2 ([2, 10]) *A subset B of a poset L is called a basis for L if for each $x \in L$, $B \cap \Downarrow x$ contains a directed subset with supremum x.*

Lemma 2.3 ([2, 10]) *A poset is continuous if and only if it has a basis. Moreover, a poset is algebraic if and only if its compact elements form a basis.*

In the context of continuous posets, there is another characterization of bases.

Proposition 2.4 *Let L be a continuous poset. A subset B is a basis if and only if given $x \ll y$*

in L, there exists $b \in B$ such that $x \ll b \ll y$.

A subset A of a poset L is Scott closed if $\downarrow A = A$ and for any directed set $D \subseteq A$, $\sup D \in A$ whenever $\sup D$ exists. The complements of the Scott closed sets form a topology, called the Scott topology and denoted by $\sigma(L)$. It is well-known that for a continuous poset the Scott topology has a base of all sets of the form $\twoheaduparrow x = \{z \in L | x \ll z\}$. The topology generated by the complements of all principal filters $\uparrow x$ (resp., principal ideals $\downarrow x$) is called the lower topology (resp., upper topology) and is denoted by $\omega(L)$ (resp., $\nu(L)$). The topology of all upper sets (resp., lower sets) is called the Alexandroff topology (resp., the dual Alexandroff topology) and is denoted by $\alpha(L)$ (resp., $\alpha^*(L)$). The common refinement $\sigma(L) \vee \omega(L)$ of the Scott topology and the lower topology is called the Lawson topology and is denoted by $\lambda(L)$.

Definition 2.5 ([7]) *Let L be a poset. The common refinement $\sigma(L) \vee \alpha^*(L)$ of the Scott topology and the dual Alexandroff topology is called the measurement topology and is denoted by $\mu(L)$.*

Proposition 2.6 ([7]) *Let L be a continuous poset. Then $\mathcal{B}_\mu = \{\twoheaduparrow x \cap \downarrow y | x, y \in L\}$ is a base for the measurement topology $\mu(L)$.*

Remark 2.7 ([7]) *By Proposition 2.6, the measurement topology coincides with the μ topology in [10] on continuous domains.*

3. The essential topology

In this section, the concept of the essential topology of dcpos in [11] is generalized to the setting of general posets and some basic properties of the essential topology are obtained.

Definition 3.1 *Let L be a poset. We use $\mathcal{P}(L)$ to denote the powerset of L. Let $\twoheaddownarrow : \mathcal{P}(L) \to \mathcal{P}(L)$ be the operator defined by $\twoheaddownarrow A = \bigcup_{x \in A} \twoheaddownarrow x$ for all $A \in \mathcal{P}(L)$. Let $\twoheaduparrow : \mathcal{P}(L) \to \mathcal{P}(L)$ be the operator defined by $\twoheaduparrow A = \bigcup_{x \in A} \twoheaduparrow x$ for all $A \in \mathcal{P}(L)$.*

Proposition 3.2 *Let L be a poset. Then for all $A, B \in \mathcal{P}(L)$ and $\{A_\alpha\}_{\alpha \in \Gamma} \subseteq \mathcal{P}(L)$:*

(1) $\twoheaddownarrow \emptyset = \emptyset$, $\twoheaduparrow \emptyset = \emptyset$;

(2) $\twoheaddownarrow (\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} \twoheaddownarrow A_\alpha$, $\twoheaduparrow (\bigcup_{\alpha \in \Gamma} A_\alpha) = \bigcup_{\alpha \in \Gamma} \twoheaduparrow A_\alpha$;

(3) $A \subseteq B \Longrightarrow \twoheaddownarrow A \subseteq \twoheaddownarrow B$, $A \subseteq B \Longrightarrow \twoheaduparrow A \subseteq \twoheaduparrow B$;

(4) $\twoheaddownarrow A \backslash \twoheaddownarrow B \subseteq \twoheaddownarrow (A \backslash B)$, $\twoheaduparrow A \backslash \twoheaduparrow B \subseteq \twoheaduparrow (A \backslash B)$;

(5) $\twoheaddownarrow (\twoheaddownarrow A) \subseteq \twoheaddownarrow A$, $\twoheaduparrow (\twoheaduparrow A) \subseteq \twoheaduparrow A$.

Proof Straightforward. □

Definition 3.3 *Let L be a poset and $A \subseteq L$. The subset A is called an e-open set if $\twoheaddownarrow A \subseteq A$. The complement of an e-open set is called an e-closed set.*

Proposition 3.4 *Let L be a poset. Then*

(1) All the e-open sets of L form a topology, called the essential topology and denoted by

$\tau_e(L)$. Moreover, the intersection of any family of e-open sets is e-open;

(2) The family of sets $\{\{x\} \cup \Downarrow x \mid x \in L\}$ is a base for $\tau_e(L)$;

(3) $F \subseteq L$ is e-closed if and only if $\Uparrow F \subseteq F$;

(4) For all $A \in \mathcal{P}(L)$, $\Downarrow A$ is e-open, $\Uparrow A$ is e-closed;

(5) Every lower set is e-open and every upper set is e-closed;

(6) The essential topology $\tau_e(L)$ is finer than the dual Alexandroff topology $\alpha^*(L)$, i.e., $\alpha^*(L) \subseteq \tau_e(L)$.

Proof (1) The proof is similar to that of [11, Proposition 2] and hence omitted.

(2) Straightforward.

(3) Assume that F is e-closed. Then $L\backslash F$ is e-open and thus $\Downarrow(L\backslash F) \subseteq L\backslash F$. Suppose that $\Uparrow F \not\subseteq F$. There is $x \in \Uparrow F$ such that $x \notin F$. This shows that there exists $y \in F$ such that $y \ll x$ and $x \in L\backslash F$. So, $y \in \Downarrow(L\backslash F) \subseteq L\backslash F$, a contradiction to $y \in F$. Therefore, $\Uparrow F \subseteq F$. Conversely, assume that $\Uparrow F \subseteq F$. We only need to show that $\Downarrow(L\backslash F) \subseteq L\backslash F$. Suppose that $\Downarrow(L\backslash F) \not\subseteq L\backslash F$. There exists $a \in \Downarrow(L\backslash F)$ such that $a \notin L\backslash F$. This shows that there exists $b \in L\backslash F$ such that $a \ll b$ and $a \in F$. So, $b \in \Uparrow F \subseteq F$, a contradiction to $b \in L\backslash F$. Therefore, $\Downarrow(L\backslash F) \subseteq L\backslash F$ and $L\backslash F$ is e-open.

(4) Follows from (3), Definition 3.3 and Proposition 3.2(5).

(5) Straightforward.

(6) Follows immediately from (5). $\square$

Proposition 3.5 *Let L be a poset. For all $A \in \mathcal{P}(L)$, we have*

(1) $\mathrm{cl}_e(A) = A \cup \Uparrow A$, where $\mathrm{cl}_e(A)$ is the closure of A in the topology $\tau_e(L)$;

(2) $\mathrm{int}_e(A) = A\backslash \Uparrow(L\backslash A)$, where $\mathrm{int}_e(A)$ is the interior of A in the topology $\tau_e(L)$;

(3) $\Uparrow \mathrm{cl}_e(A) = \mathrm{cl}_e(\Uparrow A)$.

Proof The proof is similar to that of [11, Proposition 3] and hence omitted. $\square$

Lemma 3.6 *Let L be a continuous poset. Then the operators $\Downarrow$ and $\Uparrow$ are idempotent, i.e., for all $A \in \mathcal{P}(L)$, $\Downarrow(\Downarrow A) = \Downarrow A$, $\Uparrow(\Uparrow A) = \Uparrow A$.*

Proof Follows from Proposition 3.2(5), the continuity of L and Proposition 2.1. $\square$

We can characterize the bases of a continuous poset via the essential topology.

Theorem 3.7 *Let L be a continuous poset and $B \subseteq L$. Then for all $x \in L$, $A \subseteq L$ and for all e-closed set F, the following conditions are equivalent:*

(1) B is a basis of L;

(2) $\Uparrow(\Uparrow x \cap B) = \Uparrow x$;

(3) $\Uparrow(\Uparrow A \cap B) = \Uparrow A$;

(4) $\Uparrow(F \cap B) = \Uparrow F$;

(5) $\mathrm{cl}_e(\Uparrow A \cap B) = \Uparrow A$;

(6) For all $U \in \sigma(L)$, $G \in \tau_e(L)$, $U \cap G \neq \emptyset$ implies $U \cap G \cap B \neq \emptyset$.

Proof The proof is similar to that of [11, Theorem 1] and hence omitted. □

In the context of continuous posets, we can characterize the algebraicity of the posets via the essential topology.

Theorem 3.8 *A continuous poset L is algebraic if and only if $\mathrm{cl}_e(F \cap K(L)) = \Uparrow F$ for every e-closed set F.*

Proof The proof is similar to that of [11, Proposition 8] and hence omitted. □

4. Comparisons between measurement topology and density topology

We generalize the density topology of dcpos in [11] to the setting of general posets. We make comparisons between the measurement topology and the density topology and give more properties of the measurement topology and the density topology.

Definition 4.1 *Let L be a poset. The common refinement $\sigma(L) \vee \tau_e(L)$ of the Scott topology and the essential topology is called the density topology and is denoted by $\rho(L)$.*

Proposition 4.2 *Let L be a poset. Then $\sigma(L) \subseteq \lambda(L) \subseteq \mu(L) \subseteq \rho(L)$.*

Proof The conclusion follows from Definition 2.5, Definition 4.1 and Proposition 3.4(6). □

Proposition 4.3 *Let L be a poset. Then*

(1) Every Scott open set is a clopen set in $\mu(L)$ and $\rho(L)$;

(2) The spaces $(L, \mu(L))$ and $(L, \rho(L))$ are both Hausdorff.

Proof (1) Let U be a Scott open set. It follows from Proposition 4.2 that U is $\mu(L)$-open and $\rho(L)$-open. Since $L \backslash U$ is a lower set, we have $L \backslash U \in \alpha^*(L) \subseteq \tau_e(L)$. This shows that $L \backslash U$ is $\mu(L)$-open and $\rho(L)$-open. So, every Scott open set is a clopen set in $\mu(L)$ and $\rho(L)$.

(2) Suppose that $x \neq y$ in L, and assume that $x \nleqslant y$. Then $x \in L \backslash \downarrow y$. Note that $L \backslash \downarrow y$ is a Scott (hence, $\mu(L)$) open neighbourhood of x and $\downarrow y$ is an $\alpha^*(L)$ (hence, $\mu(L)$) open neighbourhood of y. Clearly these two neighbourhoods are disjoint. So, $(L, \mu(L))$ is Hausdorff. By Proposition 4.2, $(L, \rho(L))$ is Hausdorff. □

Proposition 4.4 *Let L be a continuous poset. Then $\mathcal{B}_\rho = \{\Uparrow x \cap (\{y\} \cup \Downarrow y) | x, y \in L\}$ is a base for the density topology $\rho(L)$.*

Proof The conclusion follows from the continuity of L and Proposition 3.4(2). □

Lemma 4.5 *Let L be a continuous poset. Then for all $U \in \sigma(L)$ and all $G \in \tau_e(L)$, one has $\uparrow(U \cap G) \in \sigma(L)$. Particularly, for any $U \in \sigma(L)$ and any lower set C, one has $\uparrow(U \cap C) \in \sigma(L)$.*

Proof Let $U \in \sigma(L)$ and $G \in \tau_e(L)$. Suppose that D is a directed subset for which $\sup D$ exists and satisfies $\sup D \in \uparrow(U \cap G)$. Then there is $x \in U \cap G$ such that $x \leqslant \sup D$. By the continuity of L and the Scott openness of U, there is $t \in U$ such that $t \ll x \leqslant \sup D$. Thus, there exists

$d \in D$ such that $t \leqslant d$. Since $G \in \tau_e(L)$, we have $t \in \Downarrow x \subseteq \Downarrow G \subseteq G$. This shows that $t \in U \cap G$ and thus $d \in \uparrow(U \cap G)$. Therefore, $\uparrow(U \cap G)$ is Scott open. $\square$

Theorem 4.6 *Let L be a continuous poset.*

(1) An upper set U is $\rho(L)$-open if and only if U is Scott open;

(2) If $W \in \rho(L)$, then $\uparrow W \in \sigma(L)$;

(3) For all $x \in L$, x is compact if and only if $\{x\}$ is $\rho(L)$-open;

(4) Every $\mu(L)$-closed set is closed under directed suprema.

Proof (1) Let U be an upper set. Clearly, every Scott open set is $\rho(L)$-open. Suppose that U is $\rho(L)$-open. Let $t \in U$. By Proposition 4.4, there exist $x, y \in L$ such that $t \in \Uparrow x \cap (\{y\} \cup \Downarrow y) \subseteq U$. Hence, $t \in \uparrow(\Uparrow x \cap (\{y\} \cup \Downarrow y)) \subseteq \uparrow U = U$. By Lemma 4.5, we have $\uparrow(\Uparrow x \cap (\{y\} \cup \Downarrow y)) \in \sigma(L)$. By the arbitrariness of t, U is Scott open.

(2) Let $W \in \rho(L)$. For all $t \in \uparrow W$, there exists $a \in W$ such that $a \leqslant t$. By the continuity of L and Proposition 4.4, there exist $x, y \in L$ such that $a \in \Uparrow x \cap (\{y\} \cup \Downarrow y) \subseteq W$. Hence, $t \in \uparrow a \subseteq \uparrow(\Uparrow x \cap (\{y\} \cup \Downarrow y)) \subseteq \uparrow W$. It follows from Lemma 4.5 that $\uparrow(\Uparrow x \cap (\{y\} \cup \Downarrow y))$ is Scott open. By the arbitrariness of t, we have $\uparrow W \in \sigma(L)$.

(3) Let $x \in L$. Suppose that x is compact. It is easy to see that $\uparrow x$ is Scott open. Hence, $\{x\} = \uparrow x \cap \downarrow x$ is $\rho(L)$-open. Conversely, suppose that $\{x\}$ is $\rho(L)$-open. By (2), $\uparrow x$ is Scott open. This shows that x is compact.

(4) Let D be a directed subset of a $\mu(L)$-closed set F with existing $\sup D$. Suppose that $\sup D \notin F$. Then $\sup D \in L \backslash F$ and $L \backslash F$ is $\mu(L)$-open. By Proposition 2.6, there exist x, $y \in L$ such that $\sup D \in \Uparrow x \cap \downarrow y \subseteq L \backslash F$. By Proposition 2.1, there is $z \in L$ such that $x \ll z \ll \sup D \leqslant y$. This shows that there is $d \in D$ such that $x \ll z \leqslant d \leqslant y$. Hence, $D \cap (L \backslash F) \neq \emptyset$, a contradiction to $D \subseteq F$. $\square$

Corollary 4.7 *Let L be a continuous poset.*

(1) If X is an upper set, then $\mathrm{int}_\sigma(X) = \mathrm{int}_\lambda(X) = \mathrm{int}_\mu(X) = \mathrm{int}_\rho(X)$;

(2) If X is a lower set, then $\mathrm{cl}_\sigma(X) = \mathrm{cl}_\lambda(X) = \mathrm{cl}_\mu(X) = \mathrm{cl}_\rho(X)$.

Proof (1) Let X be an upper set. It follows from Proposition 4.2 that $\mathrm{int}_\sigma(X) \subseteq \mathrm{int}_\lambda(X) \subseteq \mathrm{int}_\mu(X) \subseteq \mathrm{int}_\rho(X)$. By Theorem 4.6(2), $\uparrow \mathrm{int}_\rho(X) \in \sigma(L)$. It follows from $\uparrow \mathrm{int}_\rho(X) \subseteq \uparrow X = X$ that $\mathrm{int}_\rho(X) \subseteq \uparrow \mathrm{int}_\rho(X) \subseteq \mathrm{int}_\sigma(X)$. So, $\mathrm{int}_\sigma(X) = \mathrm{int}_\lambda(X) = \mathrm{int}_\mu(X) = \mathrm{int}_\rho(X)$.

(2) The conclusion follows immediately from (1). $\square$

The following example shows that the $\rho(L)$-closed set on a completely distributive lattice need not be closed under directed suprema.

Example 4.8 Let $I = [0, 1]$ be the unit interval and let $S_I = \{[a, b] | a, b \in I\}$. Define a partial order " $\leqslant$ " on S_I: $\forall [a, b], [c, d] \in S_I$, $[a, b] \leqslant [c, d] \Longleftrightarrow a \leqslant c$ and $b \leqslant d$. Then

$$\sup\{[a_j, b_j] \mid j \in J\} = [\sup\{a_j\}_{j \in J}, \sup\{b_j\}_{j \in J}]$$

for any family $\{[a_j, b_j] | j \in J\}$ and $[a, b] \ll [c, d] \Longleftrightarrow a < c$ and $b < d$ for all $[a, b], [c, d] \in S_I$.

It is straightforward to prove that $(S_I, \leqslant)$ is a completely distributive lattice [12, Theorem 2.4]. Let $x = [\frac{1}{2}, \frac{3}{4}]$ and $y = [\frac{2}{3}, 1]$. Pick an increasing sequence $\{a_n\}_{n\in N^+}$ such that $a_1 > \frac{1}{2}$ and $\lim_{n\to\infty} a_n = \frac{2}{3}$. Let $D = \{[a_n, 1] | n \in N^+\}$. Then D is a directed subset of S_I. Clearly, the set $U = \Uparrow x \cap (\{y\} \cup \Downarrow y)$ is $\rho(S_I)$ open and $D \cap U = \emptyset$. Thus, $L \backslash U$ is $\rho(S_I)$ closed and $D \subseteq L\backslash U$. However, we have $\sup D = [\frac{2}{3}, 1] \in U$.

Proposition 4.9 *Let L be a continuous poset. If $\rho(L)$ is compact, then $\rho(L) = \mu(L)$.*

Proof Let L be a continuous poset with a compact $\rho(L)$ topology. By Proposition 4.2 and Proposition 4.3(2), both $\rho(L)$ and $\mu(L)$ are compact Hausdorff. It follows from the exactness of compact Hausdorff topology [13, P.181, Exercise 1] that $\rho(L) = \mu(L)$. $\square$

Proposition 4.10 *Let L be a continuous poset. If the set $A_x = \downarrow x \backslash (\{x\} \cup \Downarrow x)$ is finite for all $x \in L$, then $\rho(L) = \mu(L)$.*

Proof By Proposition 4.2, we have $\mu(L) \subseteq \rho(L)$. Suppose that W is $\rho(L)$-open. Let $z \in W$. By Proposition 4.4, there exist $x, y \in L$ such that $z \in \Uparrow x \cap (\{y\} \cup \Downarrow y) \subseteq W$. Hence, $z \in \Uparrow x \cap (\{z\} \cup \Downarrow z) \subseteq \Uparrow x \cap (\{y\} \cup \Downarrow y) \subseteq W$. By the continuity of L, we have $z = \bigvee \Downarrow z$. Then for all $t \in A_z = \downarrow z \backslash (\{z\} \cup \Downarrow z)$, there exists $x_t \ll z$ such that $x_t \not\leqslant t$. By the finiteness of A_z and the directedness of $\Downarrow z$, there is $h \in \Downarrow z$ such that $x \leqslant h$ and $x_t \leqslant h$ for all $t \in A_z$. Hence, $h \not\leqslant t$ for all $t \in A_z$. This shows that $\Uparrow h \cap \downarrow z \cap A_z = \emptyset$. Therefore, $z \in \Uparrow h \cap \downarrow z \subseteq \Uparrow x \cap (\{z\} \cup \Downarrow z) \subseteq W$. By Proposition 2.6 and the arbitrariness of z, W is $\mu(L)$-open and hence $\rho(L) \subseteq \mu(L)$. $\square$

The following example shows that the density topology does not coincide with the measurement topology on completely distributive lattices.

Example 4.11 Let $(S_I, \leqslant)$ be the poset defined in Example 4.8. Pick $x = [\frac{1}{2}, \frac{3}{4}]$ and $y = [\frac{2}{3}, 1]$. Clearly, $x, y \in S_I$ and $x \ll y$. Then $y \in \Uparrow x \cap (\{y\} \cup \Downarrow y)$ and $\Uparrow x \cap (\{y\} \cup \Downarrow y)$ is a basic $\rho(S_I)$ open set by Proposition 4.4. Suppose that $\Uparrow x \cap (\{y\} \cup \Downarrow y)$ is also $\mu(S_I)$ open. By Proposition 2.6, there is $z = [c, d] \in S_I$ such that $y \in \Uparrow z \cap \downarrow y \subseteq \Uparrow x \cap (\{y\} \cup \Downarrow y)$. This shows that $z = [c, d] \ll [\frac{2}{3}, 1] = y$. Thus, $c < \frac{2}{3}$ and $d < 1$. There exists r such that $c < r < \frac{2}{3}$. Let $h = [r, 1]$. Then $h \in \Uparrow z \cap \downarrow y$ but $h \notin \Uparrow x \cap (\{y\} \cup \Downarrow y)$, a contradiction to the assumption that $\Uparrow z \cap \downarrow y \subseteq \Uparrow x \cap (\{y\} \cup \Downarrow y)$. Therefore, $\rho(S_I) \not\subseteq \mu(S_I)$.

Proposition 4.12 *Let L be a continuous poset and $W \subseteq L$ an order convex subset. Then W is $\mu(L)$-open if and only if W is $\rho(L)$-open.*

Proof $\Longrightarrow$. Apply Proposition 4.2.

$\Longleftarrow$. Suppose that W is $\rho(L)$-open. Let $t \in W$. By Proposition 4.4, there exist $x, y \in L$ such that $t \in \Uparrow x \cap (\{y\} \cup \Downarrow y) \subseteq W$. This shows that $y \in W$ and $x \ll t \leqslant y$. By the continuity of L and Proposition 2.1, there is $s \in L$ such that $x \ll s \ll t \leqslant y$. This shows that $s \in W$. Since W is order convex and $s, y \in W$, we have $t \in \Uparrow s \cap \downarrow y \subseteq W$. By the arbitrariness of t and Proposition 2.6, W is $\mu(L)$-open. $\square$

Theorem 4.13 *Let L be a poset and consider the following conditions:*

(1) L is a continuous poset;

(2) $W = \bigcup\{\twoheaduparrow x \cap (\{y\} \cup \twoheaddownarrow y)|x, y \in W\}$ for all order convex $\rho(L)$-open set W;

(3) $W = \bigcup\{\twoheaduparrow x \cap \downarrow y|x, y \in W\}$ for all order convex $\mu(L)$-open set W.

Then (1) $\Longrightarrow$ (2) $\Longleftrightarrow$ (3). Moreover, if L is a sup semilattice, then all three conditions are equivalent.

Proof (1) $\Longrightarrow$ (2). Let W be an order convex $\rho(L)$-open set. It is easy to see that

$$\bigcup\{\twoheaduparrow x \cap (\{y\} \cup \twoheaddownarrow y)|x, y \in W\} \subseteq W.$$

Let $t \in W$. By Proposition 4.4, there exist x, $y \in L$ such that $t \in \twoheaduparrow x \cap (\{y\} \cup \twoheaddownarrow y) \subseteq W$. So, $x \ll t \leqslant y$. By the continuity of L and Proposition 2.1, there is $s \in L$ such that $x \ll s \ll t \leqslant y$. This shows that $s \in W$. Therefore, $t \in \twoheaduparrow s \cap (\{t\} \cup \twoheaddownarrow t) \subseteq \bigcup\{\twoheaduparrow x \cap (\{y\} \cup \twoheaddownarrow y)|x, y \in W\}$. By the arbitrariness of t, we have $W \subseteq \bigcup\{\twoheaduparrow x \cap (\{y\} \cup \twoheaddownarrow y) \mid x, y \in W\}$ and thus

$$W = \bigcup\{\twoheaduparrow x \cap (\{y\} \cup \twoheaddownarrow y)|x, y \in W\}.$$

(2)$\Longleftrightarrow$(3). Apply Proposition 4.12.

Let L be a sup semilattice. Then for all $x \in L$, the set $\twoheaddownarrow x$ is directed. Clearly, x is an upper bound of the set $\twoheaddownarrow x$. Let z be any upper bound of the set $\twoheaddownarrow x$. Suppose that $x \nleqslant z$. Then $x \in L\setminus \downarrow z$. Since $L\setminus \downarrow z$ is order convex $\mu(L)$-open, there exist u, $v \in L\setminus \downarrow z$ such that $x \in \twoheaduparrow u \cap \downarrow v \subseteq L\setminus \downarrow z$ by (3). This shows that $u \ll x$ but $u \nleqslant z$, a contradiction to the assumption that z is an upper bound of the set $\twoheaddownarrow x$. So, $x \leqslant z$. By the arbitrariness of z, we have $x = \bigvee \twoheaddownarrow x$. Thus, L is a continuous poset. $\square$

We can characterize bases of continuous posets via the density topology.

Theorem 4.14 *The following are equivalent for a continuous poset L and $B \subseteq L$:*

(1) B is a basis of L;

(2) B is a $\rho(L)$ dense subset of L;

(3) B is a $\mu(L)$ dense subset of L.

Proof (1) $\Longrightarrow$ (2). Let B be a basis of L and consider a nonempty basic $\rho(L)$-open set $\twoheaduparrow x \cap (\{y\} \cup \twoheaddownarrow y)$. It follows from Theorem 3.7 that $\twoheaduparrow x \cap (\{y\} \cup \twoheaddownarrow y) \cap B \neq \emptyset$. This shows that B is a $\rho(L)$ dense subset of L.

(2)$\Longrightarrow$(3). Apply Proposition 4.2.

(3)$\Longrightarrow$(1). Let B be a $\mu(L)$ dense subset of L. Given $x \ll y$ in L. By Proposition 2.1, there exist s, $t \in L$ such that $x \ll s \ll t \ll y$. Since $\twoheaduparrow s \cap \downarrow t$ is a nonempty basic $\mu(L)$-open set, we have $\twoheaduparrow s \cap \downarrow t \cap B \neq \emptyset$. Thus, there is $b \in B$ such that $x \ll s \ll b \leqslant t \ll y$. By Proposition 2.4, B is a basis of L. $\square$

Corollary 4.15 *Let L be a continuous poset. Then the following are equivalent:*

(1) $\sigma(L)$ is second countable;

(2) L has a countable basis;

(3) $\rho(L)$ is separable;

(4) $\mu(L)$ is separable.

Proof (1)$\Longrightarrow$(2). Let $\mathcal{B}$ be a countable base for $\sigma(L)$. Let $A = \{(U_1, U_2) | U_1, U_2 \in \mathcal{B}, \exists\, b \in L$ such that $U_2 \subseteq \Uparrow b \subseteq \uparrow b \subseteq U_1\}$. Clearly, the set A is countable. For all $\alpha = (U_1, U_2) \in A$, pick $b_\alpha \in L$ such that $U_2 \subseteq \Uparrow b_\alpha \subseteq \uparrow b_\alpha \subseteq U_1$. Let $B = \{b_\alpha | \alpha \in A\}$. Then B is also a countable set. Given $x \ll y$ in L. Since $\Uparrow x \in \sigma(L)$ and $\mathcal{B}$ is a countable base for $\sigma(L)$, there is $V_1 \in \mathcal{B}$ such that $y \in V_1 \subseteq \Uparrow x$. By the continuity of L and the Scott openness of V_1, there is $t \in V_1$ such that $y \in \Uparrow t \subseteq \uparrow t \subseteq V_1 \subseteq \Uparrow x$. Since $\Uparrow t \in \sigma(L)$ and $\mathcal{B}$ is a countable base for $\sigma(L)$, there is $V_2 \in \mathcal{B}$ such that $y \in V_2 \subseteq \Uparrow t \subseteq \uparrow t \subseteq V_1 \subseteq \Uparrow x$. This shows that $\beta = (V_1, V_2) \in A$. So, there is $b_\beta \in B$ such that $y \in V_2 \subseteq \Uparrow b_\beta \subseteq \uparrow b_\beta \subseteq V_1 \subseteq \Uparrow x$. By Proposition 2.4, B is a countable basis of L.

(2)$\Longrightarrow$(1). Let B be a countable basis of L. It is straightforward to prove that the family $\{\Uparrow b | b \in B\}$ is a countable base for the Scott topology $\sigma(L)$. Hence, $\sigma(L)$ is second countable.

(2)$\Longrightarrow$(3). Apply Theorem 4.14.

(3)$\Longrightarrow$(4). Apply Proposition 4.2.

(4)$\Longrightarrow$(2). Apply Theorem 4.14. $\Box$

In the context of continuous posets, we can characterize the algebraicity of the posets via the measurement topology and the density topology.

Theorem 4.16 *Let L be a continuous poset. Then the following are equivalent:*

(1) L is algebraic;

(2) The intersection of $\rho(L)$ dense sets is $\rho(L)$ dense;

(3) The intersection of $\mu(L)$ dense sets is $\mu(L)$ dense.

Proof (1)$\Longrightarrow$(2). Let L be an algebraic poset. Then $K(L)$ is the smallest basis of L. By Theorem 4.14, $K(L)$ is the smallest $\rho(L)$ dense set of L. Hence, the intersection of $\rho(L)$ dense sets contains $K(L)$ and is $\rho(L)$ dense.

(2)$\Longrightarrow$(3). Apply Theorem 4.14.

(3)$\Longrightarrow$(1). Let B be the intersection of all $\mu(L)$ dense sets of L. By (3) and Theorem 4.14, B is the smallest $\mu(L)$ dense set of L and hence the smallest basis of L. It is straightforward to prove that $B = K(L)$. So, L is algebraic. $\Box$

Acknowledgements We thank the referees for their time and comments.

References

[1] D. SCOTT. *Continuous Lattices*. Lecture Notes in Mathematics 274, Springer-Verlag, Berlin, 1972.

[2] G. GIERZ, K. HOFMANN, K. KEIMEL, et al. *Continuous Lattices and Domains*. Cambridge University Press, Cambridge, 2003.

[3] J. LAWSON. *The duality of continuous posets*. Houston J. Math., 1979, **5**(3): 357–394.

[4] J. LAWSON, Luoshan XU. *Posets having continuous intervals*. Theoret. Comput. Sci., 2004, **316**(1-3): 89–103.

[5] Xuxin MAO, Luoshan XU. *Quasicontinuity of posets via Scott topology and sobrification*. Order, 2006, **23**(4): 359–369.

[6] Luoshan XU. *Continuity of posets via Scott topology and sobrification*. Topology Appl., 2006, **153**(11): 1886–1894.

[7] Luoshan XU. *Measurement topology and full measure on posets*. Mohu Xitong yu Shuxue, 2007, **21**(1): 28–35. (in Chinese)

[8] Han ZHANG. *A note on continuous partially ordered sets*. Semigroup Forum, 1993, **47**(1): 101–104.

[9] Bin ZHAO, Yihui ZHOU. *The category of supercontinuous poset*. J. Math. Anal. Appl., 2006, **320**(2): 632–641.

[10] K. MARTIN. *A Foundation for Computation*. Thesis (Ph.D.)-Tulane University, 2000.

[11] D. RUSU, G. CIOBANU. *Essential and density topologies of continuous domains*. Ann. Pure Appl. Logic., 2016, **167**(9): 726–736.

[12] Luoshan XU. *Intrinsic topologies of the interval number system and their metric representations*. J. Yangzhou Univ. Nat. Sci. Ed., 1999, **2**(1): 1–5. (in Chinese)

[13] J. R. MUNKRES. *Topology: A First Course*. Prentice-Hall, Inc, 1975.