Journal of Mathematical Research with Applications
Jul., 2018, Vol. 38, No. 4, pp. 351–365
DOI:10.3770/j.issn:2095-2651.2018.04.003
Http://jmre.dlut.edu.cn

Relationships among Matroids Induced by Covering-Based Upper Approximation Operators

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Abstract Covering-based rough sets, as a technique of granular computing, can be a useful tool for dealing with inexact, uncertain or vague knowledge in information systems. Matroids generalize linear independence in vector spaces, graph theory and provide well established platforms for greedy algorithm design. In this paper, we construct three types of matroidal structures of covering-based rough sets. Moreover, through these three types of matroids, we study the relationships among these matroids induced by six types of covering-based upper approximation operators. First, we construct three families of sets by indiscernible neighborhoods, neighborhoods and close friends, respectively. Moreover, we prove that they satisfy independent set axioms of matroids. In this way, three types of matroidal structures of covering-based rough sets are constructed. Secondly, we study some characteristics of the three types of matroid, such as dependent sets, circuits, rank function and closure. Finally, by comparing independent sets, we study relationships among these matroids induced by six types of covering-based upper approximation operators.

Keywords covering; matroid; rough set; upper approximation operator; indiscernible neighborhood; neighborhood; close friend

MR(2010) Subject Classification 03E20; 03E72; 03F99

1. Introduction

Rough sets provide an important tool to deal with data characterized by uncertainty and vagueness. Since it was proposed by Pawlak [1], rough sets have been generalized from different viewpoints such as the similarity relation [2] or the tolerance relations [3, 4] instead of the equivalence relation, and a covering over the universe [5] instead of a partition, and the neighborhood instead of the equivalence class [6]. Covering-based rough set theory is more general and complex than rough sets [7, 8]. Currently, the connections between covering-based rough sets and other theories are attracting increasing attention. For example, covering-based rough sets have been connected with matroids [9, 10], with topology [11], and with fuzzy sets [12, 13]. Many other important contributions are presented in [14, 15].

Matroids proposed by Whitney [16] are a generalization of linear algebra and graph theory. Integrating the characteristics of linear algebra and graph theory, matroids have sound theoretical

Received March 13, 2017; Accepted May 17, 2018
Supported by the Research Foundation for Middle-aged and Young Scientist of Fujian Province (Grant No. JAT170731).
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foundations and wide applications. In theory, matroids have powerful axiomatic systems which provide a platform for connecting them with other theories, such as rough sets [17,18], generalized rough sets based on relations, covering-based rough sets and geometric lattices [19, 20]. In application, matroids have been successfully applied to diverse fields, such as combinatorial optimization, algorithm design and cryptology [21, 22]. In order to broaden the theoretical areas and application areas of covering-based rough sets and matroids, some researchers have combined them with each other. Matroids provide an interesting and natural research topic in rough set theory [23–25].

In our previous work [26], we studied the relationships between five types of covering-based upper approximation operators and closure operators and presented the necessary and sufficient conditions for five types of covering-based upper approximation operators to be closure operators. In fact, the conditions for the fifth and the sixth type of covering-based upper approximation operators to be closure operators are the same. In a word, based the above conditions, the six types of closure operators all can determine a matroid, respectively.

In particular, in [27] the modularity of matroidal structure of covering-based rough sets was studied from the viewpoint of lattices. And algebraic and topological structures of covering-based rough were established through down-sets and up-sets in [8]. Different from the above results, we mainly investigate the characteristics of matroidal structures of covering-based rough sets from the viewpoint of closure axiom and the relationships between them.

In this paper, we construct three types of matroidal structures of covering-based rough sets and study some characteristics of them. Furthermore, the relationships among these matroids induced by six types of covering-based upper approximation operators are studied through these three types of matroidal structures of covering-based rough sets. First, we construct three families of sets by using indiscernible neighborhoods, neighborhoods and close friends, respectively. And we further prove these three families of sets satisfy independent set axiom of matroids. In other words, three types of matroidal structures of covering-based rough sets are established in this way. Secondly, we study some characteristics of these three types of matroidal structures of covering-based rough sets, such as dependent sets, circuits, rank function and closure. Thirdly, through comparing independent set, we point out a one-to-one relationship between these three types of matroids and three matroids induced by the second, third and sixth types of covering-based upper approximation operators, respectively. Finally, combining with these three types of matroidal structures of covering-based rough sets, we study the relationships among these matroids induced by six types of covering-based upper approximation operators, respectively.

The remainder of this paper is organized as follows. In Section 2, we review some basic knowledge about covering-based rough sets and matroids. In Section 3, we construct three types of matroidal structures of covering-based rough sets and study their characteristics, such as dependent sets, circuits, rank function and closure sets. In Section 4, the relationships among the matroids induced by six types of covering-based upper approximation operators are studied. This paper concludes in Section 5.

2. Preliminaries

In this section, we review some fundamental definitions of classical rough sets, covering rough sets and matroids.

2.1. Classical rough sets

Let U be a finite and nonempty set, and let R be an equivalence relation on U. R generates a partition $U/R = \{[x]_R : x \in U\}$ on U, where $[x]_R$ is an equivalence class of x. We call it an elementary set of R in rough sets. For any $X \subseteq U$, we describe X by the elementary set of R, and the following two sets:

$$R_*(X) = \{x \in U | [x]_R \subseteq X\}, \quad R^*(X) = \{x \in U | [x]_R \cap X \neq \emptyset\}$$

are called the lower and upper approximations of X, respectively.

Propositions 2.1 ([1]) *Let $\emptyset$ be the empty set and $\sim X$ the complement of X in U. Pawlak's rough sets have the following properties:*

$(1L)$ $R_*(U) = U$;	$(1H)$ $R^*(U) = U$;
$(2L)$ $R_*(\emptyset) = \emptyset$;	$(2H)$ $R^*(\emptyset) = \emptyset$;
$(3L)$ $R_*(X) \subseteq X$;	$(3H)$ $X \subseteq R^*(X)$;
$(4L)$ $R_*(X \cap Y) = R_*(X) \cap R_*(Y)$;	$(4H)$ $R^*(X \cup Y) = R^*(X) \cup R^*(Y)$;
$(5L)$ $R_*(R_*(X)) = R_*(X)$;	$(5H)$ $R^*(R^*(X)) = R^*(X)$;
$(6LH)$ $R_*(X) = \sim R^*(\sim X)$;	$(7H)$ $X \subseteq Y \Rightarrow R^*(X) \subseteq R^*(Y)$;
$(7L)$ $X \subseteq Y \Rightarrow R_*(X) \subseteq R_*(Y)$;	$(8H)$ $R^*(\sim R^*(X)) = \sim R^*(X)$;
$(8L)$ $R_*(\sim R_*(X)) = \sim R_*(X)$;	$(9H)$ $\forall K \in U/R,\ R^*(K) = K$.

2.2. Covering-based rough sets

In this section, we present some basic concepts of covering-based rough sets, such as indiscernible neighborhood, minimal description and six types of covering-based upper approximation operators.

Definition 2.2 ([23]) *Let U be a universe of discourse and $\mathcal{C}$ be a family of subsets of U. If none of subsets in $\mathcal{C}$ is empty and $\cup\mathcal{C} = U$, then $\mathcal{C}$ is called a covering of U.*

In the following, we introduce the concept of covering approximation space.

Definition 2.3 ([23]) *Let U be a universe of discourse and $\mathcal{C}$ be a family of subsets of U. If none of subsets in $\mathcal{C}$ is empty and $\cup\mathcal{C} = U$, then $\mathcal{C}$ is called a covering of U.*

Definition 2.4 ([23]) *Let $\mathcal{C}$ be a covering of U. $\forall x \in U$,*

$$Md(x) = \{K \in \mathcal{C} : x \in K \wedge (\forall S \in \mathcal{C} \wedge x \in S \wedge S \subseteq K \rightarrow K = S)\}$$

is called the minimal description of x with respect to $\mathcal{C}$.

Definition 2.5 ([24]) *Let $\mathcal{C}$ be a covering of U. $\mathcal{C}$ is called unary if $\forall x \in U, |Md(x)| = 1$.*

Example 2.6 (A unary covering) *Let* $U = \{a, b, c, d\}$, $K_1 = \{a, b, c\}$, $K_2 = \{a, b\}$, $K_3 = \{a, b, c, d\}$, $\mathcal{C} = \{K_1, K_2, K_3\}$. $\mathcal{C}$ *is a unary covering of* U.

Example 2.7 (Not a unary covering) *Let* $U = \{a, b, c, d\}$, $K_1 = \{b, c\}$, $K_2 = \{a, b\}$, $K_3 = \{a, b, c, d\}$, $\mathcal{C} = \{K_1, K_2, K_3\}$. $\mathcal{C}$ *is a covering of* U, *but it is not a unary covering because* $Md(b) = \{K_1, K_2\}$.

Neighborhood, indiscernible neighborhood and close friend are important concepts of covering-based rough sets. And they have different properties.

Definition 2.8 ([24]) *Let* $(U, \mathcal{C})$ *be a covering approximation space. For all* $x \in U$, $\cap\{K \in \mathcal{C} : x \in K\}$ *is called the neighborhood of* x *with respect to* $\mathcal{C}$ *and denoted as* $N_{\mathcal{C}}(x)$.

Definition 2.9 ([24]) *Let* $(U, \mathcal{C})$ *be a covering approximation space.* $\forall x \in U, \cup\{K \in \mathcal{C} : x \in K\}$ *is called the indiscernible neighborhood of* x *and denoted as* $I_{\mathcal{C}}(x)$.

Definition 2.10 ([24]) *Let* $(U, \mathcal{C})$ *be a covering approximation space.* $\cup Md(x)$ *is called the close friends of* x *and denoted as* $CF_{\mathcal{C}}(x)$.

When the covering is clear, we omit the lowercase $\mathcal{C}$ in the minimal description. $N_{\mathcal{C}}(x), I_{\mathcal{C}}(x)$ and $CF_{\mathcal{C}}(x)$ are denoted by $N(x), I(x)$ and $CF_{\mathcal{C}}(x)$, respectively.

The reducible element of a covering is an important concept which is first proposed by Zhu and Wang in [23].

Definition 2.11 ([23]) *Let* $\mathcal{C}$ *be a covering of* U, *and* $K \in \mathcal{C}$. *If* K *is a union of some elements in* $\mathcal{C} - \{K\}$, *then we say* K *is a reducible element of* $\mathcal{C}$, *otherwise,* K *is an irreducible element of* $\mathcal{C}$.

Based on the reducible element, we know that some blocks of a covering is redundant and some blocks of the covering is essential. The reduction of a covering is used to express the essence of the covering.

Definition 2.12 ([23]) *Let* $\mathcal{C}$ *be a covering of* U. *If every element in* $\mathcal{C}$ *is an irreducible element, we say* $\mathcal{C}$ *is irreducible, otherwise* $\mathcal{C}$ *is reducible. And the family of irreducible elements of* $\mathcal{C}$ *is called the reduction of* $\mathcal{C}$, *denoted by reduct*$(\mathcal{C})$.

Definition 2.13 ([23]) *Let* $\mathcal{C}$ *be a covering of* U. *The operations* CL: $P(U) \longrightarrow P(U)$ *are defined as follows:* $\forall X \in P(U)$,

$$CL(X) = \cup\{K \in \mathcal{C} | K \subseteq X\}.$$

We call $CL(X)$ *the covering lower approximation operator.*

The operations FH, SH, TH, RH, IH *and* XH: $P(U) \longrightarrow P(U)$ *are defined as follows: for* $\forall X \in P(U)$,

$$FH(X) = CL(X) \cup \{Md(x) | x \in X - CL(X)\},$$
$$SH(X) = \cup\{K \in \mathcal{C} | K \cap X \neq \emptyset\} = \cup\{I(x) | x \in X\},$$
$$TH(X) = \cup\{Md(x) | x \in X\},$$

$$RH(X) = CL(X) \cup \{K|K \cap (X - CL(X)) \neq \emptyset\},$$
$$IH(X) = CL(X) \cup \{N(x)|x \in X - CL(X)\},$$
$$XH(X) = \{x|X \cap N(x) \neq \emptyset\}.$$

FH, SH, TH, RH, IH and XH are called the first, the second, the third, the fourth, the fifth and the sixth covering upper approximation operators with respect to the covering $\mathcal{C}$, respectively.

2.3. Matroids

Matroids largely borrow from matrix theory and graph theory. They represent the linear independence with high abstraction and serve as a useful tool for dealing with discrete data. In this subsection, matroid and some important characteristics of it such as circuit, rank function and closure are introduced.

Definition 2.14 ([16]) *A matroid M is an ordered pair $(U, \mathcal{I})$, where U (the ground set) is a finite set, and $\mathcal{I}$ (the independent sets) a family of subsets of U with the following properties (Independent set axiom):*

(I1) $\emptyset \in \mathcal{I}$;

(I2) If $I \in \mathcal{I}, I' \subseteq I$, *then* $I' \in \mathcal{I}$;

(I3) If $I_1, I_2 \in \mathcal{I}$ *and* $|I_1| < |I_2|$, *then there exists* $e \in I_2 - I_1$ *such that* $I_1 \bigcup \{e\} \in \mathcal{I}$, *where* $|I|$ *denotes the cardinality of* I.

For convenience of illustration, for any family $\mathbf{A}$ of subsets of U, we denote the following symbols.

Definition 2.15 *Let U be a finite universe and $\mathbf{A}$ a family of subsets of U. Two symbols are defined as follows:*

$$\mathrm{Min}(\mathbf{A}) = \{X \in \mathbf{A} : \forall Y \in \mathbf{A}, Y \subseteq X \Rightarrow X = Y\},$$
$$\mathrm{Opp}(\mathbf{A}) = \{X \subseteq U : X \notin \mathbf{A}\}.$$

In the following proposition, we will introduce the concept of dependent set of matroid.

Definition 2.16 ([16]) *Let $M = (U, \mathcal{I})$ be a matroid and $X \subseteq U$. If $X \notin \mathcal{I}$, then X is called a dependent set. The family of all dependent sets of M is denoted by $\mathbf{D}(M)$, where* $\mathbf{D}(M) = \mathrm{Opp}(\mathcal{I})$.

The dependent set of a matroid generalizes the linear dependence in vector spaces and the cycle in graphs. A circuit of a matroid is a minimal dependent set.

Definition 2.17 ([16]) *Let $M = (U, \mathcal{I})$ be a matroid. Any minimal dependent set in M is called a circuit of M, and we denote the family of all circuits of M by $\mathbf{C}(M)$, i.e.,* $\mathbf{C}(M) = \mathrm{Min}(\mathrm{Opp}(\mathcal{I}))$.

In matroid theory, the rank function serves as a quantitative tool. The cardinality of a maximal independent set of any subset can be expressed by the rank function.

Definition 2.18 ([16]) *Let $M = (U, \mathcal{I})$ be a matroid. The rank function $r_M : P(U) \longrightarrow Z^+$ of*

M is defined as follows: for all $X \subseteq U$,

$$r_M(X) = \max\{|I| : I \subseteq X, I \in \mathcal{I}\}.$$

Based on the rank function of a matroid, the closure operator which reflects the dependency between a set and elements is defined.

Definition 2.19 ([16]) *Let $M = (U, \mathcal{I})$ be a matroid. For all $X \subseteq U$, the closure operator $cl_M : P(U) \longrightarrow P(U)$ of M is defined as*

$$cl_M(X) = \{e \in U : r_M(X) = r_M(X \cup \{e\})\}.$$

$cl_M(X)$ is called the closure of X in M.

There are many different but equivalent ways to define a matroid. We can generate a matroid in terms of closure axiom. In other words, an operator satisfies the following four conditions if and only if it is a closure operator of a matroid.

Proposition 2.20 ([16]) *Let cl: $P(U) \longrightarrow P(U)$ be an operator. Then there exists a matroid M such that $cl = cl_M$ if and only if cl satisfies the following conditions:*

(CL1) If $\forall X \subseteq U$, then $X \subseteq cl(X)$;

(CL2) If $X \subseteq Y \subseteq U$, then $cl(X) \subseteq cl(Y)$;

(CL3) If $\forall X \subseteq U$, then $cl(cl(X)) = cl(X)$;

(CL4) If $x, y \in U$, and $y \in cl(X \cup x) - cl(X)$, then $x \in cl(X \cup y)$.

3. Matroidal structure of covering-based rough sets

In this section, we construct three types of matroidal structures of covering-based rough sets. In the following, we define three families of sets by indiscernible neighborhoods, neighborhoods and close friends of a covering, respectively. Moreover, some characteristics of the three types of matroidal structures are presented.

Definition 3.1 *Let $\mathcal{C}$ be a covering of U. Then we define three families of sets as follows:*

$$\mathcal{I}_F(\mathcal{C}) = \{X \subseteq U : \forall x, y \in X, x \neq y \to I_{\mathcal{C}}(x) \neq I_{\mathcal{C}}(y)\}.$$

$$\mathcal{I}_N(\mathcal{C}) = \{X \subseteq U : \forall x, y \in X, x \neq y \to N_{\mathcal{C}}(x) \neq N_{\mathcal{C}}(y)\}.$$

$$\mathcal{I}_{CF}(\mathcal{C}) = \{X \subseteq U : \forall x, y \in X, x \neq y \to CF_{\mathcal{C}}(x) \neq CF_{\mathcal{C}}(y)\}.$$

In the following proposition, the three families of sets will be proved to satisfy the independent set axiom.

Proposition 3.2 *Let $\mathcal{C}$ be a covering of U. Then $\mathcal{I}_F(\mathcal{C})$, $\mathcal{I}_N(\mathcal{C})$ and $\mathcal{I}_{CF}(\mathcal{C})$ satisfy (I1), (I2) and (I3) of Definition 2.14.*

Proof Firstly, we prove that $\mathcal{I}_F(\mathcal{C})$ satisfies (I1), (I2) and (I3) of Definition 2.14.

(I1) It is obvious that $\mathcal{I}_F(\mathcal{C})$ satisfies (I1).

(I2) If $I \in \mathcal{I}_F(\mathcal{C}), I' \subseteq I$, suppose $I' \notin \mathcal{I}_F(\mathcal{C})$, according to Definition 3.1, then there exist

$x, y \in I'$ such that $I_{\mathcal{C}}(x) = I_{\mathcal{C}}(y)$. Since $I' \subseteq I$, there exist $x, y \in I$ such that$I_{\mathcal{C}}(x) = I_{\mathcal{C}}(y)$, which is contradictory to $I \in \mathcal{I}_F(\mathcal{C})$. Hence $I' \notin \mathcal{I}_F(\mathcal{C})$, therefore $\mathcal{I}_F(\mathcal{C})$ satisfies (I2).

(I3) Suppose that $I_1, I_2 \in \mathcal{I}_F(\mathcal{C})$ and $|I_1| < |I_2|$. According to Definition 3.1, since $I_1, I_2 \in \mathcal{I}_F(\mathcal{C})$, we have that for all $x_1, y_1 \in I_1, x_1 \neq y_1, I_{\mathcal{C}}(x_1) \neq I_{\mathcal{C}}(y_1)$ and $x_2, y_2 \in I_2, x_2 \neq y_2, I_{\mathcal{C}}(x_2) \neq I_{\mathcal{C}}(y_2)$. Suppose that for all $u \in I_2 - I_1$. If $I_1 \cup \{u\} \notin \mathcal{I}_F(\mathcal{C})$, then there exists one and only one $x \in I_1 - I_2$ such that $I_{\mathcal{C}}(u) = I_{\mathcal{C}}(x)$. Hence $|I_2 - I_1| = |I_1 - I_2|$, which is contradictory to $|I_1| < |I_2|$. Therefore, there exists $u \in I_2 - I_1$ such that $I_1 \cup \{u\} \in \mathcal{I}_F(\mathcal{C})$. □

Similar to the above proof, it is easy to prove that $\mathcal{I}_N(\mathcal{C})$ and $\mathcal{I}_{CF}(\mathcal{C})$ also satisfy (I1), (I2) and (I3) of Definition 2.14.

As shown in the above proposition, $\mathcal{I}_F(\mathcal{C})$, $\mathcal{I}_N(\mathcal{C})$ and $\mathcal{I}_{CF}(\mathcal{C})$ are independent sets, so they can generate a matroid, respectively.

Definition 3.3 *Let $\mathcal{C}$ be a covering of U. The matroid with $\mathcal{I}_F(\mathcal{C})$ as its independent set is denoted by $M_F(\mathcal{C})$. The matroid with $\mathcal{I}_N(\mathcal{C})$ as its independent set is denoted by $M_N(\mathcal{C})$. And the matroid with $\mathcal{I}_{CF}(\mathcal{C})$ as its independent set is denoted by $M_{CF}(\mathcal{C})$. We call $M_F(\mathcal{C})$, $M_N(\mathcal{C})$ and $M_{CF}(\mathcal{C})$ the friend matroid, neighborhood matroid and close friend matroid induced by covering $\mathcal{C}$, respectively.*

An example is provided to illustrate friend matroid, neighborhood matroid and close friend matroid induced by a covering.

Example 3.4 Let $U = \{1, 2, 3, 4\}$ and covering $\mathcal{C} = \{K_1, K_2, K_3\}$, $K_1 = \{1, 3\}$, $K_2 = \{2, 3, 4\}$, $K_3 = \{1, 4\}$. Then $I(1) = \{1, 3, 4\}$, $I(2) = \{2, 3, 4\}$, $I(3) = I(4) = \{1, 2, 3, 4\}$. $N(1) = \{1\}$, $N(2) = \{2, 3, 4\}$, $N(3) = \{3\}$, $N(4) = \{4\}$. $CF(1) = \{1, 3, 4\}$, $CF(2) = \{2, 3, 4\}$, $CF(3) = CF(4) = \{1, 2, 3, 4\}$. Hence $\mathcal{I}_F(\mathcal{C}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}\}$, $\mathcal{I}_N(\mathcal{C}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, \{1,2,4\}, \{1,3,4\}, \{1,2,3,4\}\}$ and $\mathcal{I}_{CF}(\mathcal{C}) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{1,4\}, \{2,3\}, \{2,4\}, \{1,2,3\}, \{1,2,4\}\}$. Therefore, $M_F(\mathcal{C})$, $M_N(\mathcal{C})$ and $M_{CF}(\mathcal{C})$ are the friend matroid, neighborhood matroid and close friend matroid introduced by covering $\mathcal{C}$, respectively.

In the above, the three types of matroids are established from the viewpoint of rough sets. In the following, we will formulate some characteristics of the three types of matroids.

Proposition 3.5 *Let $\mathcal{C}$ be a covering of U and $M_F(\mathcal{C})$ be the friend matroid induced by covering $\mathcal{C}$. Then $\mathbf{D}(M_F(\mathcal{C})) = \{X \subseteq U : \exists x, y \in X, x \neq y \wedge I_{\mathcal{C}}(x) = I_{\mathcal{C}}(y)\}$.*

Proof According to Definition 2.16, we need only to prove $\mathbf{D}(M_F(\mathcal{C})) = \mathrm{Opp}(\mathcal{I}_F(\mathcal{C}))$. On the one hand, for all $X \in \mathbf{D}(M_F(\mathcal{C}))$, there exist $x, y \in X$ and $x \neq y$ such that $I_{\mathcal{C}}(x) = I_{\mathcal{C}}(y)$. Hence $X \notin \{X \subseteq U : \forall x, y \in X, x \neq y \rightarrow I_{\mathcal{C}}(x) \neq I_{\mathcal{C}}(y)\}$, i.e., $X \in \mathrm{Opp}(\mathcal{I}_F(\mathcal{C}))$, which implies $\mathbf{D}(M_F(\mathcal{C})) \subseteq \mathrm{Opp}(\mathcal{I}_F(\mathcal{C}))$. On the other hand, for all $X \in \mathrm{Opp}(\mathcal{I}(\mathcal{C})))$, i.e., $X \notin \mathcal{I}(\mathcal{C})) = \{X \subseteq U : \forall x, y \in X, x \neq y \rightarrow I_{\mathcal{C}}(x) \neq I_{\mathcal{C}}(y)\}$. Then there exist $x, y \in X$ and $x \neq y$ such that $I_{\mathcal{C}}(x) = I_{\mathcal{C}}(y)\}$, which implies $X \in \mathbf{D}(M_F(\mathcal{C}))$. Hence $\mathrm{Opp}(\mathcal{I}_F(\mathcal{C})) \subseteq \mathbf{D}(M_F(\mathcal{C}))$. To sum up, $\mathbf{D}(M_F(\mathcal{C})) = \mathrm{Opp}(\mathcal{I}_F(\mathcal{C}))$. □

A circuit of a matroid is a minimal dependent set, in the following proposition, we will the characteristic of the circuits of the friend matroid.

Proposition 3.6 *Let $\mathcal{C}$ be a covering of U and $M_F(\mathcal{C})$ be the friend matroid induced by covering $\mathcal{C}$. Then $\mathbf{C}(M_F(\mathcal{C})) = \{\{x,y\} : x,y \in U \wedge x \neq y \wedge I_{\mathcal{C}}(x) = I_{\mathcal{C}}(y)\}$.*

Proof According to Definition 2.17 and Proposition 3.2, we get that $\mathbf{C}(M(\mathcal{C})) = \mathrm{Min}(\mathbf{D}(M_F(\mathcal{C}))$ and $\mathbf{D}(M_F(\mathcal{C})) = \{X \subseteq U : \exists x,y \in X, x \neq y \wedge I_{\mathcal{C}}(x) = I_{\mathcal{C}}(y)\}$. Then it is obvious that $\mathbf{C}(M_F(\mathcal{C})) = \{\{x,y\} : x,y \in U \wedge x \neq y \wedge I_{\mathcal{C}}(x) = I_{\mathcal{C}}(y)\}\}$. □

In all the characteristics of a matroid introduced in this paper, the rank function of a matroid is the one and only one numeric characteristic. In the following proposition, we will investigate the rank function of friend matroid.

Proposition 3.7 *Let $\mathcal{C}$ be a covering of U and $M_F(\mathcal{C})$ be the friend matroid induced by covering $\mathcal{C}$. Then for all $X \subseteq U$, $r_{M_F(\mathcal{C})}(X) = |\{I_{\mathcal{C}}(x) : x \in X\}|$.*

Proof According to Definition 3.1, we have for all $I \in \mathcal{I}(\mathcal{C}))$, $|I| = |I(x) : x \in I|$. According to Definition 2.15, we can obtain $r_{M_F(\mathcal{C})}(X) = |\{I_{\mathcal{C}}(x) : x \in X\}|$. □

The closure of a subset is a set of all elements depending on the subset in matroids. In other words, the closure of a subset is all those elements when added to the subset, the rank is the same. We will formulate the closure operator of friend matroid in the following proposition.

Proposition 3.8 *Let $\mathcal{C}$ be a covering of U and $M_F(\mathcal{C})$ be the friend matroid induced by covering $\mathcal{C}$. Then for all $X \subseteq U$, $cl_{M_F(\mathcal{C})}(X) = \{e \in U : \exists x \in X \text{ s.t. } I_{\mathcal{C}}(x) = I_{\mathcal{C}}(e)\}$.*

Proof We only need to prove that

$$\{e \in U : \exists x \in X \text{ s.t. } I(x) = I(e)\} = \{e \in U : r_{M_F(\mathcal{C})} = r_{M_F(\mathcal{C})}(X \cup \{e\})\}.$$

In fact, for all $e \in \{e \in U : \exists x \in X \text{ s.t. } I(x) = I(e)\} \Leftrightarrow$ there exists a circuit $C = \{x,e\} \in \mathbf{C}(M_F(\mathcal{C}))$ such that

$$e \in C \Leftrightarrow r_{M_F(\mathcal{C})}(X) = r_{M_F(\mathcal{C})}(X \cup \{e\})\} \Leftrightarrow e \in \{e \in U : r_{M_F(\mathcal{C})}(X) = r_{M_F(\mathcal{C})}(X \cup \{e\})\}. \ \square$$

Because the three types of matroidal structure induced by a covering are similar, we can use the same method to study neighborhood matroid and close friend matroid introduced by a covering. And we obtain the following results.

Proposition 3.9 *Let $\mathcal{C}$ be a covering of U and $M_N(\mathcal{C})$ be the neighborhood matroid induced by covering $\mathcal{C}$. Then*

(1) For all $X \subseteq U$, X is a dependent set of $M_N(\mathcal{C})$ if and only if $\forall x,y \in X, N_{\mathcal{C}}(x) = N_{\mathcal{C}}(y)$.

(2) For all $X \subseteq U$, X is a circuit of $M_N(\mathcal{C})$ if and only if $|X| = 2$ and $\forall x,y \in X, N_{\mathcal{C}}(x) = N_{\mathcal{C}}(y)$.

(3) For all $X \subseteq U$, $r_{M_N(\mathcal{C})}(X) = |\{N_{\mathcal{C}}(x) : x \in X\}|$.

(4) For all $X \subseteq U$, $cl_{M_N(\mathcal{C})}(X) = \{e \in U : \exists x \in X \text{ s.t. } N_{\mathcal{C}}(x) = N_{\mathcal{C}}(e)\}$.

Proposition 3.10 *Let $\mathcal{C}$ be a covering of U and $M_{CF}(\mathcal{C})$ be the close friend matroid induced by covering $\mathcal{C}$. Then*

(1) For all $X \subseteq U$, X is a dependent set of $M_{CF}(\mathcal{C})$ if and only if $\forall x, y \in X, CF_{\mathcal{C}}(x) = CF_{\mathcal{C}}(y)$.

(2) For all $X \subseteq U$, X is a circuit of $M_{CF}(\mathcal{C})$ if and only if $|X| = 2$ and $\forall x, y \in X, CF_{\mathcal{C}}(x) = CF_{\mathcal{C}}(y)$.

(3) For all $X \subseteq U$, $r_{M_{CF}(\mathcal{C})}(X) = |\{CF_{\mathcal{C}}(x) : x \in X\}|$.

(4) For all $X \subseteq U$, $cl_{M_{CF}(\mathcal{C})}(X) = \{e \in U : \exists x \in X \ s.t. \ CF_{\mathcal{C}}(x) = CF_{\mathcal{C}}(e)\}$.

The proofs of Propositions 3.9 and 3.10 are similar to those of Propositions 3.5–3.8. Hence we omit the proofs of them.

4. Relationship among these matroids induced by six types of covering-based upper approximation operators

In [26], we present the necessary and sufficient conditions for five types of covering-based upper approximation operators to be closure operators of matroids. In fact, the condition for the fifth and sixth types of covering-based upper approximation operators to be closure operators of matroids are the same. According to the closure axiom of matroids and the above necessary and sufficient conditions, these six types of covering-based upper approximation operators can induce a matroid, respectively. In this section, we discuss the relationships among these six types of matroids induced by six types of covering-based rough sets through comparing their independent sets.

In the following, we recall the necessary and sufficient conditions for six types of covering-based upper approximation operators to be closure operators of matroids.

Proposition 4.1 ([26]) *Let $\mathcal{C}$ be a covering of U. Then*

(1) FH is the closure operator of a matroid if and only if reduct($\mathcal{C}$) *forms a partition on U.*

(2) SH is the closure operator of a matroid if and only if $\{I(x) : x \in U\}$ forms a partition on U.

(3) TH is the closure operator of a matroid if and only if $\{$CFriends$(x) : x \in U\}$ *forms a partition on U.*

(4) RH is the closure operator of a matroid if and only if RH and $\mathcal{C}$ satisfy:

(i) For all $K_1, K_2 \in \mathcal{C}$, if $K_1 \neq K_2$ and $K_1 \cap K_2 \neq \emptyset$, then $\forall x \in K_1 \cap K_2, \{x\} \in \mathcal{C}$.

(ii) For all $y \in RH(\{x\})$, if $\{x\} \notin \mathcal{C}$, then $\{y\} \notin \mathcal{C}$.

(5) IH is the closure operator of a matroid if and only if $\{N(x) : x \in U\}$ forms a partition on U.

Remark 4.2 XH is the closure operator of a matroid if and only if $\{N(x) : x \in U\}$ forms a partition on U.

Based on the above results, if FH, SH, TH, RH, IH and XH form the closure operator of a matroid, respectively according to Proposition 2.20, they can determine a matroid, respectively.

Moreover, according to [16], we know the independent sets of the matroids induced by these six types of covering-based upper approximation operators can be represented as follows:

(1) $\mathcal{I}_{FH} = \{I \subseteq E: \ \forall x \in I, x \notin FH(I - \{x\})\}$.

(2) $\mathcal{I}_{SH} = \{I \subseteq E: \ \forall x \in I, x \notin SH(I - \{x\})\}$.

(3) $\mathcal{I}_{TH} = \{I \subseteq E: \ \forall x \in I, x \notin TH(I - \{x\})\}$. $(*)$

(4) $\mathcal{I}_{RH} = \{I \subseteq E: \ \forall x \in I, x \notin RH(I - \{x\})\}$.

(5) $\mathcal{I}_{IH} = \{I \subseteq E: \ \forall x \in I, x \notin IH(I - \{x\})\}$.

(6) $\mathcal{I}_{XH} = \{I \subseteq E: \ \forall x \in I, x \notin XH(I - \{x\})\}$.

Through comparing the above independent sets, we study the relationships among these matroids induced by six types of upper approximation operators in the following propositions.

First, the relationships among $\mathcal{I}_{SH}, \mathcal{I}_{TH}$, $\mathcal{I}_{IH}$, $\mathcal{I}_F$, $\mathcal{I}_N$ and $\mathcal{I}_{CF}$ will be obtained when SH, TH and IH form closure operators. Moreover, we also formulate $\mathcal{I}_{SH}, \mathcal{I}_{TH}$ and $\mathcal{I}_{IH}$.

The condition under which $\mathcal{I}_{SH} = \mathcal{I}_F$ is presented as follows.

Proposition 4.3 *Let $\mathcal{C}$ be a covering of U. If $\{I(x) : x \in U\}$ induced by $\mathcal{C}$ forms a partition on U. Then $\mathcal{I}_{SH} = \mathcal{I}_F$.*

Proof Suppose $\{I(x) : x \in U\}$ induced by $\mathcal{C}$ forms a partition on U, then SH is the closure operator of matroid $M = (U, \mathcal{I}_{SH})$. In fact, on the one hand, if $I \in \mathcal{I}_{SH}$, i.e., $\forall x \in I, x \notin SH(I - \{x\}) = \cup_{y \in (I - \{x\})} I(y)$, then for $\forall x, y \in I, x \neq y$, we have $I(x) \neq I(y)$, i.e., $I \in \mathcal{I}_F$. On the other hand, if $I \in \mathcal{I}_F$, namely $\forall x, y \in I, x \neq y$, $I(x) \neq I(y)$. If $\forall x \in I, x \notin SH(I - \{x\})$, then there exists $y \in I$ such that $I(x) = I(y)$, which is a contradiction. Therefore, $\mathcal{I}_{SH} = \mathcal{I}_F$. $\square$

The condition under which $\mathcal{I}_{TH} = \mathcal{I}_{CF}$ is presented as follows.

Proposition 4.4 *Let $\mathcal{C}$ be a covering of U. If $\{CF(x) : x \in U\}$ induced by $\mathcal{C}$ forms a partition on U. Then $\mathcal{I}_{TH} = \mathcal{I}_{CF}$.*

Proof Suppose $\{CF(x) : x \in U\}$ induced by $\mathcal{C}$ forms a partition on U. Then TH is the closure operator of matroid $M = (U, \mathcal{I}_{TH})$. On the one hand, for all $I \in \mathcal{I}_{TH}$, $\forall x \in I, x \notin TH(I - \{x\}) = \cup_{y \in (I - \{x\})} CF(y)$ which implies that $\forall x, y \in I, x \neq y \to CF(x) \neq CF(y)$. Hence $I \in \mathcal{I}_{CF}$ and $\mathcal{I}_{TH} \subseteq \mathcal{I}_{CF}$. On the other hand, for all $I \in \mathcal{I}_{CF}$, we have $\forall x, y \in X, x \neq y \to CF(x) \neq CF(y)$. Since $\{CF(x) : x \in U\}$ induced by $\mathcal{C}$ forms a partition on U. Hence $\forall x \in I, x \notin TH(I - \{x\}) = \cup_{y \in (I - \{x\})} CF(y)$, i.e., $I \in \mathcal{I}_{TH}$. Therefore, $\mathcal{I}_{CF} \subseteq \mathcal{I}_{TH}$. $\square$

The following proposition presents the condition under which $\mathcal{I}_{IH} = \mathcal{I}_N$.

Proposition 4.5 *Let $\mathcal{C}$ be a covering of U. If $\{N(x) : x \in U\}$ induced by $\mathcal{C}$ forms a partition on U. Then $\mathcal{I}_{IH} = \mathcal{I}_N$.*

Proof Suppose $\{N(x) : x \in U\}$ induced by $\mathcal{C}$ forms a partition on U, then IH is the closure operator of $M = (U, \mathcal{I}_{IH})$. In fact, $I \in \mathcal{I}_{IH} \Longleftrightarrow \forall x \in I, x \notin IH(I - \{x\}) = \cup_{y \in (I - \{x\})} N(y) \Longleftrightarrow \forall x, y \in I, x \neq y \to N(x) \neq N(y) \Longleftrightarrow I \in \mathcal{I}_F$. Therefore, $\mathcal{I}_{IH} = \mathcal{I}_F$. $\square$

Combined with the above results, the relationship among these matroids induced by six types of covering-based upper approximation operators respectively will be shown through comparing their independent sets. First, based on Remark 4.2, we present the relationship between $\mathcal{I}_{IH}$ and $\mathcal{I}_{XH}$.

Proposition 4.6 *Let $\mathcal{C}$ be a covering on U. If $\{N(x) : x \in U\}$ induced by $\mathcal{C}$ forms a partition on U. Then $\mathcal{I}_{IH} = \mathcal{I}_{XH}$.*

Proof According to the Definition of $\mathcal{I}_{IH}$, $\mathcal{I}_{XH}$, IH and XH, it is straightway. □

In order to study the relationships among the first, third and fourth types of covering-based upper approximation operators, we introduce the following four lemmas.

Lemma 4.7 ([26]) *$\mathcal{C}$ is unary and $\{CF(x) : x \in U\}$ forms a partition on U if and only if* reduct($\mathcal{C}$) *forms a partition on U.*

Lemma 4.8 ([25]) *Let $\mathcal{C}$ be a covering of U, and TH and RH be the third and fourth types of upper approximation operators, respectively. If $TH = RH$, then $\mathcal{C}$ is unary.*

Lemma 4.9 ([25]) *Let $\mathcal{C}$ be a covering on U, and TH and RH be the third and fourth types of upper approximation operators, respectively. $TH = RH$ if and only if for all $K_1, K_2 \in \mathcal{C}$, $K_1 \neq K_2$ and $\forall x \in K_1 \cap K_2 \neq \emptyset$, $\{x\} \in \mathcal{C}$.*

Lemma 4.10 ([25]) *$TH = FH$ if and only if $\mathcal{C}$ is a unary covering.*

Based on the above four lemmas, the following proposition points out that $\mathcal{I}_{FH} = \mathcal{I}_{TH}$ when FH and TH form a closure operator of a matroid, respectively.

Proposition 4.11 *Let $\mathcal{C}$ be a covering on U. If FH and TH induced by $\mathcal{C}$ are closure operators of matroids. Then $\mathcal{I}_{FH} = \mathcal{I}_{TH}$.*

Proof If FH is a closure operator of a matroid. Then reduct($\mathcal{C}$) induced by $\mathcal{C}$ forms a partition on U. According to Lemmas 4.7 and 4.10, we have $TH = FH$ and $\mathcal{I}_{FH} = \mathcal{I}_{TH}$. □

In the following result, we will show that $\mathcal{I}_{RH}$ is equal to $\mathcal{I}_{FH}$ when RH forms a closure operator of a matroid. Hence we only discuss the relationships between $\mathcal{I}_{FH}$ and other independent sets.

Proposition 4.12 *Let $\mathcal{C}$ be a covering on U. If RH induced by $\mathcal{C}$ is a closure operator of a matroid, then $\mathcal{I}_{RH} = \mathcal{I}_{FH}$.*

Proof According to Proposition 4.1, if RH induced by $\mathcal{C}$ is a closure operator of a matroid, then for all $K_1, K_2 \in \mathcal{C}$, $K_1 \neq K_2$ and $\forall x \in K_1 \cap K_2 \neq \emptyset$, $\{x\} \in \mathcal{C}$. According to Lemmas 4.9 and 4.10, we have that $RH = FH$, which implies that FH also forms a closure operator of a matroid. Hence $\mathcal{I}_{RH} = \mathcal{I}_{FH}$. □

In order to illustrate, we denote the following remark.

Remark 4.13 Let U be a finite set and $\mathcal{F}_1, \mathcal{F}_2$ be two families of subsets on U. If for $\forall A \in \mathcal{F}_1$, there exist $B \in \mathcal{F}_2$ such $A \subset B$, we say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$, denoted by $\mathcal{F}_2 \preceq \mathcal{F}_1$.

Based on the above results, the relationships among the six types of matroids will be discussed by studying the relationships among their independent sets as follows.

Proposition 4.14 *Let $\mathcal{C}$ be a covering on U. If FH and SH induced by $\mathcal{C}$ are closure operators of matroids, then $\mathcal{I}_{SH} \subseteq \mathcal{I}_{FH}$.*

Proof Suppose FH and SH induced by $\mathcal{C}$ are closure operators of matroids. Then $\{I(x) : x \in U\}$ and reduct($\mathcal{C}$) induced by $\mathcal{C}$ form a partition on U. According to Lemmas 4.7 and 4.10, we have $\mathcal{C}$ is unary, $\{CF(x) : x \in U\}$ forms a partition on U and $\mathcal{I}_{FH} = \mathcal{I}_{TH} = \mathcal{I}_{CF}$. Since $CF(x) = \cup Md(x) \subseteq \cup_{x\in K\in\mathcal{C}} K = I(x)$, $\{CF(x) : x \in U\}$ is finer than $\{I(x) : x \in U\}$, in other words, reduct($\mathcal{C}$) is finer than $\{I(x) : x \in U\}$. Based on the result, we have $\forall I \in \mathcal{I}_{SH}$, then $I \in \mathcal{I}_{FH}$. Hence $\mathcal{I}_{SH} \subseteq \mathcal{I}_{FH}$. □

Example 4.15 Let $U = \{1, 2, 3\}$ and $\mathcal{C} = \{\{1\}, \{2\}, \{1, 2\}, \{3\}\}$. reduct($\mathcal{C}$) $= \{\{1\}, \{2\}, \{3\}\}$, $I(1) = I(2) = \{1, 2\}$ and $I(3) = \{3\}$. Hence $\mathcal{I}_{FH} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\mathcal{I}_{SH} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}$. Therefore, $\mathcal{I}_{SH} \subseteq \mathcal{I}_{FH}$.

Proposition 4.16 *Let $\mathcal{C}$ be a covering on U. If FH and IH induced by $\mathcal{C}$ are closure operators of matroids, then $\mathcal{I}_{FH} = \mathcal{I}_{IH}$.*

Proof Suppose FH and IH induced by $\mathcal{C}$ are closure operators of matroids. Then $\mathcal{C}$ is unary, $\{CF(x) : x \in U\}$ and $\{N(x) : x \in U\}$ form a partition on U. Since $\mathcal{C}$ is unary and $N(x) = \cap Md(x)$, $CF(x) = \cup Md(x) = \cap Md(x) = N(x)$ and $\{CF(x) : x \in U\} = \{N(x) : x \in U\}$. According to the definition of $\mathcal{I}_{FH}$ and $\mathcal{I}_{IH}$, we can obtain $\mathcal{I}_{FH} = \mathcal{I}_{IH}$. □

Example 4.17 (Continued from Example 4.15) We have that reduct($\mathcal{C}$) $= \{\{1\}, \{2\}, \{3\}\}$, $N(1) = \{1\}$, $N(2) = \{2\}$ and $N(3) = \{3\}$. Hence $\mathcal{I}_{FH} = \mathcal{I}_{IH} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$. Therefore, $\mathcal{I}_{FH} = \mathcal{I}_{IH}$.

Proposition 4.18 *Let $\mathcal{C}$ be a covering on U. If SH and TH induced by $\mathcal{C}$ are closure operators of matroids, then $\mathcal{I}_{SH} \subseteq \mathcal{I}_{TH}$.*

Proof If SH and TH induced by $\mathcal{C}$ are closure operators of matroids, then $\{I(x) : x \in U\}$ and $\{CF(x) : x \in U\}$ form a partition on U. Since $CF(x) = \cup Md(x) \subseteq \cup_{x\in K\in\mathcal{C}} K = I(x)$, $\{CF(x) : x \in U\}$ is finer than $\{I(x) : x \in U\}$. Based on this result, we have $\mathcal{I}_{SH} \subseteq \mathcal{I}_{TH}$. □

Example 4.19 Let $U = \{1, 2, 3\}$ and $\mathcal{C} = \{\{1\}, \{2\}, \{1, 2\}, \{3\}\}$. $CF(1) = \{1\}$, $CF(2) = \{2\}$, $CF(3) = \{3\}$, $I(1) = I(2) = \{1, 2\}$ and $I(3) = \{3\}$. Hence $\mathcal{I}_{TH} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ and $\mathcal{I}_{SH} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 3\}, \{2, 3\}\}$. Therefore, $\mathcal{I}_{SH} \subseteq \mathcal{I}_{TH}$.

Example 4.20 Let $U = \{1, 2, 3, 4\}$ and $\mathcal{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\}\}$. $I(1) = I(2) = I(3) = \{1, 2, 3\}$, $I(4) = \{4\}$, $CF(1) = CF(2) = CF(3) = \{1, 2, 3\}$ and $CF(4) = \{4\}$. Then $\mathcal{I}_{SH} = \mathcal{I}_{TH}$.

Proposition 4.21 *Let $\mathcal{C}$ be a covering on U. If SH and IH induced by $\mathcal{C}$ are closure operators of matroids, then $\mathcal{I}_{SH} \subseteq \mathcal{I}_{IH}$.*

Proof If SH and TH induced by $\mathcal{C}$ are closure operators of matroids, then $\{I(x) : x \in U\}$ and $\{N(x) : x \in U\}$ form a partition on U. Since $N(x) = \cap Md(x) \subseteq \cup_{x \in K \in \mathcal{C}} K = I(x)$, $\{N(x) : x \in U\}$ is finer than $\{I(x) : x \in U\}$. Based on the fact, we have that $\mathcal{I}_{SH} \subseteq \mathcal{I}_{IH}$. □

Example 4.22 Let $U = \{1, 2, 3, 4\}$ and $\mathcal{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\}\}$. We have that $N(1) = \{1\}, N(2) = \{2\}, N(3) = \{3\}$, $N(4) = \{4\}$, $I(1) = I(2) = I(3) = \{1, 2, 3\}$ and $I(4) = \{4\}$. Hence $\mathcal{I}_{SH} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$ and $\mathcal{I}_{IH} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$. Therefore, $\mathcal{I}_{SH} \subseteq \mathcal{I}_{IH}$.

Proposition 4.23 *Let $\mathcal{C}$ be a covering on U. If TH and IH induced by $\mathcal{C}$ are closure operators of matroids, then $\mathcal{I}_{TH} \subseteq \mathcal{I}_{IH}$.*

Proof If TH and IH induced by $\mathcal{C}$ are closure operators of matroids, then $\{\text{CFriends}(x) : x \in U\}$ and $\{N(x) : x \in U\}$ form a partition on U. Since for all $x \in U$, $N(x) = \cap Md(x) \subseteq \cup Md(x) = CF(x)$, $\{N(x) : x \in U\}$ is finer than $\{CF(x) : x \in U\}$. Based on this fact, we have that $\mathcal{I}_{TH} \subseteq \mathcal{I}_{IH}$. □

Example 4.24 (Continued from Example 4.20) Let $U = \{1, 2, 3, 4\}$ and $\mathcal{C} = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{4\}\}$. We have that $N(1) = \{1\}$, $N(2) = \{2\}$, $N(3) = \{3\}$, $N(4) = \{4\}$, $I(1) = I(2) = I(3) = \{1, 2, 3\}$ and $I(4) = \{4\}$. Hence $\mathcal{I}_{SH} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 4\}, \{2, 4\}, \{3, 4\}\}$ and $\mathcal{I}_{IH} = \{\emptyset, \{1\}, \{2\}, \{3\}, \{4\}, \{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}, \{1, 2, 3, 4\}$. Therefore, $\mathcal{I}_{SH} \subseteq \mathcal{I}_{IH}$.

Based on the above results, we have the following two remarks.

Remark 4.25 If FH ($FH \neq TH$), SH and IH are closure operators of matroids, then $\mathcal{I}_{SH} \subseteq \mathcal{I}_{FH} = \mathcal{I}_{IH}$.

Remark 4.26 If SH, TH ($TH \neq FH$) and IH are closure operators of matroids, then $\mathcal{I}_{SH} = \mathcal{I}_{TH} \subseteq \mathcal{I}_{IH}$.

5. Conclusions

In this paper, we constructed three types of matroidal structures of covering-based rough sets by indiscernible neighborhoods, neighborhoods and close friends, respectively and studied some characteristics of these three types of matroid. We also studied the relationships between these three types of matroid and these matroids induced by the six types of covering-based upper approximation operators. Moreover, through our study, the relationships among these matroids induced by the six types of covering-based upper approximation operators are attribute to the relationships among three types of matroidal structures of covering-based rough sets. The relationships among them are presented as follows

When the following covering-based upper approximation operators are closure operators	The relationship among their independence
FH and SH	$\mathcal{I}_{SH} \subseteq \mathcal{I}_{FH}$
FH and TH	$\mathcal{I}_{FH} = \mathcal{I}_{TH}$
FH and RH	$\mathcal{I}_{RH} = \mathcal{I}_{FH}$
SH and TH	$\mathcal{I}_{SH} \subseteq \mathcal{I}_{TH}$
SH and IH	$\mathcal{I}_{SH} \subseteq \mathcal{I}_{IH}$
TH and IH	$\mathcal{I}_{TH} \subseteq \mathcal{I}_{IH}$
IH and XH	$\mathcal{I}_{IH} = \mathcal{I}_{XH}$
$FH(FH \neq TH)$, SH and IH	$\mathcal{I}_{SH} \subseteq \mathcal{I}_{FH} = \mathcal{I}_{IH}$
SH,TH $(TH \neq FH)$ and IH	$\mathcal{I}_{SH} = \mathcal{I}_{TH} \subseteq \mathcal{I}_{IH}$

Table 1 The relationships among the six types of covering-based upper approximation operators

References

[1] Z. PAWLAK. *Rough sets*. International Journal of Computer and Information Sciences, 1982, **11**: 341–356.

[2] R. SLOWINSKI, D. VANDERPOOTEN. *A generalized definition of rough approximations based on similarity*. IEEE Transactions on Knowledge and Data Engineering, 2000, **12**: 331–336.

[3] G. CATTANEO. *Abstract Approximation Spaces for Rough Theories*. Physica, Heidelberg, 1998.

[4] A. SKOWRON, J. STEPANIUK. *Tolerance approximation spaces*. Fund. Inform., 1996, **27**(2-3): 245–253.

[5] W. ZAKOWSKI. *Approximations in the space* (u, π). Demonstratio Math., 1983, **16**(3): 761–769.

[6] Feng ZHU, Feiyue WANG. *A new type of covering rough sets*. In: IEEE International Conference on Intelligent Systems, 2007, **12**: 444–449.

[7] Keyue QIN, Yan GAO, Zheng PEI. *On covering rough sets*. International Conference on Rough Sets and Knowledge Technology, 2007, **4481**: 34–41.

[8] Shiping WANG, Feng ZHU. *Structure of covering-based rough sets*. Journal of Mathematical and Computer Sciences, 2011, **49**: 509–512.

[9] Shiping WANG, Qingxin ZHU, Feng ZHU, et al. *Matroidal structure of rough sets and its characterization to attribute reduction*. Knowledge-Based Systems, 2012, **36**: 155–161.

[10] Feng ZHU, Shiping WANG. *Matroidal approaches to generalized rough sets based on relations*. International Journal of Machine Learning and Cybernetics, 2011, **2**(4): 273–279.

[11] Lirun SU, Feng ZHU. *Dependence space of topology and its application to attribute reduction*. International Journal of Machine Learning and Cybernetics, 2016, **1**: 1–8.

[12] B. DAVVAZ. *Roughness based on fuzzy ideals*. Inform. Sci., 2006, **176**(16): 2417–2437.

[13] Guilong LIU, Ying SAI. *Invertible approximation operators of generalized rough sets and fuzzy rough sets*. Inform. Sci., 2010, **180**(11): 2221–2229.

[14] Fan MIN, Feng ZHU. *Attribute reduction with test cost constraint*. Journal of Electronic Science and Technology of China, 2011, **9**:97–102.

[15] Yiyu YAO. *Constructive and algebraic methods of theory of rough sets*. Inform. Sci., 1998, **109**: 21–47.

[16] J. BONIN, J. OXLEY. *Matroid Theory*. Graduate Texts in Mathematics, 1996, **197**: 234–260.

[17] Xiaonan LI, Sanyang LIU. *Matroidal approaches to rough set theory via closure operators*. International Journal of Approximate Reasoning, 2012, **53**: 513–527.

[18] Yanfang LIU, Feng ZHU. *Characteristic of partition-circuit matroid through approximation number*. Eprint Arxiv., 2012, **61**(1): 314–319.

[19] Degang CHEN, Wenxiu ZHANG, Daniel YEUNG, et al. *Rough approximations on a complete completely distributive lattice with applications to generalized rough sets*. Inform. Sci., 2006, **176**(13): 1829–1848.

[20] Hongru LI, Wenxiu ZHANG, Hong WANG. *Classification and reduction of attributes in concept lattices*. Granular Computing, 2006, **20**: 142–147.

[21] J. EDMONDS. *Matroids and the greedy algorithm*. Math. Programming, 1971, **1**(1): 127–136.

[22] E. LAWLER. *Combinatorial Optimization: Networks and Matroids*. Dover Publications, 2011, **84(3)**: 461–463.

[23] Feng ZHU, Feiyue WANG. *Reduction and axiomization of covering generalized rough sets*. Inform. Sci., 2003, **152**: 217–230.

[24] Feng ZHU, Feiyue WANG. *Relationships among three types of covering rough sets*. Granular Computing, 2006, **13**: 43–48.

[25] Feng ZHU, Feiyue WANG. *The fourth type of covering-based rough sets*. Inform. Sci., 2012, **201**(19): 80–92.

[26] Lirun SU, Feng ZHU. *Condition for covering-based upper approximation operators to be closure operators of matroids*. Journal of Granular Computing, Rough Sets and Intelligent Systems, 2013, **3**: 144–159.

[27] Lirun SU, Feng ZHU. *Closed-set lattice and modular matroid induced by covering-based rough sets*. International Journal of Machine Learning and Cybernetics, 2014, **7**: 1–11.