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Semicommutativity of Amalgamated Rings

Handan KOSE[1], **Yousum KURTULMAZ**[2], **Burcu UNGOR**[3], **Abdullah HARMANCI**[4,*]

1. *Department of Mathematics, Ahi Evran University, Kirsehir, Turkey;*
2. *Department of Mathematics, Bilkent University, Ankara, Turkey;*
3. *Department of Mathematics, Ankara University, Ankara, Turkey;*
4. *Department of Mathematics, Hacettepe University, Ankara, Turkey*

Abstract In this paper, we study some cases when an amalgamated construction $A \bowtie^f I$ of a ring A along an ideal I of a ring B with respect to a ring homomorphism f from A to B, is prime, semiprime, semicommutative, nil-semicommutative and weakly semicommutative.

Keywords semicommutative ring; nil-semicommutative ring; weakly semicommutative ring; amalgamated construction

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1. Introduction

Throughout this paper all rings are associative with identity unless otherwise stated. Let A and B be commutative rings with a ring homomorphism $f : A \to B$ and I be an ideal of B. The amalgamation of A with B along an ideal I of B with respect to f (denoted by $A \bowtie^f I$) was introduced and studied in [1–3]. In [4], clean properties of amalgamated rings in commutative case were studied. Also in [5], for a ring R and an ideal I, a case was studied when an amalgamated duplication $R \bowtie I$ of R along an ideal I is quasi-Frobenius. Some homological properties of amalgamated duplication of a ring along an ideal were investigated in [6]. Bezout properties of amalgamated rings were studied in [7]. In the commutative case of rings, most of properties of amalgamated duplications are investigated. Namely, Gorenstein global dimension of an amalgamated duplication of a coherent ring along a regular principal ideal was observed in [8] and it is proved that for a coherent ring R which contains a nonunit regular element x, $\mathrm{wGgldim}(R \bowtie^f xR) = \mathrm{wGgldim}(R)$, and $\mathrm{Ggldim}(R \bowtie^f xR) = \mathrm{Ggldim}(R)$, and in [9], among others, it was shown that for a CM local ring R, $R \bowtie I$ is Gorenstein if and only if I is a canonical ideal of R.

Let A and B be two rings (not necessarily commutative) with identity, I an ideal of B and $f : A \to B$ a ring homomorphism. In this setting, we consider the following subring of $A \times B$

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* Corresponding author
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E-mail address: handan.kose@ahievran.edu.tr (Handan KOSE); kurtulmazy@gmail.com (Yosum KURTULMAZ); bungor@science.ankara.edu.tr (Burcu UNGOR); harmanci@hacettepe.edu.tr (Abdullah HARMANCI)

(endowed with the usual componentwise operations):

$$A \bowtie^f I := \{(a, f(a) + i) \mid a \in A, i \in I\}$$

which is called amalgamated construction of A with B along I with respect to f. This construction is a generalization of the amalgamated duplication of a ring along an ideal introduced and studied in [1–3, 9]. Also in [10], the ideal extensions were defined and investigated for noncommutative rings. In case $A = B$, I is an ideal and f is the identity homomorphism of A, then amalgamated construction $A \bowtie^f I$ of A along I with respect to f is isomorphic to the ideal extension $\mathbb{E}(A; I)$ of A by I in [10]. Motivated by these works, in this note we study primeness, semiprimeness, semicommutativity, nil-semicommutativity and weakly semicommutativity of amalgamated construction $A \bowtie^f I$ of a ring A with a ring B along an ideal I of B with respect to a ring homomorphism f from A to B.

A ring R is called semicommutative if for any a, $b \in R$, $ab = 0$ implies $aRb = 0$ (this ring is also called a zero insertion(ZI) ring in [11–13]). The ring R is semicommutative if and only if any right (left) annihilator over R is an ideal of R by [14, Lemma 1] or [15, Lemma 1.2]. Every commutative ring is semicommutative. Therefore, if A and B are commutative, then the ring $A \times B$ is commutative, and so is $A \bowtie^f I$ as a subring of $A \times B$. A ring R is called nil-semicommutative [16] if $ab = 0$ implies $aRb = 0$ for every nilpotent elements a, $b \in R$. Every semicommutative ring is nil-semicommutative. Another version of semicommutativity is weakly semicommutativity. In [13] and [17], weakly semicommutative rings were investigated. The ring R is called weakly semicommutative if for any a, $b \in R$, $ab = 0$ implies arb is nilpotent for any $r \in R$. Clearly, semicommutative rings are weakly semicommutative. There is no implication between nil-semicommutative rings and weakly semicommutative rings.

In what follows, by $\mathbb{Z}$ and $\mathbb{Z}_n$ we denote, respectively, the ring of integers and the ring of integers modulo n for a positive integer n and $\mathrm{nil}(R)$ will stand for the set of all nilpotent elements of a ring R.

2. Reduced, prime and semiprime properties of amalgamated rings

We start this section by the following proposition which characterizes when the amalgamated construction $A \bowtie^f I$ is a reduced ring. This proposition also generalizes [1, Proposition 5.4] which is proved for commutative rings. Recall that a ring R is called reduced if it has no nonzero nilpotent elements.

Proposition 2.1 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. Then the following conditions are equivalent:*

(1) $A \bowtie^f I$ is a reduced ring.

(2) A is a reduced ring and $\mathrm{nil}(B) \cap I = (0)$.

In particular, if A and B are reduced, then $A \bowtie^f I$ is reduced.

Proof (1) $\Rightarrow$ (2). Let $a \in A$ with $a^n = 0$ for some positive integer n. Then $(a, f(a))^n = 0$ in

$A \bowtie^f I$. By (1), $(a, f(a)) = 0$ or $a = 0$. Let $b \in \mathrm{nil}(B) \cap I$. There exists a positive integer t such that $b^t = 0$. So $(0, f(0) + b)^t = 0$. Hence $(0, b) = 0$ or $b = 0$.

(2) $\Rightarrow$ (1). Let $(a, f(a) + x) \in A \bowtie^f I$ with $(a, f(a) + x)^s = 0$ for some positive integer s. Then $0 = (a^s, (f(a) + x)^s)$. Hence $a^s = 0$ and $(f(a) + x)^s = 0$. By (2), $a = 0$. Since $x \in \mathrm{nil}(B) \cap I$, $x = 0$. So $(a, f(a) + x) = 0$. Thus $A \bowtie^f I$ is reduced. The rest is clear. □

The proof of the following lemma is obvious. We record it for an easy reference.

Lemma 2.2 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. For any $x \in I$, $\{(0, f(0) + xy) \mid y \in B\}$ is a right ideal of $A \bowtie^f I$ and for any $b \in B$, $\{(0, f(0) + by) \mid y \in I\}$ is a right ideal of $A \bowtie^f I$.*

Recall that a ring R is called prime if for any ideals (right or left) I and J of R, $IJ = 0$ implies $I = 0$ or $J = 0$, equivalently, for any r, $s \in R$, $rRs = 0$ implies $r = 0$ or $s = 0$, and R is called semiprime if it has no nonzero nilpotent ideals, equivalently, $aRa = 0$ implies $a = 0$ for any $a \in R$. Obviously, every prime ring is semiprime. A proper ideal I of a ring R is called semiprime if R/I is a semiprime ring.

Theorem 2.3 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. Assume that B is a semicommutative ring and f is a monomorphism and $\mathrm{Im}(f) \cap I = 0$. Then $A \bowtie^f I$ is a prime ring if and only if A and $f(A) + I$ are prime rings.*

Proof Necessity. Assume that $A \bowtie^f I$ is a prime ring. Let a, $b \in A$ with $aAb = 0$. Then $f(a)f(b) = 0$. Semicommutativity of B implies $f(a)Bf(b) = 0$. We use this fact to have $(a, f(a) + 0)(c, f(c) + y)(b, f(b) + 0) = 0$ in $A \bowtie^f I$ for all $c \in A$ and $y \in I$. By assumption, $(a, f(a)) = 0$ or $(b, f(b)) = 0$. So $a = 0$ or $b = 0$. Hence A is prime. To prove $f(A) + I$ is prime, let $f(a) + x$, $f(b) + y \in f(A) + I$. Assume that $(f(a) + x)(f(A) + I)(f(b) + y) = 0$. Then for all $c \in A$ and $z \in I$,

$$(f(a) + x)(f(c) + z)(f(b) + y) = 0. \qquad (*)$$

By $(*)$, we have $f(acb) \in I$ and by hypothesis, $acb = 0$. Thus $(a, f(a)+x)(c, f(c)+z)(b, f(b)+y) = 0$ for all $c \in A$ and $z \in I$. Primeness of $A \bowtie^f I$ implies $(a, f(a) + x) = 0$ or $(b, f(b) + y) = 0$. Hence $f(a) + x = 0$ or $f(b) + y = 0$. It follows that $f(A) + I$ is prime.

Sufficiency. Suppose that A and $f(A) + I$ are prime rings. To prove $A \bowtie^f I$ is prime, let $(a, f(a) + x)$, $(b, f(b) + y) \in A \bowtie^f I$. Assume that $(a, f(a) + x)(A \bowtie^f I)(b, f(b) + y) = 0$. Then $aAb = 0$ and $(f(a) + x)(f(A) + I)(f(b) + y) = 0$. By supposition, $a = 0$ or $b = 0$ and $f(a) + x = 0$ or $f(b) + y = 0$. We consider some cases. In the cases ($a = 0$ and $f(a) + x = 0$) or ($b = 0$ and $f(b) + y = 0$), the proof is clear. Assume that $a = 0$ and $f(b) + y = 0$. Then $f(b) = -y \in \mathrm{Im}(f) \cap I$ implies $f(b) = 0$ and $y = 0$. By hypothesis, $b = 0$. So $(b, f(b) + y) = 0$. Similarly, the case $b = 0$ and $f(a) + x = 0$ implies that $(a, f(a) + x) = 0$. So $A \bowtie^f I$ is prime. □

Theorem 2.4 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. Assume that B is a semicommutative ring. If $A \bowtie^f I$ is a prime ring, then A is a prime ring and nil$(B) \cap I = (0)$.*

Proof To prove A is prime, let a, $b \in A$ with $aAb = 0$. Then $f(a)f(b) = 0$. By semicommutativity of B, we have $f(a)Bf(b) = 0$. Hence $(a, f(a) + 0)(A \bowtie^f I)(b, f(b) + 0) = 0$. It implies that $a = 0$ or $b = 0$. Hence A is prime. Let $x \in \mathrm{nil}(B) \cap I$ with $x^n = 0$ for some positive integer n. By semicommutativity of B, we have $x^{n-1}Bx^{n-1} = 0$. Then $(0, f(0) + x^{n-1})(A \bowtie^f I)(0, f(0) + x^{n-1}) = \{(0, f(0) + x^{n-1}(f(a) + y)x^{n-1}) \mid a \in A, y \in I\} = 0$. Hence $x^{n-1} = 0$. By continuing in this way, we have $x = 0$. □

There are rings not satisfying the converse statement in Theorem 2.4 as the following example shows.

Example 2.5 Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & 0 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $I = \begin{bmatrix} 0 & 0 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ be the ideal of B and $f : A \to B$ be a ring homomorphism defined by $f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$. Then A is a prime ring, B is semicommutative and $\mathrm{nil}(B) \cap I = (0)$. And

$$A \bowtie^f I = \left\{ (0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}) \right\}$$

is not prime.

Theorem 2.6 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. Then the following hold.*

(1) If B is a semicommutative ring, f is a monomorphism, $\mathrm{Im}(f) \cap I = (0)$ and $A \bowtie^f I$ is a semiprime ring, then A and $f(A) + I$ are semiprime rings.

(2) If A and $f(A) + I$ are semiprime rings, then $A \bowtie^f I$ is semiprime.

Proof (1) Assume that $A \bowtie^f I$ is a semiprime ring. Let $a \in A$ with $aAa = 0$. Semicommutativity of B implies $f(a)Bf(a) = 0$. We use this fact to have $(a, f(a)+0)(c, f(c)+y)(a, f(a)+0) = 0$ in $A \bowtie^f I$ for all $c \in A$ and $y \in I$. By assumption, $(a, f(a)) = 0$. So $a = 0$. Hence A is semiprime. To prove $f(A) + I$ is semiprime, let $f(a) + x \in f(A) + I$. Assume that for any $c \in A$ and $z \in I$,

$$(f(a) + x)(f(c) + z)(f(a) + x) = 0. \tag{$**$}$$

Then $(**)$ implies $f(a)f(a) = -(xf(a) + f(a)x + x^2) \in \mathrm{Im}(f) \cap I$. Hence $f(a)f(a) = 0$. By semicommutativity of B and being f monomorphism, we have $aAa = 0$. Since A is semiprime, we have $a = 0$. From $(**)$, we have $x^4 = 0$. Hence $x^2Bx^2B = 0$. Let $X = \{(0, f(0) + x^2b) \mid b \in B\}$. Then X is a right ideal of $A \bowtie^f I$ with $X^2 = 0$. By hypothesis, $X = 0$. So $x^2 = 0$. Again by semicommutativity of B, we have $xBx = 0$. We define $Y = \{(0, f(0) + xb) \mid b \in B\}$. Then Y is a right ideal of $A \bowtie^f I$ with $Y^2 = 0$. By hypothesis $Y = 0$. So $x = 0$. Hence $f(a) + x = 0$ and so $f(A) + I$ is semiprime.

(2) Suppose that A and $f(A) + I$ are semiprime. To prove $A \bowtie^f I$ is semiprime, let $(a, f(a) + x) \in A \bowtie^f I$. Assume that $(a, f(a) + x)(A \bowtie^f I)(a, f(a) + x) = 0$. Then $aAa = 0$ and $(f(a) + x)(f(A) + I)(f(a) + x) = 0$. By supposition, $a = 0$ and $f(a) + x = 0$. This completes the proof. □

Proposition 2.7 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. Then the following hold.*

(1) If A and B are semiprime rings, B is semicommutative, then $A \bowtie^f I$ is semiprime.

(2) If I is a semiprime ideal of B, B is semicommutative and $A \bowtie^f I$ is semiprime, then A and B are semiprime.

Proof (1) Assume that A and B are semiprime rings. To prove $A \bowtie^f I$ is semiprime, let $(a, f(a)+x) \in A \bowtie^f I$ with $(a, f(a)+x)(A \bowtie^f I)(a, f(a)+x) = 0$ in $A \bowtie^f I$. Then $aAa = 0$ and $(f(a)+x)(f(A)+I)(f(a)+x) = 0$. By assumption, $a = 0$ and $f(a) = 0$. Hence $x(f(A)+I)x = 0$ and so $x^2 = 0$. By semicommutativity and the semiprimeness of B, we have $x = 0$. Thus $f(a) + x = 0$.

(2) Suppose that I is a semiprime ideal of B, B is semicommutative and $A \bowtie^f I$ is semiprime. Let $a \in A$ with $aAa = 0$. Then $(a, f(a) + 0)(A \bowtie^f I)(a, f(a) + 0) = 0$. Hence $(a, f(a) + 0) = 0$. Thus $a = 0$. To prove B is semiprime, let $x \in B$ with $xBx = 0$. By assumption, $x \in I$. From $xBx = 0$, we have $(0, f(0)+x)(A \bowtie^f I)(0, f(0)+x) = 0$. Again by assumption, $(0, f(0)+x) = 0$ and so $x = 0$. □

Theorem 2.8 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. Assume that B is a semicommutative ring. Then the following conditions are equivalent:*

(1) $A \bowtie^f I$ is a semiprime ring.

(2) A is a semiprime ring and $\mathrm{nil}(B) \cap I = (0)$.

Proof (1) $\Rightarrow$ (2). Let $a \in A$ with $aAa = 0$. Then $f(a)f(a) = 0$. By semicommutativity of B, we have $f(a)Bf(a) = 0$. So $(a, f(a) + 0)(A \bowtie^f I)(a, f(a) + 0) = (aAa, f(a)(f(A) + I)f(a)) = 0$ in $A \bowtie^f I$. By (1), $(a, f(a)+0) = 0$. Hence $a = 0$ and so A is semiprime. To prove $\mathrm{nil}(B)\cap I = (0)$, let $b \in \mathrm{nil}(B) \cap I$. As in the proof of Theorem 2.4, it can be shown that $b = 0$.

(2) $\Rightarrow$ (1). To prove $A \bowtie^f I$ is semiprime, let $(a, f(a) + x) \in A \bowtie^f I$. Assume that $(a, f(a) + x)(A \bowtie^f I)(a, f(a) + x) = 0$. Then $aAa = 0$ and $(f(a) + x)(f(A) + I)(f(a) + x) = 0$. Hence $a = 0$ and $xIx = 0$, in particular $x^3 = 0$. So $x \in \mathrm{nil}(B)\cap I$ or $x = 0$. Hence $(a, f(a)+x) = 0$. Thus $A \bowtie^f I$ is semiprime. □

Proposition 2.9 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. Assume that $f^{-1}(I) \cap \mathrm{nil}(A) = (0)$ and $f(A) + I$ is a semiprime ring. Then $A \bowtie^f I$ is a semiprime ring.*

Proof To prove $A \bowtie^f I$ is semiprime, let $(a, f(a)+x) \in A \bowtie^f I$. Assume that $(a, f(a)+x)(A \bowtie^f I)(a, f(a) + x) = 0$. Then $aAa = 0$ and $(f(a) + x)(f(A) + I)(f(a) + x) = 0$. From semiprimeness of $f(A) + I$, we have $f(a) + x = 0$. The equation $aAa = 0$ gives rise a to be nilpotent in A from which we have $a \in f^{-1}(I) \cap \mathrm{nil}(A)$. It follows that $a = 0$, so $f(a) = 0$ and $x = 0$. Therefore $(a, f(a) + x) = 0$ in $A \bowtie^f I$. Hence $A \bowtie^f I$ is a semiprime ring. □

3. Semicommutativity of amalgamated rings

Our next theorem states necessary and sufficient conditions under which the amalgamated construction $A \bowtie^f I$ is a semicommutative ring. An ideal I of a ring R is called semicommutative if it is considered as a semicommutative ring without identity.

Theorem 3.1 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. Then the following hold.*

(1) If $A \bowtie^f I$ is semicommutative, then so is A.

(2) If A and $f(A) + I$ are semicommutative, then so is $A \bowtie^f I$.

(3) Assume that $I \cap S \neq \emptyset$ where S is the set of regular central elements of B. Then $A \bowtie^f I$ is a semicommutative ring if and only if $f(A) + I$ and A are semicommutative rings.

(4) Assume that $f^{-1}(I) \cap$ nil$(A) = (0)$. If $f(A) + I$ is a semicommutative ring, then $A \bowtie^f I$ is a semicommutative ring.

(5) If $f(A) + I$ is a semicommutative ring and f is a monomorphism, then A and I are semicommutative.

Proof (1) Assume that $A \bowtie^f I$ is semicommutative. Let $x,\ y \in A$ such that $xy = 0$. Then $(x, f(x))(y, f(y)) = 0$ in $A \bowtie^f I$. By assumption, $(x, f(x))(A \bowtie^f I)(y, f(y)) = 0$. Hence $xAy = 0$.

(2) Let $(a, f(a)+x)$, $(b, f(b)+y) \in A \bowtie^f I$. Assume that $(a, f(a)+x)(b, f(b)+y) = 0$. Then $ab = 0$ nd $(f(a)+x)(f(b)+y) = 0$. By hypothesis, $aAb = 0$ and $(f(a)+x)(f(A)+I)(f(b)+y) = 0$. Hence $arb = 0$ for any $r \in A$, $(f(a) + x)(f(c) + z)(f(b) + y) = 0$ for any $f(c) + z \in f(A) + I$. So $(a, f(a) + x)(c, f(c) + z)(b, f(b) + y) = 0$. Thus $A \bowtie^f I$ is semicommutative.

(3) Assume that $I \cap S \neq \emptyset$ where S is the set of regular central elements of B and $A \bowtie^f I$ is a semicommutative ring. By (1), A is semicommutative. To prove that $f(A) + I$ is semicommutative, let $f(a) + x,\ f(b) + y \in f(A) + I$ with $(f(a) + x)(f(b) + y) = 0$. For $0 \neq s \in I \cap S$, $(0, f(0) + s(f(a) + x))(0, f(0) + s(f(b) + y)) = 0$. By hypothesis, $(0, f(0) + s(f(a) + x))(0, f(0) + s(f(c)+z))(0, f(0)+s(f(b)+y)) = 0$ for all $f(c)+z \in f(A)+I$. So $s^3(f(a)+x)(f(c)+z)(f(b)+y) = 0$ since s is central. Regularity of s implies that $(f(a) + x)(f(c) + z)(f(b) + y) = 0$. So $f(A) + I$ is semicommutative. The converse is clear by (2).

(4) Assume that $f^{-1}(I) \cap \mathrm{nil}(A) = (0)$ and $f(A) + I$ is a semicommutative ring. To prove the semicommutativity of $A \bowtie^f I$, we first prove the semicommutativity of A. For if $a,\ b \in A$ and $ab = 0$, then $f(a)f(b) = 0$. By hypothesis, for each $c \in A$, $f(a)f(c)f(b) = 0$. For each $c \in A$, $bacba = 0$ since $(bacba)^2 = 0$ and $f(bacba) = 0$ and $bacba \in f^{-1}(I) \cap \mathrm{nil}(A) = (0)$. Then $(acb)^3 = 0$. This and $f(acb) = 0$ imply $acb = 0$. Hence A is semicommutative. By (2), the semicommutativity of $f(A) + I$ and A imply that of $A \bowtie^f I$.

(5) Suppose that $f(A) + I$ is a semicommutative ring. Let $a, b \in A$ with $ab = 0$. Then $f(a)f(b) = 0$ in $f(A) + I$. By hypothesis, $f(a)(f(A) + I)f(b) = 0$. In particular, $f(a)(f(c) + 0)f(b) = 0$. So $f(acb) = 0$ for all $c \in A$. Hence $acb = 0$ for all $c \in A$ since f is a monomorphism. Thus A is semicommutative. The rest is clear since every subring of a semicommutative ring is

semicommutative. □

The converse implication in (1) of Theorem 3.1 does not hold in general.

Example 3.2 Let $A = \mathbb{Z}_2$ and $X = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$, $Y = \begin{bmatrix} \mathbb{Z}_2 & 0 \\ \mathbb{Z}_2 & \mathbb{Z}_2 \end{bmatrix}$ and $B = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$, $I = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$ and let e_{ij} denote the matrix unit in B, that is, e_{ij} is a 4×4 matrix whose entries are all 0 except the (i, j) entry, that it is 1. Let $f : A \to B$ be a ring homomorphism defined by $f(a) = aI_4$ where I_4 is the identity matrix of B. Then A is semicommutative, B is not semicommutative. Let $a = e_{11} + e_{12} + e_{33} + e_{44}$, $b = e_{12} + e_{22}$, $c = e_{21} \in f(A) + I$. Then $ab = 0$ but $acb \neq 0$. Hence $f(A) + I$ is not semicommutative. Let $x = (1, f(1) + e_{11} + e_{21})$, $y = (0, e_{11} + e_{21})$, $z = (1, f(1) + e_{22}) \in A \bowtie^f I$. Then $xy = 0$ but $xzy = (0, e_{21})$. Hence $A \bowtie^f I$ is not semicommutative.

4. Nil-Semicommutativity of amalgamated rings

In this section, we investigate nil-semicommutativity of amalgamated rings. In [16], a ring R is called nil-semicommutative if for every a, $b \in \mathrm{nil}(R)$, $ab = 0$ implies $aRb = 0$. Every semicommutative ring is nil-semicommutative. We study the conditions under which $A \bowtie^f I$ is nil-semicommutative. We start with the following example for motivation.

Example 4.1 Let $A = \mathbb{Z}_2$ and $B = \begin{bmatrix} \mathbb{Z}_2 & \mathbb{Z}_2 \\ 0 & \mathbb{Z}_2 \end{bmatrix}$ be the rings and $I = \begin{bmatrix} 0 & \mathbb{Z}_2 \\ 0 & 0 \end{bmatrix}$ be the ideal of B and $f : A \to B$ be a ring homomorphism defined by $f(a) = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ where $a \in \mathbb{Z}_2$. Then f is a monomorphism, B is not semicommutative but nil-semicommutative and $\mathrm{nil}(B) \cap I \neq (0)$. Also

$$A \bowtie^f I = \left\{ (0, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}), (0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}), (1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}), (1, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}) \right\},$$

$$f(A) + I = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\}.$$

Then A, $A \bowtie^f I$ and $f(A) + I$ are nil-semicommutative.

An ideal I of a ring R is called nil-semicommutative if it is considered as a nil-semicommutative ring without identity.

Theorem 4.2 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. Then the following hold.*

(1) If $A \bowtie^f I$ is a nil-semicommutative ring, then so is A.

(2) If A and $f(A) + I$ are nil-semicommutative rings, then so is $A \bowtie^f I$.

(3) If $f^{-1}(I) = (0)$ and $A \bowtie^f I$ is nil-semicommutative, then $f(A)+I$ is nil-semicommutative.

(4) Assume that $\mathrm{nil}(B) \cap I = (0)$. Then $A \bowtie^f I$ is a nil-semicommutative ring if and only if A is a nil-semicommutative ring.

(5) Assume that f is a monomorphism and B is semicommutative. If $f(A)+I$ is a nil-semicommutative ring, then $A \bowtie^f I$ is a nil-semicommutative ring.

(6) If $f(A)+I$ is a nil-semicommutative ring and f is a monomorphism, then the rings A and I are nil-semicommutative.

Proof (1) Assume that $A \bowtie^f I$ is a nil-semicommutative ring. Let a, $b \in \mathrm{nil}(A)$ with $ab = 0$. Then $(a, f(a))$ and $(b, f(b))$ are nilpotent and $(a, f(a))(b, f(b)) = 0$ in $A \bowtie^f I$. Hence $(a, f(a))(c, f(c)+x)(b, f(b)) = 0$ for all $c \in A$ and $x \in I$, in particular, $acb = 0$ for every $c \in A$. Thus A is nil-semicommutative.

(2) Suppose that A and $f(A)+I$ are nil-semicommutative rings. Let $(a, f(a)+x)$, $(b, f(b)+y)$ be nilpotent and $(a, f(a)+x)(b, f(b)+y) = 0$ in $A \bowtie^f I$. Then a, b are nilpotent, $ab = 0$ and $aAb = 0$; and $f(a)+x$ and $f(b)+y$ are nilpotent, $(f(a)+x)(f(c)+z)(f(b)+y) = 0$ for all $f(c)+z \in f(A)+I$. Then $(a, f(a)+x)(c, f(c)+z)(b, f(b)+y) = (acb, (f(a)+x)(f(c)+z)(f(b)+y)) = 0$ for all $(c, f(c)+z) \in A \bowtie^f I$. Hence $A \bowtie^f I$ is nil-semicommutative.

(3) Assume that $f^{-1}(I) = (0)$ and $A \bowtie^f I$ is nil-semicommutative. Let a, $b \in A$ and x, $y \in I$. Assume that $f(a)+x$ and $f(b)+y$ are nilpotent and $(f(a)+x)(f(b)+y) = 0$. Say $(f(a)+x)^s = 0$ and $(f(b)+y)^t = 0$ where s and t are positive integers. Then a^s, b^t, $ab \in f^{-1}(I)$. Hence a and b are nilpotent and $ab = 0$. Then $(a, f(a)+x)(b, f(b)+y) = 0$. Clearly, $(a, f(a)+x)$ and $(b, f(b)+y)$ are nilpotent. By assumption, $(a, f(a)+x)(A \bowtie^f I)(b, f(b)+y) = 0$. It follows that $(f(a)+x)(f(A)+I)(f(b)+y) = 0$. Hence $f(A)+I$ is nil-semicommutative.

(4) Assume that $\mathrm{nil}(B) \cap I = (0)$. If $A \bowtie^f I$ is a nil-semicommutative ring, by (1), A is a nil-semicommutative ring. Conversely, assume that A is a nil-semicommutative ring. Let $(a, f(a)+x)$, $(b, f(b)+y)$ be nilpotent with $(a, f(a)+x)(b, f(b)+y) = 0$ in $A \bowtie^f I$, so there exist positive integers m, n such that $(a, f(a)+x)^n = 0$, $(b, f(b)+y)^m = 0$. Then $a^n = 0$ and $b^m = 0$ and $ab = 0$ in A; and $(f(a)+x)^n = 0$ and $(f(b)+y)^m = 0$ and $(f(a)+x)(f(b)+y) = 0$ in $f(A)+I$. Then $f(b)If(a) = 0$ since $f(b)If(a) \subseteq \mathrm{nil}(B) \cap I = (0)$. Also $f(a^{n-1})If(a^{n-1}) = 0$ since $f(a^{n-1})If(a^{n-1}) \subseteq \mathrm{nil}(B) \cap I = (0)$. Hence $(f(a^{n-1})z)^2 = 0$ for each $z \in I$. Continuing in this way, $f(a)I = 0$. Similarly, $If(a) = 0$, $f(b)I = 0$ and $If(b) = 0$. Also, if r, $s \in I$ with $rs = 0$, then we claim $rBs = 0$. Then $(sr)^2 = 0$, and so $sr \in \mathrm{nil}(B) \cap I = 0$, hence $sr = 0$. For any $t \in B$, $srt = 0$. Then $(rts)^2 = 0$. This implies that $rts \in \mathrm{nil}(B) \cap I = 0$. Thus $rBs = 0$. For any $f(c)+z \in f(A)+I$ we have $(f(a)+x)(f(c)+z)(f(b)+y) = f(acb) + f(a)zf(b) + xf(c)f(b) + f(a)f(c)y + f(a)zy + xf(c)y + xzf(b) + xzy = 0$ since as noted, $aAb = 0$, $If(a) = 0$, $f(a)I = 0$, $f(b)I = 0$, $If(b) = 0$ and $xBy = 0$.

(5) Assume that f is a monomorphism, B is semicommutative and $f(A)+I$ is nil-semicommutative. Let $(a, f(a)+x)$ and $(b, f(b)+y)$ be nilpotent in $A \bowtie^f I$ with $(a, f(a)+x)(b, f(b)+y) = 0$. Then $ab = 0$ and $(f(a)+x)(f(b)+y) = 0$. So $f(a)f(b) = 0$. Semicommutativity of B implies $f(a)Bf(b) = 0$ and $(f(a)+x)B(f(b)+y) = 0$. In particular, $f(a)f(A)f(b) = 0$ and $(f(a)+x)(f(A)+I)(f(b)+y) = 0$. Since f is a monomorphism, $aAb = 0$. It follows that $(a, f(a)+x)(A \bowtie^f I)(b, f(b)+y) = 0$. Hence $A \bowtie^f I$ is nil-semicommutative.

(6) Assume that $f(A)+I$ is a nil-semicommutative ring. Let a, b be nilpotent in A with

$ab = 0$. Then $f(a)$ and $f(b)$ are nilpotent and $f(a)f(b) = 0$ in $f(A) + I$. By assumption, $f(a)(f(A) + I)f(b) = 0$. In particular, for any $c \in A$, $f(acb) = 0$. Then $acb = 0$ for $c \in A$ since f is a monomorphism. Hence A is nil-semicommutative. The rest is clear. □

5. Weakly semicommutativity of amalgamated rings

In this section, weakly semicommutativity of amalgamated rings is investigated under some conditions. In [13], weakly semicommutative rings were defined and studied. A ring R is called weakly semicommutative if for any a, $b \in R$, $ab = 0$ implies arb is nilpotent for any $r \in R$. Clearly, semicommutative rings are weakly semicommutative. We first mention an easy result that subrings of weakly semicommutative rings are weakly semicommutative.

Lemma 5.1 *Every subring and every isomorphic copy of a weakly semicommutative ring are weakly semicommutative.*

An ideal I of a ring R is called weakly semicommutative if it is considered as a weakly semicommutative ring without identity.

Theorem 5.2 *Let A and B be a pair of rings, $f : A \to B$ be a ring homomorphism and I be a proper ideal of B. Then the following hold.*

(1) If $A \bowtie^f I$ is weakly semicommutative, then so is A.

(2) If A and $f(A) + I$ are weakly semicommutative, then so is $A \bowtie^f I$.

(3) Assume that $I \cap S \neq \emptyset$ where S is the set of regular central elements of B. Then $A \bowtie^f I$ is a weakly semicommutative ring if and only if $f(A) + I$ and A are weakly semicommutative rings.

(4) Assume that $f(A) \cap I = (0)$ and f is a monomorphism. If $A \bowtie^f I$ is weakly semicommutative, then $f(A) + I$ is weakly semicommutative.

(5) Assume that f is a monomorphism. If $f(A)+I$ is weakly semicommutative, then $A \bowtie^f I$, A and I are weakly semicommutative.

(6) Assume that $f^{-1}(I) \subseteq \text{nil}(A)$. If $f(A) + I$ is weakly semicommutative, then $A \bowtie^f I$, A and I are weakly semicommutative.

Proof (1) Assume that $A \bowtie^f I$ is weakly semicommutative. Let a, $b \in A$ with $ab = 0$. Then $(a, f(a))(b, f(b)) = 0$ in $A \bowtie^f I$. By assumption, $(a, f(a))(A \bowtie^f I)(b, f(b))$ is nil. Hence aAb is nil. Thus A is weakly semicommutative.

(2) Suppose that A and $f(A)+I$ are weakly semicommutative. Let $(a, f(a)+x), (b, f(b)+y) \in A \bowtie^f I$ with $(a, f(a) + x)(b, f(b) + y) = 0$. Then $ab = 0$ and $(f(a) + x)(f(b) + y) = 0$. By supposition, atb is nilpotent for each $t \in A$ and $(f(a)+x)(f(c)+z)(f(b)+y)$ is nilpotent for each $c \in A$ and $z \in I$. If $(atb)^r = 0$ and $((f(a)+x)(f(c)+z)(f(b)+y))^s = 0$ for some positive integers r and s, let $m = \max\{r, s\}$. Then $((a, f(a) + x)(c, f(c) + z)(b, f(b) + y))^m = 0$. So $A \bowtie^f I$ is weakly semicommutative.

(3) Let $I \cap S \neq \emptyset$ where S is the set of regular central elements of B. To complete the proof of (3), by (1) and (2), if $A \bowtie^f I$ is a weakly semicommutative ring, we show that $f(A) + I$

is weakly semicommutative. So let $(f(a)+x)(f(b)+y)=0$ in $f(A)+I$ and $0 \neq s \in I \cap S$. Then $(0, s(f(a)+x))(0, s(f(b)+y))=0$. Hence $(0, s(f(a)+x))(c, f(c)+z)(0, s(f(b)+y))$ is nilpotent in $A \bowtie^f I$, for all $f(c)+z \in f(A)+I$. The element s being central implies that $s^2(f(a)+x)(f(c)+z)(f(b)+y)$ is nilpotent for all $f(c)+z \in f(A)+I$. Since s is regular, $(f(a)+x)(f(c)+z)(f(b)+y)$ is nilpotent for all $f(c)+z \in f(A)+I$. Thus $f(A)+I$ is weakly semicommutative.

(4) Assume that $f(A) \cap I=(0)$, f is a monomorphism and $A \bowtie^f I$ is weakly semicommutative. To prove $f(A)+I$ is weakly semicommutative, let $f(a)+x$, $f(b)+y \in f(A)+I$ with $(f(a)+x)(f(b)+y)=0$. Then $f(a)f(b) \in f(A) \cap I$. By assumption, $f(ab)=0$ and so $ab=0$. Hence $(a, f(a)+x)(b, f(b)+y)=0$. Weakly semicommutativity of $A \bowtie^f I$ implies that $(a, f(a)+x)(A \bowtie^f I)(b, f(b)+y)$ is nil. It follows that $(f(a)+x)(f(A)+I)(f(b)+y)$ is nil. So $f(A)+I$ is weakly semicommutative.

(5) Assume that f is a monomorphism and $f(A)+I$ is a weakly semicommutative ring. Let $(a, f(a)+x)$, $(b, f(b)+y) \in A \bowtie^f I$ with $(a, f(a)+x)(b, f(b)+y)=0$. Then $ab=0$ and $(f(a)+x)(f(b)+y)=0$. So $f(a)f(b)=0$. By assumption, we have $f(a)(f(c)+z)f(b)$ and $(f(a)+x)(f(c)+z)(f(b)+y)$ are nilpotent for each $c \in A$ and $z \in I$. Then $f(a)f(c)f(b)$ is nilpotent for each $c \in A$. Again by assumption, acb is nilpotent for each $c \in A$. It follows that $(a, f(a)+x)(A \bowtie^f I)(b, f(b)+y)$ is a nil subset of $A \bowtie^f I$. Hence $A \bowtie^f I$ is weakly semicommutative. On the other hand, $f(A)$ and I are weakly semicommutative as subrings of $f(A)+I$ by Lemma 5.1 and A is weakly semicommutative as it is isomorphic to $f(A)$.

(6) Assume that $f^{-1}(I) \subseteq \operatorname{nil}(A)$ and $f(A)+I$ is weakly semicommutative. To prove $A \bowtie^f I$ is weakly semicommutative, let $(a, f(a)+x)$, $(b, f(b)+y) \in A \bowtie^f I$ with $(a, f(a)+x)(b, f(b)+y)=0$. Then $ab=0$ and $(f(a)+x)(f(b)+y)=0$. So $f(a)f(b)=0$. By assumption, $f(a)f(A)f(b)$ and $(f(a)+x)(f(A)+I)(f(b)+y)$ are nil subsets of $f(A)+I$. On the other hand, for any $c \in A$, $f(acb)^n=0$ for some positive integer n. Hence $(acb)^n \in f^{-1}(I)$. Since $f^{-1}(I) \subseteq \operatorname{nil}(A)$, $(acb)^n$ therefore acb is nilpotent. Thus $(a, f(a)+x)(A \bowtie^f I)(b, f(b)+y)$ is a nil set. The rest is clear. This completes the proof. □

The following example shows that the converse implication of (1) in Theorem 5.2 does not hold in general. Also the statement "$f(A)+I$ is weakly semicommutative" in (2) of Theorem 5.2 is not superfluous.

Example 5.3 Let $A=\mathbb{Z}_3$, $X=\begin{bmatrix}\mathbb{Z}_3 & \mathbb{Z}_3\\ \mathbb{Z}_3 & \mathbb{Z}_3\end{bmatrix}$, $Y=\begin{bmatrix}\mathbb{Z}_3 & 0\\ \mathbb{Z}_3 & \mathbb{Z}_3\end{bmatrix}$ and $B=\begin{bmatrix}X & 0\\ 0 & Y\end{bmatrix}$, $I=\begin{bmatrix}X & 0\\ 0 & 0\end{bmatrix}$ and $f: A \to B$ be a ring homomorphism defined by $f(a)=aI_4$ where I_4 is the identity matrix of B. Then A is weakly semicommutative but $A \bowtie^f I$ is not weakly semicommutative. Let $a=(1, f(1)+2e_{11}+e_{21}+e_{22}+2e_{33}+2e_{44})$, $b=(0, e_{11}+e_{12}+e_{21}+e_{22})$, $c=(0, e_{11}+2e_{12}+2e_{21}+2e_{22}) \in A \bowtie^f I$. Then $ab=0$ but $acb=(0, 2e_{21}+2e_{22})$ is not nilpotent in $A \bowtie^f I$. Thus $A \bowtie^f I$ is not weakly semicommutative. Let $x=2e_{11}+e_{12}+e_{33}+e_{44}$, $y=e_{11}+e_{12}+e_{21}+e_{22}$, $z=e_{11}+2e_{12}+2e_{21}+2e_{22} \in f(A)+I$. Then $xy=0$ but $xzy=2e_{11}+2e_{12}$ is not nilpotent in $f(A)+I$. Thus $f(A)+I$ is not weakly semicommutative.

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