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Minimal Prime Ideals and Units in 2-Primal Ore Extensions

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Abstract Let R be an (α,δ)-compatible ring. It is proved that R is a 2-primal ring if and only if for every minimal prime ideal $\mathscr{P}$ in $R[x;\alpha,\delta]$ there exists a minimal prime ideal P in R such that $\mathscr{P}=P[x;\alpha,\delta]$, and that $f(x)\in R[x;\alpha,\delta]$ is a unit if and only if its constant term is a unit and other coefficients are nilpotent.

Keywords 2-primal ring; (α,δ)-compatible ring; Ore extension

MR(2010) Subject Classification 16S36; 16N40

1. Introduction

Throughout this paper all rings are associative with identity. Let R be a ring, $\mathrm{End}(R,+)$ the ring of additive endomorphisms of R, and Φ a subset of $\mathrm{End}(R,+)$. According to Lam et al. [1], an ideal I of R is said to be a Φ-ideal if $\varphi(I)\subseteq I$ for all $\varphi\in\Phi$, and a Φ-ideal $P\neq R$ is called a Φ-prime ideal if for any Φ-ideals I and J such that $IJ\subseteq P$, we have either $I\subseteq P$ or $J\subseteq P$. Let Ω denote the multiplicative semigroup with unity generated in $\mathrm{End}(R,+)$ by Φ. Obviously, an ideal I of R is a Φ-ideal if and only if I is an Ω-ideal, and so I is a Φ-prime ideal if and only if I is an Ω-prime ideal. In some circumstance, one may assume that Φ additionally satisfies (H_1): Φ is closed under composition and $id_R\in\Phi$, i.e., $\Omega=\Phi$ and (H_2): For any $a\in R$, $\sum_{\varphi\in\Phi}R\varphi(a)R$ is a Φ-ideal of R. For instance, if α is an endomorphism and δ is an α-derivation of R, i.e., δ is an additive endomorphism such that $\delta(ab)=\delta(a)b+\alpha(a)\delta(b)$ for any $a,b\in R$, then the multiplicative semigroup Φ with unity generated in $\mathrm{End}(R,+)$ by α and δ satisfies both (H_1) and (H_2) (see [1, Example 1.3]). We denote by $R[x;\alpha,\delta]$ the Ore extension whose elements are all polynomials over R, the addition is defined as usual and the multiplication subject to the relation $xa=\alpha(a)x+\delta(a)$ for any $a\in R$. Ore extensions appear in several natural contexts, including skew and differential polynomial rings, group algebras of polycyclic groups, universal enveloping algebras of solvable Lie algebras, and coordinate rings of quantum groups. In the sequel the set of nilpotent elements, the prime radical, the Jacobson radical of a ring R are denoted by $N(R)$, $N_*(R)$, $J(R)$, respectively. For a subset S of R, the symbol $\mathbf{C}(S)$ stands for the complement of S in R.

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Let R be a ring, α an endomorphism, and δ an α-derivation of R. Following Annin [2], R is said to be α-compatible in case $ab = 0 \Leftrightarrow a\alpha(b) = 0$ for any $a, b \in R$. Similarly, a ring R in [3] is called δ-compatible if $ab = 0$ implies $a\delta(b) = 0$. Moreover, if R is both α-compatible and δ-compatible, then it is said to be (α, δ)-compatible. A ring R is called reduced if $N(R) = 0$, and R is called 2-primal if $N(R) = N_*(R)$. Clearly, any reduced ring is a 2-primal ring. The attraction of 2-primal rings lies in the structure of their prime ideals. It is well known that a ring R is 2-primal if and only if every minimal prime ideal P of R is completely prime, i.e., R/P is a domain [4]. More importantly, 2-primal rings provide a sort of bridge between commutative and noncommutative ring theory [5]. Nasr-Isfahani [6] studied Ore extensions of 2-primal rings. It is proved that if R is an (α, δ)-compatible ring, then R is a 2-primal ring if and only $R[x; \alpha, \delta]$ is a 2-primal ring if and only if every minimal (α, δ)-prime ideal in R is a completely prime ideal. However the existing literature does not discuss the relationship between minimal prime ideals in R and the ones in $R[x; \alpha, \delta]$. The main objective of this paper is to study this problem. It is proved for an (α, δ)-compatible ring R that R is 2-primal ring if and only if for every minimal prime ideal $\mathscr{P}$ in $R[x; \alpha, \delta]$ there exists a minimal prime ideal P in R such that $\mathscr{P} = P[x; \alpha, \delta]$, and that $f(x) \in R[x; \alpha, \delta]$ is a unit if and only if its constant term is a unit and other coefficients are nilpotent. It turns out that $J(R[x; \alpha, \delta]) = N_*(R[x; \alpha, \delta])$ and the stable range of $R[x; \alpha, \delta]$ is not equal to one.

2. Minimal prime ideals in 2-primal Ore extensions

Let R be a ring, α an endomorphism and δ an α-derivation of R. For integers i, j with $0 \le i \le j$, we use the symbol $f_i^j \in \mathrm{End}(R, +)$ to denote the map which is the sum of all possible words in α, δ built with i letters α and j-i letters δ. For example, $f_0^0 = 1$, $f_j^j = \alpha^j$, $f_0^j = \delta^j$, $f_{j-1}^j = \alpha^{j-1}\delta + \alpha^{j-2}\delta\alpha + \cdots + \delta\alpha^{j-1}$, and so forth [1].

Lemma 2.1 *Let R be an (α, δ)-compatible ring and $a_1, a_2, \ldots, a_n$ in R. We have the following*

(1) $x^n r = \sum_{i=0}^n f_i^n(r)x^i$ *in* $R[x; \alpha, \delta]$ *for any* $r \in R$.

(2) $a_1 a_2 \cdots a_n = 0$ *if and only if* $\alpha^{k_1}(a_1)\alpha^{k_2}(a_2)\cdots\alpha^{k_n}(a_n) = 0$ *for any integer* $k_i \ge 0$.

(3) $a_1 a_2 \cdots a_n = 0$ *implies* $f_{i_1}^{j_1}(a_1) f_{i_2}^{j_2}(a_2) \cdots f_{i_n}^{j_n}(a_n) = 0$.

Proof (1) It is true in any Ore extension by [1, Lemma 4.1]. (2) This is [7, Lemma 3.1]. (3) It is a restatement of [8, Corollary 2.1]. $\square$

Lemma 2.1(3) implies that if $a_1 a_2 \cdots a_n = 0$, then $\delta^{k_1}(a_1)\delta^{k_2}(a_2)\cdots\delta^{k_n}\ (a_n) = 0$.

Let $\Phi \subseteq \mathrm{End}(R, +)$ satisfy (H_1) and (H_2). A subset M of a ring R is called a Φ-m-system if for any $a, b \in M$ there exist $\varphi_1, \varphi_2 \in \Phi$ and $r \in R$ such that $\varphi_1(a)r\varphi_2(b) \in M$. It is known that a Φ-ideal P is a Φ-prime ideal if and only if $\mathbf{C}(P)$ is a Φ-m-system [1].

Lemma 2.2 *Any Φ-prime ideal P of a ring R contains a minimal Φ-prime ideal.*

Proof We apply Zorn's Lemma to the family of Φ-prime ideals contained in P. It suffices to show that, for any chain of Φ-prime ideals $\{P_i | i \in I\}$, the intersection $P' = \bigcap P_i$ is a Φ-prime ideal. Clearly, P' is a Φ-ideal. Let $a, b \notin P'$. Then we have $a \notin P_i$ and $b \notin P_j$ for some $i, j \in I$. If, say, $P_i \subseteq P_j$, then $a, b \notin P_i$. Since P_i is a Φ-prime ideal, $\mathbf{C}(P_i)$ is a Φ-m-system. There exist $\varphi_1, \varphi_2 \in \Phi$ and $r \in R$ such that $\varphi_1(a)r\varphi_2(b) \notin P_i$, and so $\varphi_1(a)r\varphi_2(b) \notin P'$. This means that $\mathbf{C}(P')$ is a Φ-m-system, i.e., P' is a Φ-prime ideal. $\square$

According to [9], an endomorphism α of a ring R is said to be rigid if $a\alpha(a) = 0$ implies $a = 0$ for any $a \in R$, and a ring R in [10] is called α-rigid if there exists a rigid endomorphism α of R. It is known by [9, Theorem 3.3] that if R is an α-rigid ring, then $R[x; \alpha, \delta]$ is reduced.

Recall that if $f : R \to S$ is an epimorphism of rings with kernel K, then there exists a one-to-one correspondence between the set of all prime ideals in R that contain K and the set of all prime ideals in S, given by $P \mapsto f(P)$. In particular, P is a minimal prime ideal containing K in R if and only if $f(P)$ is a minimal prime ideal is S.

Lemma 2.3 *If R is a 2-primal (α, δ)-compatible ring and P a minimal prime ideal of R, then $\alpha(P)$, $\alpha^{-1}(P)$, and $\delta(P)$ are all contained in P. Thus α induces an endomorphism $\bar{\alpha}$, and δ induces an $\bar{\alpha}$-derivation $\bar{\delta}$ of $\overline{R} = R/P$ via $\bar{\alpha}(\bar{a}) = \overline{\alpha(a)}$, $\bar{\delta}(\bar{a}) = \overline{\delta(a)}$ for any $a \in R$. Moreover the ring $\overline{R}$ is an $\bar{\alpha}$-rigid ring, and so it is an $(\bar{\alpha}, \bar{\delta})$-compatible ring.*

Proof The Hypothesis implies $N(R) = N_*(R)$, and so $R/N_*(R)$ is a reduced ring. Let $\widehat{R} = R/N_*(R)$ and $\hat{r} = r + N(R)$ for any $r \in R$. Define $\hat{\alpha}, \hat{\delta}$: $R \to R$ via $\hat{\alpha}(\hat{a}) = \widehat{\alpha(a)}$, and $\hat{\delta}(\hat{a}) = \widehat{\delta(a)}$, respectively for any $a \in R$. Lemma 2.1 and the (α, δ)-compatibility of R imply that $N_*(R)$ is an (α, δ)-ideal, so $\hat{\alpha}$ and $\hat{\delta}$ make sense. It is easy to check that $\hat{\alpha}$ is an ring endomorphism and $\hat{\delta}$ is an $\hat{\alpha}$-derivation of $\widehat{R}$. We claim that $\widehat{R}$ is an $\hat{\alpha}$-rigid ring. To see this, let $\hat{a}\hat{\alpha}(\hat{a}) = \hat{0}$ for any $a \in R$. Then we have $\hat{a}\widehat{\alpha(a)} = \hat{0} = \widehat{a\alpha(a)}$. This means $a\alpha(a) \in N_*(R)$, and so $a^2 \in N_*(R)$ by Lemma 2.1(2). It yields that $a \in N_*(R)$ since R is a 2-primal ring, i.e., $\hat{a} = \hat{0}$, as required. Since P is a minimal prime ideal of R, $\widehat{P}$ is a minimal prime ideal in the reduced ring $\widehat{R}$. It follows that $\hat{\alpha}(\widehat{P}), \hat{\alpha}^{-1}(\widehat{P}) \subseteq \widehat{P}$ by [9, Lemma 3.2]. This implies that $\alpha(P)$ and $\alpha^{-1}(P)$ are contained in P. It is proved in [9, Theorem 3.3, p.295] that if R is an α-rigid ring and Q is a minimal prime ideal of R, then one has $\delta(Q) \subseteq Q$. Using this fact, we have $\hat{\delta}(\widehat{P}) \subseteq \widehat{P}$. This implies $\delta(P) \subseteq P$. Moreover, [11, Lemma 4] implies that $\widehat{R}$ is an $(\hat{\alpha}, \hat{\delta})$-compatible ring. Finally, let $\overline{R} = R/P$. Because $\alpha(P)$ and $\delta(P)$ are in P, $\bar{\alpha}$ and $\bar{\delta}$, as defined in Lemma 2.3, make sense. Applying the fact that P is a completely prime ideal in R, and $\alpha^{-1}(P) \subseteq P$, it is easily proved that $\overline{R}$ is an $\bar{\alpha}$-rigid ring, and so it is an $(\bar{\alpha}, \bar{\delta})$-compatible ring by [11, Lemma 4]. $\square$

Combining Lemma 2.3 with Lemma 2.1(1), it can be concluded that if P is a minimal prime ideal of a 2-primal (α, δ)-compatible ring R, then $P[x; \alpha, \delta]$ is an ideal of $R[x; \alpha, \delta]$.

Lemma 2.4 *If R is a reduced (α, δ)-compatible ring, then an ideal P of R is a minimal prime ideal if and only if P is a minimal (α, δ)-prime ideal.*

Proof Assume that P is a minimal prime ideal of R. Then P is an (α, δ)-ideal by Lemma 2.3, and P is also a completely prime ideal since a reduced ring is 2-primal. Thus for any $a, b \notin P$, we have $ab \notin P$. This means $\alpha(a)1\alpha(b) = \alpha(ab) \notin P$ by Lemma 2.3. Let Φ be the multiplicative semigroup with unity generated in $\mathrm{End}(R, +)$ by α and δ. The above argument implies that

$\mathbf{C}(P)$ is a Φ-m-system. Hence P is a Φ-prime ideal, i.e., P is an (α,δ)-prime ideal. If P is not minimal, then there exists a minimal (α,δ)-prime ideal $P'\subset P$. Now [6, Lemma 2.7] implies that P' is a completely prime ideal, this contradicts the minimality of P. Conversely, assume that P is a minimal (α,δ)-prime ideal of R. Then P is a completely prime ideal by [6, Lemma 2.7]. If P is not a minimal prime ideal, then there exists a minimal prime ideal $P'\subset P$. On the other hand, the above argument implies that P' is a minimal (α,δ)-prime ideal, an obvious contradiction. $\square$

Lemma 2.5 *If R is a reduced (α,δ)-compatible ring, then $\mathscr{P}$ is a minimal prime ideal in $R[x;\alpha,\delta]$ if and only if there exists a minimal prime ideal P in R such that $\mathscr{P}=P[x;\alpha,\delta]$.*

Proof First we show that R is an α-rigid ring. For any $a\in R$, if $a\alpha(a)=0$ then we have $a^2=0$ by Lemma 2.1(2), this gives $a=0$ since R is reduced. It follows that $R[x;\alpha,\delta]$ is a reduced ring by [9, Theorem 3.3]. Assume that $\mathscr{P}$ is a minimal prime ideal in $R[x;\alpha,\delta]$. Then $\mathscr{P}$ is a completely prime ideal by [4, Proposition 1.11]. Let $P'=R\bigcap\mathscr{P}$. It is easy to see that P' is an ideal of R. If $a,b\in P'$ satisfy $ab\in P'$, then $ab\in\mathscr{P}$, and so $a\in\mathscr{P}$ or $b\in\mathscr{P}$. This implies that P' is a completely prime ideal of R. There exists a minimal prime ideal P of R such that $P\subseteq P'$ and R/P is a domain. We claim that $\mathscr{P}=P[x;\alpha,\delta]$. In fact, from Lemmas 2.1(1) and 2.3, there is no difficulty to check that $P[x;\alpha,\delta]$ is an ideal of $R[x;\alpha,\delta]$. By Lemma 2.3, $\overline{R}=R/P$ is an $(\bar\alpha,\bar\delta)$-compatible ring, and so $\overline{R}[x;\bar\alpha,\bar\delta]$ makes sense, where the meaning of $\bar\alpha$ and $\bar\delta$ are the same as in Lemma 2.3. Now we define a ring epimorphism $\varrho:R[x;\alpha,\delta]\to\overline{R}[x;\bar\alpha,\bar\delta]$ via $\varrho(a_0+a_1x+\cdots+a_nx^n)=\bar{a_0}+\bar{a_1}x+\cdots+\bar{a_n}x^n$. Clearly, ϱ is an additive map. For any $f(x)=\sum_{i=0}^m a_ix^i, g(x)=\sum_{j=0}^n b_jx^j\in R[x;\alpha,\delta]$, then one has $f(x)g(x)=\sum_{k=0}^{m+n}(\sum_{s+t=k}(\sum_{i=s}^m a_if_s^i(b_t)))x^k$ (see [8, p.704]) from which it is easy to check $\varrho(f(x)g(x))=\varrho(f(x))\varrho(g(x))$. Observing $\ker\varrho=P[x;\alpha,\delta]$, there exists a ring isomorphism $R[x;\alpha,\delta]/P[x;\alpha,\delta]\cong\overline{R}[x;\bar\alpha,\bar\delta]$. Since the right hand side is a domain (cf. Corollary 3.3), we conclude that $P[x;\alpha,\delta]$ is a completely prime ideal of $R[x;\alpha,\delta]$. On the other hand, since $P\subseteq\mathscr{P}$, it is easy to see $P[x;\alpha,\delta]\subseteq\mathscr{P}$, and thus $\mathscr{P}=P[x;\alpha,\delta]$ by the minimality of $\mathscr{P}$.

Conversely, assume that P is a minimal prime ideal of R. Then $P[x;\alpha,\delta]$ is a completely prime ideal by the argument of the above paragraph. We claim that $P[x;\alpha,\delta]$ is a minimal prime ideal of $R[x;\alpha,\delta]$. Otherwise, there exists a minimal prime ideal $\mathscr{P}$ in $R[x;\alpha,\delta]$ such that $\mathscr{P}\subset P[x;\alpha,\delta]$. Similarly to the proof in the above paragraph, there exists a minimal prime ideal P' in R such that $\mathscr{P}=P'[x;\alpha,\delta]$. This means $P'[x;\alpha,\delta]\subset P[x;\alpha,\delta]$. Thus we have $P'\subset P$, contradicting the minimality of P. This completes the proof. $\square$

Theorem 2.6 *If R is a 2-primal (α,δ)-compatible ring, then $\mathscr{P}$ is a minimal prime ideal in $R[x;\alpha,\delta]$ if and only if there exists a minimal prime ideal P in R such that $\mathscr{P}=P[x;\alpha,\delta]$.*

Proof The hypothesis implies that $R[x;\alpha,\delta]$ is a 2-primal ring by [6, Theorem 2.10]. Assume that $\mathscr{P}$ is a minimal prime ideal in $R[x;\alpha,\delta]$. Clearly, in $R[x;\alpha,\delta]/N_*(R[x;\alpha,\delta])$, $\overline{\mathscr{P}}$ is a minimal prime ideal. Since $R[x;\alpha,\delta]$ is a 2-primal ring, one has $N_*(R[x;\alpha,\delta])=N_*(R)[x;\alpha,\delta]$ by [6, Corollary 2.11]. In view of the proof in Lemma 2.3, α induces an endomorphism via $\hat\alpha(\hat a)=\widehat{\alpha(a)}$, and δ induces an $\hat{\alpha}$-derivation of the reduced ring $\widehat{R} = R/N_*(R)$ via $\hat{\delta}(\hat{a}) = \widehat{\delta(a)}$ for any $a \in R$, such that $\widehat{R}$ is an $(\hat{\alpha}, \hat{\delta})$-compatible ring. The canonical ring homomorphism $\pi : R[x; \alpha, \delta] \to R/N_*(R)[x; \hat{\alpha}, \hat{\delta}]$ via $a_0 + a_1x + \cdots + a_nx^n \mapsto \hat{a}_0 + \hat{a}_1x + \cdots + \hat{a}_nx^n$ with $\ker(\pi) = N_*(R)[x; \alpha, \delta] = N_*(R[x; \alpha, \delta])$ induces a ring isomorphism ρ: $R[x; \alpha, \delta]/N_*(R[x; \alpha, \delta]) \cong R/N_*(R)[x; , \hat{\alpha}, \hat{\delta}]$. Thus $\rho(\overline{\mathscr{P}})$ is a minimal prime ideal in $R/N_*(R)[x; \hat{\alpha}, \hat{\delta}]$, and so there exists minimal prime ideal P of R such that $\rho(\overline{\mathscr{P}}) = P/N_*(R)[x; \hat{\alpha}, \hat{\delta}]$ by Lemma 2.5. This means $\overline{\mathscr{P}} = P[x; \alpha, \delta]/N_*(R[x; \alpha, \delta])$ by the virtue of ρ, i.e., $\mathscr{P} = P[x; \alpha, \delta]$.

For the converse, suppose that P is a minimal prime ideal of R. Then $\widehat{P}$ is a minimal prime ideal of the reduced ring $\widehat{R} = R/N_*(R)$, and so $\widehat{P}[x; \hat{\alpha}, \hat{\delta}]$ is a minimal prime ideal in $\widehat{R}[x; \hat{\alpha}, \hat{\delta}]$ by Lemma 2.5. From the ring isomorphism ρ, we know that $P[x; \alpha, \delta]/N_*(R[x; \alpha, \delta])$ is a minimal prime ideal in $R[x; \alpha, \delta]/N_*(R[x; \alpha, \delta])$. This means that $P[x; \alpha, \delta]$ is a minimal prime ideal in $R[x; \alpha, \delta]$. The proof is completed. □

Corollary 2.7 *If R is a 2-primal (α, δ)-compatible ring and P_i $(i \in \Lambda)$ are all minimal prime ideals of R, then $N_*(R[x; \alpha, \delta]) = \bigcap_{i \in \Lambda} P_i[x; \alpha, \delta]$.*

Proof It is a direct consequence of Lemma 2.2 and Theorem 2.6. □

Corollary 2.8 *Let R be an (α, δ)-compatible ring. Then $R[x; \alpha, \delta]$ is a 2-primal ring if and only if for every minimal prime ideal $\mathscr{P}$ in $R[x; \alpha, \delta]$ there exists a minimal prime ideal P in R such that $\mathscr{P} = P[x; \alpha, \delta]$.*

Proof The only if part follows by Theorem 2.6. Conversely, for any minimal prime ideal $\mathscr{P}$ in $R[x; \alpha, \delta]$, then $\mathscr{P} = P[x; \alpha, \delta]$ for some minimal prime ideal P in R by hypothesis. This means that $R[x; \alpha, \delta]/\mathscr{P} \cong R/P[x; \bar{\alpha}, \bar{\delta}]$ is a domain, and so $\mathscr{P}$ is a completely prime ideal. It follows that $R[x; \alpha, \delta]$ is a 2-primal ring by [4, Proposition 1.11]. □

3. Units in 2-primal Ore extensions

The main objective of this section is to determine all units in Ore extensions of 2-primal (α, δ)-compatible rings, generalizing the main results on 2-primal polynomial rings in [12].

Lemma 3.1 Let R be a reduced (α, δ)-compatible ring. If $f(x) = a_0 + a_1x + \cdots + a_mx^m, g(x) = b_0 + b_1x + \cdots + b_nx^n \in R[x; \alpha, \delta]$ satisfy $f(x)g(x) = c \in R$, then $a_0b_0 = c$ and $a_ib_j = 0$ for $i + j > 0$.

Proof First we claim that all $a_ib_j = 0$ for $i+j > 0$. Assume this is not true. Then we can choose $f(x) = \sum_{i=0}^m a_ix^i, g(x) = \sum_{j=0}^n b_jx^j$ with $m + n$ minimal such that $f(x)g(x) = c \in R$ but the conclusions of the claim are not satisfied. The case of $m + n = 0$ obviously gives a contradiction and hence we may suppose that $m + n > 0$. The leading coefficient $a_m\alpha^m(b_n) = 0$ of $f(x)g(x)$ and the (α, δ)-compatibility show that $a_mb_n = 0$, and hence $b_na_m = 0$, since R is reduced. This leads to $b_nf(x)g(x) = b_nc$ and by the minimality of the degree of the counter-example. It can be concluded that this cannot be a counter-example so that for any i, j such that $i + j > 0$

we have $b_n a_i b_j = 0$ and so $(b_n a_i)^2 = 0$. Since R is reduced, we get $a_i b_n = 0$ for $i > 0$. The (α, δ)-compatibility leads to $a_i x^i b_n = 0$, in other words $f(x)(g(x) - b_n x^n) = c$. The minimality of $m + n$ as the degree of a counter-example leads to the conclusion. This proves the claim. Now applying the formula in [8, p.704], we have $f(x)g(x) = \sum_{k=0}^{m+n}(\sum_{s+t=k}(\sum_{i=s}^{m} a_i f_s^i(b_t)))x^k$. Let $c_k = \sum_{s+t=k}(\sum_{i=s}^{m} a_i f_s^i(b_t))$. Then $c_0 = a_0 f_0^0(b_0) + a_1 f_0^1(b_0) + \cdots + a_m f_0^m(b_0)$. The claim implies $a_1 b_0 = \cdots = a_m b_0 = 0$, and so $a_1 f_0^1(b_0) = \cdots = a_m f_0^m(b_0) = 0$ by Lemma 2.1(3). Thus we conclude that $c_0 = a_0 b_0 = f(x)g(x) = c$, as desired. $\Box$

The next two corollaries are the direct results of Lemma 3.1.

Corollary 3.2 *An (α, δ)-compatible ring R is a domain if and only if $R[x; \alpha, \delta]$ is a domain.*

Corollary 3.3 *Let R be a reduced (α, δ)-compatible ring. If $f(x) = a_0 + a_1 x + \cdots + a_m x^m$ is a unit in $R[x; \alpha, \delta]$, then a_0 is a unit and $a_i = 0$ for all $i \geq 1$.*

Theorem 3.4 *If R is a 2-primal (α, δ)-compatible ring, then $f(x) \in R[x; \alpha, \delta]$ is a unit if and only if its constant term is a unit in R and other coefficients are nilpotent.*

Proof Since R is 2-primal and (α, δ)-compatible, $R[x; \alpha, \delta]$ is a 2-primal ring and $N_*(R[x; \alpha, \delta])$ $= N_*(R)[x; \alpha, \delta]$ by [6, Corollary 2.11]. With help of the proof in Lemma 2.3, we know that $\widehat{R} = R/N_*(R)$ is a reduced $(\hat{\alpha}, \hat{\delta})$-compatible ring where $\hat{\alpha}(\hat{a}) = \widehat{\alpha(a)}$, $\hat{\delta}(\hat{a}) = \widehat{\delta(a)}$ for any $a \in R$. There exists a ring homomorphism $R[x; \alpha, \delta] \to \widehat{R}[x; \hat{\alpha}, \hat{\delta}]$ via $f(x) = a_0 + \cdots + a_m x^m \to \hat{f}(x) = \hat{a}_0 + \cdots + \hat{a}_m x^m$. If $f(x) \in R[x; \alpha, \delta]$ is a unit, then $\hat{f}(x)$ is a unit in $\widehat{R}[x; \hat{\alpha}, \hat{\delta}]$. Applying Corollary 3.3, $\hat{a}_0$ is a unit and $\hat{a}_i = \hat{0}$ for each $i \geq 1$, and so a_0 is a unit in R and a_i is nilpotent for $i \geq 1$. The converse is clear, since $f(x)$ is a sum of a unit and an element in $N_*(R[x; \alpha, \delta])$. $\Box$

Theorem 3.5 *If R is a 2-primal (α, δ)-compatible ring, then $J(R[x; \alpha, \delta]) = N_*(R[x; \alpha, \delta])$.*

Proof Let $f(x) = a_0 + a_1 x + \cdots + a_m x^m \in J(R[x; \alpha, \delta])$. Then $1 + f(x)x$ is a unit, this implies $a_i \in N_*(R)$ for all $i \geq 0$ by Theorem 3.4. Thus we have $f(x) \in N_*(R[x; \alpha, \delta])$ by [6, Corollary 2.11], and so $J(R[x; \alpha, \delta] \subseteq N_*(R[x; \alpha, \delta]))$. The reverse inclusion is clearly. $\Box$

A ring R is said to have stable range one, denoted by $S_r(R) = 1$, if for any $a, b \in R$ whenever $aR + bR = R$, there exists $r \in R$ such that $a + br$ is a unit. It is known that R has stable range one if and only if so does R/I for any ideal $I \subseteq J(R)$ (see [13, pp.319–321]).

Theorem 3.6 *If R is a reduced (α, δ)-compatible ring, then $S_r(R[x; \alpha, \delta]) \neq 1$.*

Proof Assume the conclusion does not hold. Then $x(-x) + 1 + x^2 = 1$ implies that there exists $f(x) \in R[x; \alpha, \delta]$ such that $x + (1 + x^2)f(x) = u(x)$ is a unit. Write $f(x) = a_0 + a_1 x + \cdots + a_n x^n$. If $n = 0$, then $x + a_0 + x^2 a_0$ is a unit. Applying Lemma 2.1, we have $x^2 a_0 = f_0^2(a_0) + f_1^2(a_0)x + f_2^2(a_0)x^2$. Thus $u(x) = a_0 + f_0^2(a_0) + (1 + f_1^2(a_0))x + \alpha^2(a_0)x^2$ is a unit. By Corollary 3.3, we have $\alpha^2(a_0) = 0$, and so $a_0 = 0$ by the α-compatibility of R. This means 1 is nilpotent, a contradiction. Thus we may assume that $n \geq 1$ and $a_n \neq 0$. Now the leading coefficient of $u(x) = x + f(x) + x^2 f(x)$ is $\alpha^2(a_n)$. Since $u(x)$ is a unit, we have $\alpha^2(a_n) = 0$ by Corollary 3.3, and thus $a_n = 0$ by the α-compatibility of R, which is a desired contradiction. $\square$

Corollary 3.7 *If R is a 2-primal (α, δ)-compatible ring, then $S_r(R[x; \alpha, \delta]) \neq 1$.*

Proof By hypothesis, $\widehat{R} = R/N_*(R)$ is a reduced ring. Define $\hat{\alpha}$, $\hat{\delta}$: $\widehat{R} \to \widehat{R}$ via $\hat{\alpha}(\hat{a}) = \widehat{\alpha(a)}$, and $\hat{\delta}(\hat{a}) = \widehat{\delta(a)}$, respectively, for any $a \in R$. Then $\hat{\alpha}$ is an endomorphism and $\hat{\delta}$ is an $\hat{\alpha}$-derivation of $\widehat{R}$ such that $\widehat{R}$ is an $\hat{\alpha}$-rigid ring by the proof of Lemma 2.3. This means that $\widehat{R}$ is an $(\hat{\alpha}, \hat{\delta})$-compatible ring by [11, Lemma 4], and so $\widehat{R}[x; \hat{\alpha}, \hat{\delta}]$ is a reduced ring by [9, Theorem 3.3]. Since there exists a canonical ring isomorphism $R[x; \alpha, \delta]/N_*(R)[x; \alpha, \delta] \cong \widehat{R}[x; \hat{\alpha}, \hat{\delta}]$, and $N_*(R)[x; \alpha, \delta] = N_*(R[x; \alpha, \delta])$ (see [6, Corollary 2.11]), we conclude that $S_r(R[x; \alpha, \delta]) \neq 1$ by Theorem 3.6. This completes the proof of the corollary. $\square$

References

[1] T. Y. LAM, A. LEROY, J. MATCZUK. *Primeness semiprimeness and the prime radical of Ore extensions*. Comm. Algebra, 1997, **25**(8): 2459–2506.

[2] S. ANNIN. *Associated primes over skew polynomial rings*. Comm. Algebra, 2002, **30**(5): 2511–2528.

[3] E. HASHEMI, A. MOUSSAVI. *Polynomial extensions of quasi-Baer rings*. Acta Math. Hungar., 2005, **107**(3): 207–224.

[4] G. Y. SHIN. *Prime ideals and sheaf representation of a pseudo symmetric rings*. Trans. Amer. Math. Soc., 1973, **184**: 43–60.

[5] G. MARKS. *A taxonomy of 2-primal rings*. J. Algebra, 2003, **266**(2): 494–520.

[6] A. R. NASR-ISFAHANI. *Ore extensions of 2-primal rings*. J. Algebra Appl., 2014, **13**(3): 1–7.

[7] Weixing CHEN, Shuying CUI. *On weakly semicommutative rings*. Comm. Math. Res., 2011, **27**(2): 179–192.

[8] Lunqun OUYANG, Huanyin CHEN. *On weak symmetric rings*. Comm. Algebra, 2010, **38**: 697–713.

[9] J. KREMPA. *Some examples of reduced rings*. Algebra Colloq., 1996, **3**(4): 289–300.

[10] C. Y. HONG, N. K. KIM, T. K. KWAK. *On skew Armendariz rings*. Comm. Algebra, 2003, **31**(1): 103–122.

[11] C. Y. HONG, N. K. KIM, T. K. KWAK. *Ore extensions of Baer and p.p.-rings*. J. Pure Appl. Algebra, 2000, **151**(3): 215–226.

[12] Weixing CHEN. *Units in polynomial rings over 2-primal rings*. Southeast Asian Bull. Math., 2006, **30**: 1049–1053.

[13] L. N. VASERSTEIN. *Bass's first stable range condition*. J. Appl. Algebra, 1984, **34**: 319–330.