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One-Signed Periodic Solutions of First-Order Functional Difference Equations with Parameter

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Abstract In this paper, the authors obtain the existence of one-signed periodic solutions of the first-order functional difference equation

$$\Delta u(n)=a(n)u(n)-\lambda b(n)f(u(n-\tau(n))),\quad n\in\mathbb{Z}$$

by using global bifurcation techniques, where $a,b:\mathbb{Z}\to[0,\infty)$ are T-periodic functions with $\sum_{n=1}^{T}a(n)>0$, $\sum_{n=1}^{T}b(n)>0$; $\tau:\mathbb{Z}\to\mathbb{Z}$ is T-periodic function, $\lambda>0$ is a parameter; $f\in C(\mathbb{R},\mathbb{R})$ and there exist two constants $s_2<0<s_1$ such that $f(s_2)=f(0)=f(s_1)=0$, $f(s)>0$ for $s\in(0,s_1)\cup(s_1,\infty)$, and $f(s)<0$ for $s\in(-\infty,s_2)\cup(s_2,0)$.

Keywords one-signed periodic solutions; existence; functional difference equations; bifurcation from infinity

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1. Introduction

Let T be a positive integer, $\mathbb{R}$ denote the real number set, and $\mathbb{Z}$ be the integer set.

In this paper, we investigate the existence of one-signed periodic solutions of the following functional difference equation

$$\Delta u(n)=a(n)u(n)-\lambda b(n)f(u(n-\tau(n))),\quad n\in\mathbb{Z},\tag{1.1}$$

where $a,b:\mathbb{Z}\to[0,\infty)$ are T-periodic functions with $\sum_{n=1}^{T}a(n)>0$, $\sum_{n=1}^{T}b(n)>0$, $\tau:\mathbb{Z}\to\mathbb{Z}$ is T-periodic function, $\lambda>0$ is a parameter. In recent years, there has been considerable interest in the existence of positive solutions of the following differential equation

$$x'(t)=\tilde{a}(t)x(t)-\lambda\tilde{b}(t)\tilde{f}(x(t-\tau(t))),\tag{1.2}$$

where $\tilde{a},\tilde{b}\in C(\mathbb{R},[0,\infty))$ are ω-periodic functions with $\int_0^\omega\tilde{a}(t)\mathrm{d}t>0$, $\int_0^\omega\tilde{b}(t)\mathrm{d}t>0$, τ is a continuous ω-periodic function, $\lambda>0$ is a parameter. (1.2) has been proposed as a model for a variety of physiological processes and conditions including production of blood cells, respiration, and

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cardiac arrhythmias. Thus, the existence of periodic solutions of first-order differential equations and first-order difference equations has been discussed by many authors, see, for example [1–12] and the references therein.

Recently, Raffoul [11] dealt with the equation (1.1), determined values of λ for which there exist T-periodic positive solutions by using Krasnosel'skii's fixed point theorem under the assumptions:

(H1) $f \in C([0,\infty),[0,\infty))$, and $f(s) > 0$ for $s > 0$.

(H2) $a, b : \mathbb{Z} \to [0,\infty)$ are T-periodic functions with $\sum_{n=1}^{T} a(n) > 0$, $\sum_{n=1}^{T} b(n) > 0$, $\tau : \mathbb{Z} \to \mathbb{Z}$ is T-periodic function.

(H3) There exist $f_0, f_\infty \in (0,\infty)$, such that

$$f_0 = \lim_{|s|\to 0} \frac{f(s)}{s}, \quad f_\infty = \lim_{|s|\to\infty} \frac{f(s)}{s}.$$

He proved the following

Theorem 1.1 ([11]) *Assume that (H1)–(H3) hold and λ satisfies*

$$\frac{1}{\sigma B f_\infty} < \lambda < \frac{1}{C f_0} \quad \textit{or} \quad \frac{1}{\sigma C f_0} < \lambda < \frac{1}{B f_\infty}.$$

Then the equation (1.1) has a positive periodic solution. Here

$$C = \max_{n\in\mathbb{Z}} \sum_{s=0}^{T-1} \tilde{G}(n,s)b(s), \;\; B = \min_{n\in\mathbb{Z}} \sum_{s=0}^{T-1} \tilde{G}(n,s)b(s), \;\; \sigma = \prod_{i=1}^{T} (1+a(i))^{-1}.$$

In 2011, by using the Dancer global bifurcation theorem, Ma et al. [12] studied the positive periodic solution of the following generalized form of (1.1)

$$\Delta u(n) = a(n)g(u(n))u(n) - \lambda b(n) f(u(n-\tau(n))), \quad n \in \mathbb{Z}, \tag{1.3}$$

under the assumptions (H1), (H2) and g satisfies

(A1) $g \in C([0,\infty),[0,\infty))$, and there exist positive constants l, L, such that $0 < l \le g \le L < \infty$.

They obtained the following

Theorem 1.2 ([12]) *Let (H1), (H2) and (A1) hold. Assume that*

(A2) There exist $f_0, \underline{f}_\infty, \bar{f}^\infty \in (0,\infty)$ such that

$$\underline{f}_\infty = \liminf_{s\to\infty} \frac{f(s)}{s}, \quad \bar{f}^\infty = \limsup_{|s|\to\infty} \frac{f(s)}{s}.$$

Then the equation (1.3) has a positive T-periodic solution if either

$$\frac{\lambda^\diamond(L)}{\underline{f}_\infty} < \lambda < \frac{\lambda^\diamond(g(0))}{f_0} \quad \textit{or} \quad \frac{\lambda^\diamond(g(0))}{f_0} < \lambda < \frac{\lambda^\diamond(l)}{\bar{f}^\infty},$$

where $\lambda^\diamond(c)$ is the first eigenvalue of the linear eigenvalue problem

$$\Delta u(n) = ca(n)u(n) - \lambda b(n) f(u(n-\tau(n))), \quad n \in \mathbb{Z}$$

and c is a positive constant.

However, Raffoul [11] and Ma [12] only focused their attentions on the fact that $f(s) > 0$, $s \in (0, \infty)$. Of course, the natural question is what would happen if f is allowed to have some zeros in $\mathbb{R} \setminus \{0\}$? But it is a difficult job to study the global behavior of the components of nodal solutions of (1.3) under the condition:

(H4) $f \in C(\mathbb{R}, \mathbb{R})$, there exist two constants $s_2 < 0 < s_1$ such that $f(s_2) = f(0) = f(s_1) = 0$, $f(s) > 0$ for $s \in (0,\ s_1) \cup (s_1, \infty)$, $f(s) < 0$ for $s \in (-\infty, s_2) \cup (s_2, 0)$.

Moreover, we know from the proof of Theorem 1.2 that the assumptions (H1), (H2), (A1) and (A2) imply that the component from the trivial solution at $(\frac{\lambda^{\diamond}(c)}{f_0}, 0)_p$ and the component from infinity at $(\frac{\lambda^{\diamond}(c)}{f_\infty}, \infty)_p$ are coincident (notice that we use $(\frac{\lambda^{\diamond}(c)}{f_0}, 0)_p$ and $(\frac{\lambda^{\diamond}(c)}{f_\infty},\ \infty)_p$ to denote the ‘ point ’ in some product spaces, and use (a, b) to denote the usual open interval in this work). However, in Section 2, we will prove that these two components are disjoint under the assumptions (H2)–(H4), in which the essential role is played by the fact whether f possesses zeros in $\mathbb{R} \setminus \{0\}$.

Therefore, in this paper, we are devoted to studying the global behavior of the components of one-signed solutions of (1.1) under the condition (H4), and our main results are sharp.

The rest of this paper is organized as follows: in Section 2, we give some notations and the main results. In Section 3, we are devoted to proving the main results, and we illustrate the results with an appropriate example.

2. Statement of the main results

Let

$$E = \{u : \mathbb{Z} \to \mathbb{R} \,|\, u(n+T) = u(n)\}$$

be the Banach space with the norm $\|u\| = \max_{n \in \mathbb{T}} |u(n)|$, where $\mathbb{T} = \{0, 1, \ldots, T-1\}$.

By a positive solution of (1.1) we mean a pair (λ, u), where $\lambda > 0$ and u is a solution of (1.1) with $u > 0$ in $\mathbb{Z}$.

It is well known that (1.1) is equivalent to

$$u(n) = \lambda \sum_{s=n}^{n+T-1} G(n,s) b(s) f(u(s - \tau(s))) =: (\mathcal{A}u)(n), \quad n \in \mathbb{Z}, \tag{2.1}$$

where

$$G(n,s) = \frac{\prod_{i=n}^{s} (1 + a(i))^{-1}}{1 - \prod_{i=1}^{T} (1 + a(i))^{-1}}, \quad s \in \{n, n+1, \ldots, n+T\}. \tag{2.2}$$

Notice that $0 < \prod_{i=1}^{T} (1 + a(i))^{-1} < 1$, we have that

$$\frac{\sigma}{1-\sigma} \le G(n,s) \le \frac{1}{1-\sigma}.$$

Define K be a cone in E by

$$K = \{u \in E \,|\, u(n) \ge 0,\ u(n) \ge \sigma \|u\|\}.$$

Proposition 2.1 *$\mathcal{A}(K) \subset K$, and $\mathcal{A} : K \to K$ is completely continuous.*

Proof For any $u \in K$, it follows that

$$\begin{aligned}\|\mathcal{A}u\| &= \max_{n\in\mathbb{Z}} \mathcal{A}u(n) = \max_{n\in\mathbb{Z}} \lambda \sum_{s=n}^{n+T-1} G(n,s)b(s)f(u(s-\tau(s)))\\ &\le \frac{\lambda}{1-\sigma}\sum_{s=n}^{n+T-1} b(s)f(u(s-\tau(s))),\\ \mathcal{A}u(n) &= \lambda \sum_{s=n}^{n+T-1} G(n,s)b(s)f(u(s-\tau(s)))\\ &\ge \frac{\lambda\sigma}{1-\sigma}\sum_{s=n}^{n+T-1} b(s)f(u(s-\tau(s))) \ge \sigma\|\mathcal{A}u\|.\end{aligned}$$

So $\mathcal{A}(K) \subset K$. By Areza-Ascoli theorem, $\mathcal{A}: K \to K$ is completely continuous. □

Next, let us consider the spectrum of the linear eigenvalue problem

$$\Delta u(n) = a(n)u(n) - \lambda b(n)u(n-\tau(n)),\ n \in \mathbb{Z}. \tag{2.3}$$

Lemma 2.2 *Assume that (H2) holds. Then the equation (2.3) has a unique eigenvalue λ_1, which is positive and simple, and the corresponding eigenfunction $\varphi(\cdot)$ is of one sign.*

Proof The equation (2.3) is equivalent to

$$u(n) = \lambda \sum_{s=n}^{n+T-1} G(n,s)b(s)u(s-\tau(s)) =: (\mathcal{T}u)(n), \quad n \in \mathbb{Z}.$$

By the same method as the one used to prove Proposition 2.1, it is easy to prove that $\mathcal{T}$ is a strong positive operator. By the Krein-Rutman Theorem [13, Theorem 19.3], the spectrum radius $r(\mathcal{T}) > 0$ and subsequently, the problem (2.3) has a unique eigenvalue λ_1, which is positive and simple, and the corresponding eigenfunction $\varphi(\cdot)$ is of one sign. □

Define $L : E \to E$ by setting

$$(Lu)(n) := -\Delta u(n) + a(n)u(n), \quad u \in E. \tag{2.4}$$

It is easy to verify that $\sum_{n=1}^{T} a(n) = 0$ is a resonant condition, so $\sum_{i=1}^{T} a(i) > 0$ is sufficient to prove that L^{-1} exists. This together with Proposition 2.1 leads to $L^{-1} : E \to E$ is compact and continuous.

Let $\zeta,\ \xi \in C(\mathbb{R},\ \mathbb{R})$ be such that

$$f(s) = f_0 s + \zeta(s), \quad f(s) = f_\infty s + \xi(s). \tag{2.5}$$

Clearly,

$$\lim_{|s|\to 0} \frac{\zeta(s)}{s} = 0, \quad \lim_{|s|\to\infty} \frac{\xi(s)}{s} = 0. \tag{2.6}$$

Let us consider

$$Lu(n) - \lambda b(n) f_0 u(n-\tau(n)) = \lambda b(n)\zeta(u(n-\tau(n))) \tag{2.7}$$

as a bifurcation problem from the trivial solution $u \equiv 0$, and

$$Lu(n) - \lambda b(n) f_\infty u(n-\tau(n)) = \lambda b(n)\xi(u(n-\tau(n))) \tag{2.8}$$

as a bifurcation problem from infinity. We note that (2.7) and (2.8) are the same, and each of them is equivalent to (1.1).

Let $\mathbb{E} = \mathbb{R} \times E$ under the product topology. We add the points $\{(\lambda, \infty)_p \mid \lambda \in \mathbb{R}\}$ to our space $\mathbb{E}$. Let S^+ denote the set of positive functions in E and $S^- = -S^+$, and $S = S^- \cup S^+$. They are disjoint and open in E. Finally, let $\Phi^\pm = \mathbb{R} \times S^\pm$ and $\Phi = \mathbb{R} \times S$.

Remark 2.3 It is worth remarking that if u is a nontrivial solution of (1.1) and a, b and f satisfy (H2)–(H4), then $u \in S^\nu$ for some $\nu = \{+, -\}$. To see this, define

$$q(n) = \begin{cases} \frac{f(u(n))}{u(n)}, & u(n) \neq 0, \\ f_0, & u(n) = 0. \end{cases}$$

Thus the equation (1.1) is equivalent to

$$\Delta u(n) = a(n)u(n) - \lambda b(n) q(n - \tau(n)) u(n - \tau(n)), \quad n \in \mathbb{Z}. \tag{2.9}$$

Obviously, $b(\cdot) q(\cdot - \tau(\cdot))$ satisfies (H2). It follows from Lemma 2.2 that the nontrivial solution $u \in S^\nu$ for some $\nu \in \{+, -\}$. $\square$

The result of Rabinowitz [14] for (2.7) can be stated as follows: for each $\nu \in \{+, -\}$, there exists a continuum $\mathcal{C}^\nu$ of solutions of (2.7) joining $(\frac{\lambda_1}{f_0}, 0)_p$ to infinity, and $\mathcal{C}^\nu \backslash \{(\frac{\lambda_1}{f_0}, 0)_p\} \subset \Phi^\nu$.

The result of Rabinowitz [15] for (2.8) can be stated as follows: for each $\nu \in \{+, -\}$, there exists a continuum $\mathcal{D}^\nu$ of solutions of (2.8) meeting $(\frac{\lambda_1}{f_\infty}, \infty)_p$, and $\mathcal{D}^\nu \backslash \{(\frac{\lambda_1}{f_\infty}, \infty)_p\} \subset \Phi^\nu$.

Our main results are

Theorem 2.4 *Assume that (H2)–(H4) hold, and let*

(H5) f satisfies the Lipschitz condition in $[s_2, s_1]$.

Then

(i) For $(\lambda, u) \in \mathcal{C}^+ \cup \mathcal{C}^-$, $s_2 < u(n) < s_1$, $n \in \mathbb{T}$;

(ii) For $(\lambda, u) \in \mathcal{D}^+ \cup \mathcal{D}^-$, we have that either $\max_{n\in\mathbb{T}} u(n) > s_1$ or $\min_{n\in\mathbb{T}} u(n) < s_2$.

Corollary 2.5 *Let (H2)–(H5) hold. If $f_0 < f_\infty$, then*

(i) If $\lambda \in (\frac{\lambda_1}{f_\infty}, \frac{\lambda_1}{f_0}]$, then (1.1) has at least two solutions u_∞^+ and u_∞^-, such that u_∞^+ is positive on $\mathbb{T}$ and u_∞^- is negative on $\mathbb{T}$;

(ii) If $\lambda \in (\frac{\lambda_1}{f_0}, \infty)$, then (1.1) has at least four solutions u_∞^+, u_∞^-, u_0^+, and u_0^-, such that u_∞^+, u_0^+ are positive on $\mathbb{T}$ and u_∞^-, u_0^- are negative on $\mathbb{T}$.

Corollary 2.6 *Let (H2)–(H5) hold. If $f_\infty < f_0$, then*

(i) If $\lambda \in (\frac{\lambda_1}{f_0}, \frac{\lambda_1}{f_\infty}]$, then (1.1) has at least two solutions u_0^+ and u_0^-, such that u_0^+ is positive on $\mathbb{T}$ and u_0^- is negative on $\mathbb{T}$;

(ii) If $\lambda \in (\frac{\lambda_1}{f_\infty}, \infty)$, then (1.1) has at least four solutions u_∞^+, u_∞^-, u_0^+, and u_0^-, such that u_∞^+, u_0^+ are positive on $\mathbb{T}$ and u_∞^-, u_0^- are negative on $\mathbb{T}$.

3. Proof of the main results

To prove Theorem 2.4, we need the following proposition.

Proposition 3.1 *(i) The discrete first-order boundary value problem*

$$\Delta u(n) - a(n)u(n) + h(n) = 0, \quad n \in \mathbb{T},\ u(0) = u(T) \tag{3.1}$$

has a unique solution for all $h \in l^1(0,T)$ if and only if $\prod_{i=0}^{T-1}(1+a(i)) \neq 1$.

(ii) Let $\prod\limits_{i=0}^{T-1}(1+a(i)) > 1$ hold. If $h \geq 0$ and $h(\cdot) \not\equiv 0$ on $\mathbb{T}$, then the solution u of (3.1) is positive on $\mathbb{T}$.

Proof (i) The equation $\Delta u(n) - a(n)u(n) = 0$ has a solution $u(n) = C\prod_{i=0}^{n-1}(1+a(i))$, where C is a constant. If $u(n)$ is a nontrivial solution, then by $u(0) = C$, $u(T) = C\prod_{i=0}^{T-1}(1+a(i))$, we deduce that $\prod_{i=0}^{T-1}(1+a(i)) = 1$.

On the other hand, from $\prod_{i=0}^{T-1}(1+a(i)) = 1$, we can get that $\Delta u(n) - a(n)u(n) = 0$ has a nontrivial solution $u(n) = C\prod_{i=0}^{n-1}(1+a(i))$, where $C \in \mathbb{R} \setminus \{0\}$.

(ii) The difference equation (3.1) can be rewritten to the form

$$u(n+1)\prod_{i=0}^{n}(1+a(i))^{-1} - u(n)\prod_{i=0}^{n-1}(1+a(i))^{-1} + \prod_{i=0}^{n}(1+a(i))^{-1}h(n) = 0.$$

By summing the above equation from $s = n$ to $s = n+T-1$ we obtain

$$u(n) = \sum_{s=n}^{n+T-1} G(n,s)h(s).$$

Since $\prod_{i=0}^{T-1}(1+a(i)) > 1$, it follows that $G(n,s) > 0$, $s \in \{n, n+1, \ldots, n+T\}$. If $h \geq 0$ and $h(\cdot) \not\equiv 0$ on $\mathbb{T}$, then $u(n) > 0$ on $\mathbb{T}$. $\Box$

Proof of Theorem 2.4 Suppose on the contrary that there exists $(\lambda, u) \in \mathcal{C}^+ \cup \mathcal{C}^- \cup \mathcal{D}^+ \cup \mathcal{D}^-$ such that either

$$\max\{u(n) \,|\, n \in \mathbb{T}\} = s_1,$$

or

$$\min\{u(n) \,|\, n \in \mathbb{T}\} = s_2.$$

We divide the proof into two cases.

Case 1 $\max\{u(n) \,|\, n \in \mathbb{T}\} = s_1$.

In this case, we know that

$$0 \leq u(n) \leq s_1, \quad 0 \leq u(n-\tau(n)) \leq s_1, \quad n \in \mathbb{T}.$$

Let us consider the equation (1.1). By (H2), (H4) and (H5), there exists $m \geq 0$ such that $b(n)f(s) + ms$ is strictly increasing in s for $s \in [s_2, s_1]$. Then (1.1) can be rewritten to the form

$$Lu + \lambda m u(n-\tau(n)) = \lambda[b(n)f(u(n-\tau(n))) + mu(n-\tau(n))]$$

since $Ls_1 - a(n)s_1 = 0 = f(s_1)$,

$$Ls_1 - a(n)s_1 + \lambda m s_1 = \lambda[b(n)f(s_1) + ms_1].$$

Subtracting, it follows that

$$L(s_1 - u) + \lambda m(s_1 - u(n - \tau(n))) - a(n)s_1 \geq 0,$$

i.e.,

$$L(s_1 - u) + \lambda m s_1 \geq 0, \quad n \in \mathbb{T}$$

and

$$s_1 - u(0) = s_1 - u(T) > 0.$$

From Proposition 3.1, we deduce that $s_1 > u(n)$, $n \in \mathbb{T}$, which contradicts $\max\{u(n) \,|\, n \in \mathbb{T}\} = s_1$. Hence,

$$u(n) < s_1, \quad n \in \mathbb{T}.$$

Case 2 $\min\{u(n) \,|\, n \in \mathbb{T}\} = s_2$.

In this case, we know that

$$s_2 \leq u(n) \leq 0, \quad s_2 \leq u(n - \tau(n)) \leq 0, \quad n \in \mathbb{T}.$$

Let us consider the equation (1.1). By (H2), (H4) and (H5), there exists $m \geq 0$ such that $b(n)f(s) + ms$ is strictly increasing on s for $s \in [s_2, s_1]$. Then we have

$$Lu + \lambda m u(n - \tau(n)) = \lambda[b(n)f(u(n - \tau(n))) + mu(n - \tau(n))]$$

and since $Ls_2 - a(n)s_2 = 0 = f(s_2)$, we have

$$Ls_2 - a(n)s_2 + \lambda m s_2 = \lambda[b(n)f(s_2) + ms_2].$$

Subtracting, we get that

$$L(s_2 - u) + \lambda m(s_2 - u(n - \tau(n))) - a(n)s_2 \leq 0,$$

i.e.,

$$L(s_2 - u) + \lambda m s_2 \leq 0, \quad n \in \mathbb{T}$$

and

$$s_2 - u(0) = s_2 - u(T) < 0.$$

From Proposition 3.1, we deduce that $s_2 - u(n) < 0$, $n \in \mathbb{T}$, this contradicts $\min\{u(n) \,|\, n \in \mathbb{T}\} = s_2$. Therefore, $s_2 < u(n)$, $n \in \mathbb{T}$. $\Box$

Next, we prove Corollaries 2.5 and 2.6.

Proof of Corollaries 2.5 and 2.6 Since boundary value problem

$$-\Delta u(n) + a(n)u(n) = 0, \quad u(0) = u(T) \tag{3.2}$$

has a unique solution $u \equiv 0$, we get

$$(\mathcal{C}^+ \cup \mathcal{C}^- \cup \mathcal{D}^+ \cup \mathcal{D}^-) \subset \{(\lambda, u)_p \in \mathbb{R} \times E \,|\, \lambda \geq 0\}.$$

Take $\Lambda \in \mathbb{R}$ as an interval such that $\Lambda \cap \{\frac{\lambda_1}{f_\infty}\} = \{\frac{\lambda_1}{f_\infty}\}$ and $\mathcal{M}$ is a neighborhood of $(\frac{\lambda_1}{f_\infty}, \infty)_p$ whose projection on $\mathbb{R}$ lies in Λ and whose projection on E is bounded away from 0. Then by [15, Theorem 1.6 and Corollary 1.8], we have that for each $\nu \in \{+, -\}$, either

(1) $\mathcal{D}^\nu \setminus \mathcal{M}$ is bounded in $\mathbb{R} \times E$ in which case $\mathcal{D}^\nu \setminus \mathcal{M}$ meets $\{(\lambda, 0)_p \mid \lambda \in \mathbb{R}\}$, or

(2) $\mathcal{D}^\nu \setminus \mathcal{M}$ is unbounded.

Moreover, if (1) occurs and $\mathcal{D}^\nu \setminus \mathcal{M}$ has a bounded projection on $\mathbb{R}$, then $\mathcal{D}^\nu \setminus \mathcal{M}$ meets $(\frac{\lambda_k}{f_\infty},\ \infty)_p$ where $\lambda_k \neq \lambda_1$ is another eigenvalue of (2.3).

Obviously, Theorem 2.4 (ii) implies that (1) does not occur. So $\mathcal{D}^+ \setminus \mathcal{M}$ is unbounded.

Remark 2.3 guarantees that $\mathcal{D}^+$ is a component of solutions of (2.8) in S^+ which meets $(\frac{\lambda_1}{f_\infty},\ \infty)_p$, and consequently $\mathrm{Proj}_{\mathbb{R}}(\mathcal{D}^+ \setminus \mathcal{M})$ is unbounded. Thus

$$\mathrm{Proj}_{\mathbb{R}}(\mathcal{D}^+) \supset (\frac{\lambda_1}{f_\infty}, +\infty). \tag{3.3}$$

Similarly, we get

$$\mathrm{Proj}_{\mathbb{R}}(\mathcal{D}^-) \supset (\frac{\lambda_1}{f_\infty}, +\infty). \tag{3.4}$$

By Theorem 2.4, for any $(\lambda,\ u) \in (\mathcal{C}^+ \cup \mathcal{C}^-)$,

$$\|u\| < \max\{s_1,\ |s_2|\} := s^*. \tag{3.5}$$

(3.5) and (2.7) imply that

$$\|u\| < \max\{s^*,\ \|a\|_\infty s^* + \lambda\|b\|_\infty \max_{|s|\le s^*} |f(s)|\},$$

which means that the sets $\{(\lambda,\ u) \in \mathcal{C}^+ \mid \lambda \in [0,\ d]\}$ and $\{(\lambda,\ u) \in \mathcal{C}^- \mid \lambda \in [0,\ d]\}$ are bounded for any fixed $d \in (0,\ \infty)$. This together with the fact that $\mathcal{C}^+$ and $\mathcal{C}^-$ join $(\frac{\lambda_1}{f_0},\ 0)_p$ to infinity yields respectively that

$$\mathrm{Proj}_{\mathbb{R}}(\mathcal{C}^+) \supset (\frac{\lambda_1}{f_0}, +\infty) \tag{3.6}$$

and

$$\mathrm{Proj}_{\mathbb{R}}(\mathcal{C}^-) \supset (\frac{\lambda_1}{f_0}, +\infty). \tag{3.7}$$

Combining (3.3), (3.4), (3.6) and (3.7), we conclude the desired results. $\square$

Remark 3.2 The methods used in the proof of Theorem 2.4, Corollaries 2.5 and 2.6 have been used in the study of other kinds of boundary value problems, see [16–18] and the references therein.

Remark 3.3 The conditions in Corollaries 2.5 and 2.6 are sharp. Let us take

$$a(n) \equiv a > 0,\ \lambda = a,\ b(n) = 1,\ f(s) = s + h(s),\ \tau(n) \equiv 0.$$

Let

$$h(s) = \begin{cases} -\frac{2s}{s^2+1}, & s \in (-\infty, -1) \cup (1, +\infty), \\ -\frac{2s^3}{s^2+1}, & s \in [-1,\ 1], \end{cases}$$

and consider problem

$$\Delta u(n) = a(n)u(n) - a[u(n) + h(u(n))],\quad n \in \mathbb{T}, \qquad u(0) = u(T). \tag{3.8}$$

It is easy to see that $\lambda_1 = a$, $f_0 = f_\infty = 1$. Since

$$\frac{\lambda_1}{f_\infty} = a = \frac{\lambda_1}{f_0},$$

the conditions of Corollaries 2.5 and 2.6 are not valid. In this case, (3.8) has no nontrivial solution. In fact, if u is a nontrivial solution of (3.8), then

$$0 = \sum_{n=0}^{T-1} \Delta u(n) = a \sum_{n=0}^{T-1} h(u(n)) \neq 0,$$

which is a contradiction.

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