Journal of Mathematical Research with Applications
Jul., 2018, Vol. 38, No. 4, pp. 411–417
DOI:10.3770/j.issn:2095-2651.2018.04.008
Http://jmre.dlut.edu.cn

Existence of Positive Solutions for a Class of Kirchhoff Type Systems

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Abstract In this paper, we are interested in the existence of positive solutions for the Kirchhoff type problems

$$\begin{cases} -(a_1+b_1M_1(\int_\Omega|\nabla u|^p\mathrm{d}x))\triangle_p u=\lambda f(u,v), & \text{in } \Omega,\\ -(a_2+b_2M_2(\int_\Omega|\nabla v|^q\mathrm{d}x))\triangle_q v=\lambda g(u,v), & \text{in } \Omega,\\ u=v=0, & \text{on } \partial\Omega, \end{cases}$$

where $1<p,q<N, Mi: R_0^+\to R^+$ $(i=1,2)$ are continuous and increasing functions. λ is a parameter, $f,g\in C^1((0,\infty)\times(0,\infty))\times C([0,\infty)\times[0,\infty))$ are monotone functions such that $f_s,f_t,g_s,g_t\ge 0$, and $f(0,0)<0, g(0,0)<0$ (semipositone). Our proof is based on the sub- and super-solutions techniques.

Keywords positive solutions; existence; Kirchhoff type systems

MR(2010) Subject Classification 35J92; 35J57

1. Introduction and main result

In this article, we are interested in the existence of positive solutions for a class of Kirchhoff type systems of the form

$$\begin{cases} -(a_1+b_1M_1(\int_\Omega|\nabla u|^p\mathrm{d}x))\triangle_p u=\lambda f(u,v), & \text{in } \Omega,\\ -(a_2+b_2M_2(\int_\Omega|\nabla v|^q\mathrm{d}x))\triangle_q v=\lambda g(u,v), & \text{in } \Omega,\\ u=v=0, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $p,q>1$, $\lambda>0$, a_i, b_i $(i=1,2)$ are real parameters and $b_i\ne 0$. The functions M_i $(i=1,2)$, $f(u,v)$ and $g(u,v)$ satisfy the following conditions:

(H_1) $M_i:\mathbb{R}_0^+\to\mathbb{R}^+$ $(i=1,2)$ are continuous and increasing functions and there exist constants m_i $(i=1,2)$ such that $M_i(t)\ge m_i>-\frac{a_i}{b_i}$ for all $t\in\mathbb{R}_0^+$, where $\mathbb{R}_0^+=[0,+\infty)$.

(H_2) $f,g\in C^1((0,\infty)\times(0,\infty))\times C([0,\infty)\times[0,\infty))$ are monotone functions such that f_s, f_t, g_s, $g_t\ge 0$, and $f(0,0)<0$, $g(0,0)<0$ (semipositone).

(H_3) There exist $r>\alpha>0$ such that $f(s,t)\ge 0$, $g(s,t)\ge 0$ for $s,t\in(\alpha,r]$.

Received August 9, 2017; Accepted March 1, 2018
Supported by the National Natural Science Foundation of China (Grant Nos. 11571093; 11471164).
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Since the first two equations in (1.1) contain an integral over Ω, it is no longer a pointwise identity; therefore it is often called nonlocal problem. This problem models several physical and biological systems, where u describes a process which depends on the average of itself, such as the population density [1]. Moreover, problem (1.1) is related to the stationary version of the Kirchhoff equation

$$\rho\frac{\partial^2 u}{\partial t^2}-\Big(\frac{P_0}{h}+\frac{E}{2L}\int_0^L|\frac{\partial u}{\partial x}|^2\mathrm{d}x\Big)\frac{\partial^2 u}{\partial x^2}=0 \tag{1.2}$$

presented by Kirchhoff in 1883 (see [2]). This equation is an extension of the classical d'Alembert's wave equation by considering the effects of the changes in the length of the string during the vibrations. The parameters in (1.2) have the following meanings: L is the length of the string, h is the area of the cross-section, E is the Young modulus of the material, ρ is the mass density, and P_0 is the initial tension.

In recent years, problems involving Kirchhoff type operators have been studied in many papers, we refer to [3–13], in which the authors have used variational method and topological method to get the existence of solution. In this paper, motivated by the ideas introduced in [14–18] and the properties of Kirchhoff type operators in [19–21], we study problem (1.1) in the semipositone case. Using the sub and supersolutions techniques, we prove the existence of a positive solution for the problem without assuming any condition on f and g at infinity. Using an argument inspired by [18], we obtain the following main results which complement and extend the corresponding results in [19, 22].

In order to state precisely our main result, we first consider the following eigenvalue problem for the r-Laplace operator $-\triangle_r u$:

$$\begin{cases} -\triangle_r u=\lambda|u|^{r-2}u, & \text{in } \Omega,\\ u=0, & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

Let $\phi_{1,r}\in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue $\lambda_{1,r}$ of (1.3) such that $\phi_{1,r}>0$ in Ω and $\|\phi_{1,r}\|_\infty=1$. It can be shown that $\frac{\partial\phi_{1,r}}{\partial\eta}<0$ on $\partial\Omega$ and hence, depending on Ω, there exist positive constants m,δ,σ such that

$$\begin{cases} |\nabla\phi_{1,r}|^r-\lambda_{1,r}\phi_{1,r}^r\ge m, & \text{in } \overline{\Omega}_\delta,\\ \phi_{1,r}\ge\sigma, & \text{in } \Omega\backslash\overline{\Omega}_\delta, \end{cases} \tag{1.4}$$

where $\overline{\Omega}_\delta:=\{x\in\Omega: d(x,\partial\Omega)\le\delta\}$. We will also consider the unique solution $e\in C^1(\overline{\Omega})$ of the boundary value problem

$$\begin{cases} -\triangle_r e=1, & \text{in } \Omega,\\ e=0, & \text{on } \partial\Omega \end{cases} \tag{1.5}$$

to discuss our result. It is known that $e>0$ in Ω and $\frac{\partial e}{\partial\eta}<0$ on $\partial\Omega$. For our main result we also assume that there exist positive constants $l_1,l_2\in(\alpha,r]$ such that

(H_4) $\frac{\lambda_{1,p}}{f(l_1,l_2)}<\frac{m}{|f(0,0)|}$, $\frac{\lambda_{1,q}}{g(l_1,l_2)}<\frac{m}{|g(0,0)|}$.

(H_5) $\lambda_{1,p}(\frac{p}{p-1})^{p-1}<\frac{\sigma^p}{\|e\|_\infty^{p-1}}$, $\lambda_{1,q}(\frac{q}{q-1})^{q-1}<\frac{\sigma^q}{\|e\|_\infty^{q-1}}$.

Our main result is given by the following theorem.

Theorem 1.1 *Under assumptions (H_1)–(H_5), there exist two positive constants λ_* and λ^* such that (1.1) has a positive solution for all $\lambda \in (\lambda_*, \lambda^*)$.*

2. Preliminaries

We will prove our result by using the method of sub and supersolutions. We refer the readers to recent paper [18, 19] on the topic. A pair of functions (ψ_1, ψ_2) is said to be a subsolution of problem (1.1) if it is in $W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ such that

$$\Big(a_1 + b_1 M_1\Big(\int_\Omega |\nabla \psi_1|^p \mathrm{d}x\Big)\Big) \int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla w \mathrm{d}x \le \lambda \int_\Omega f(\psi_1, \psi_2) w \mathrm{d}x, \quad \forall w \in W,$$

and

$$\Big(a_2 + b_2 M_2\Big(\int_\Omega |\nabla \psi_2|^q \mathrm{d}x\Big)\Big) \int_\Omega |\nabla \psi_2|^{q-2} \nabla \psi_2 \cdot \nabla w \mathrm{d}x \le \lambda \int_\Omega g(\psi_1, \psi_2) w \mathrm{d}x, \quad \forall w \in W,$$

where $W := \{w \in C_0^\infty(\Omega) : w \ge 0 \text{ in } \Omega\}$. A pair of functions $(\phi_1, \phi_2) \in W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ is said to be a supersolution of (1.1) if

$$\Big(a_1 + b_1 M_1\Big(\int_\Omega |\nabla \phi_1|^p \mathrm{d}x\Big)\Big) \int_\Omega |\nabla \phi_1|^{p-2} \nabla \phi_1 \cdot \nabla w \mathrm{d}x \ge \lambda \int_\Omega f(\phi_1, \phi_2) w \mathrm{d}x, \quad \forall w \in W,$$

and

$$\Big(a_2 + b_2 M_2\Big(\int_\Omega |\nabla \phi_2|^q \mathrm{d}x\Big)\Big) \int_\Omega |\nabla \phi_2|^{q-2} \nabla \phi_2 \cdot \nabla w \mathrm{d}x \ge \lambda \int_\Omega g(\phi_1, \phi_2) w \mathrm{d}x, \quad \forall w \in W.$$

The following result plays an important role in our arguments. Its detail proof is similar to the proof in [18, 19], so we will omit it.

Lemma 2.1 *Assume that $M : \mathbb{R}_0^+ \to \mathbb{R}^+$ is continuous and increasing function, and there exists $m_0 > -\frac{a}{b}$ ($b \ne 0$) such that $M(t) \ge m_0$ for all $t \in \mathbb{R}_0^+$. If the functions $u, v \in W_0^{1,r}(\Omega)$ satisfy*

$$\begin{aligned}&\Big(a + bM\Big(\int_\Omega |\nabla u|^r \mathrm{d}x\Big)\Big) \int_\Omega |\nabla u|^{r-2} \nabla u \cdot \nabla \varphi \mathrm{d}x \\ &\le \Big(a + bM\Big(\int_\Omega |\nabla v|^r \mathrm{d}x\Big)\Big) \int_\Omega |\nabla v|^{r-2} \nabla v \cdot \nabla \varphi \mathrm{d}x\end{aligned}$$

for all $\varphi \in W_0^{1,r}(\Omega)$, $\varphi \ge 0$, then $u \le v$ in Ω.

From Lemma 2.1, we can establish the following basic principle of the sub and supersolutions method for nonlocal systems. Indeed, we consider the following system

$$\begin{cases} -(a_1 + b_1 M_1(\int_\Omega |\nabla u|^p \mathrm{d}x)) \triangle_p u = h(x, u, v), & \text{in } \Omega, \\ -(a_2 + b_2 M_2(\int_\Omega |\nabla v|^q \mathrm{d}x)) \triangle_q v = k(x, u, v), & \text{in } \Omega, \\ u = v = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$ and $h, k : \overline{\Omega} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfy the following conditions:

(HK1) $h(x, s, t)$ and $k(x, s, t)$ are Caratheodory functions and they are bounded if s, t belong to bounded sets.

(HK2) There exists a function $g : \mathbb{R} \to \mathbb{R}$ being continuous, nondecreasing with $g(0) = 0$,

$0 \le g(s) \le C(1+|s|^{\min\{p,q\}-1})$ for some $C>0$, and applications $s \mapsto h(x,s,t)+g(s)$ and $t \mapsto k(x,s,t)+g(t)$ are nondecreasing for a.e. $x \in \Omega$.

If $u, v \in L^\infty(\Omega)$ with $u(x) \le v(x)$ for a.e. $x \in \Omega$, we denote by $[u,v]$ the set $\{w \in L^\infty(\Omega) : u(x) \le w(x) \le v(x)$ for a.e. $x \in \Omega\}$. Using Lemma 2.1, we can establish a version of the abstract lower and upper-solution method for our class of the operators as follows.

Proposition 2.2 *Let $M_1, M_2 : \mathbb{R}_0^+ \to \mathbb{R}^+$ be two functions satisfying the condition (H$_1$) and the functions h, k satisfy the conditions (HK1) and (HK2). Assume that $(\underline{u}, \underline{v})$, $(\overline{u}, \overline{v})$ are respectively, a weak subsolution and a weak supersolution of system (2.1) with $\underline{u}(x) \le \overline{u}(x)$ and $\underline{v}(x) \le \overline{v}(x)$ for a.e. $x \in \Omega$. Then there exists a minimal weak solution (u_*, v_*) (respectively, a maximal weak solution (u^*, v^*)) for system (2.1) in the set $[\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$. In particular, every weak solution $(u,v) \in [\underline{u}, \overline{u}] \times [\underline{v}, \overline{v}]$ of (2.1) satisfies $u_*(x) \le u(x) \le u^*(x)$ and $v_*(x) \le v(x) \le v^*(x)$ for a.e. $x \in \Omega$.*

3. Proof of main result

In this section, we prove Theorem 1.1 by using the sub and super-solutions method.

Proof of Theorem 1.1 First we construct a positive subsolution of problem (1.1). For this purpose, let $\lambda_{1,r}$, $\phi_{1,r}$ $(r = p, q)$, and δ, m, σ, Ω_δ be as described in Section 2. We now shall verify that $(\psi_1, \psi_2) = (l_1 \sigma^{\frac{p}{1-p}} \phi_{1,p}^{\frac{p}{p-1}},\ l_2 \sigma^{\frac{q}{1-q}} \phi_{1,q}^{\frac{q}{q-1}})$ in which $l_1, l_2 \in (\alpha, r]$, is a subsolution of problem (1.1). Let $w \in W$. Since

$$\nabla \psi_1 = \frac{p}{p-1} l_1 \sigma^{\frac{p}{1-p}} \phi_{1,p}^{\frac{1}{p-1}} \nabla \phi_{1,p},$$

we deduce by (H$_1$) that

$$\begin{aligned}
&\Big(a_1 + b_1 M_1\Big(\int_\Omega |\nabla \psi_1|^p \mathrm{d}x\Big)\Big) \int_\Omega |\nabla \psi_1|^{p-2} \nabla \psi_1 \cdot \nabla w \mathrm{d}x \\
&= (\frac{pl_1}{p-1})^{p-1} \sigma^{-p} \Big(a_1 + b_1 M_1\Big(\int_\Omega |\nabla \psi_1|^p \mathrm{d}x\Big)\Big) \int_\Omega \phi_{1,p} |\nabla \phi_{1,p}|^{p-2} \nabla \phi_{1,p} \cdot \nabla w \mathrm{d}x \\
&= (\frac{pl_1}{p-1})^{p-1} \sigma^{-p} \Big(a_1 + b_1 M_1\Big(\int_\Omega |\nabla \psi_1|^p \mathrm{d}x\Big)\Big) \int_\Omega |\nabla \phi_{1,p}|^{p-2} \nabla \phi_{1,p} \cdot [\nabla(\phi_{1,p} w) - w \nabla \phi_{1,p}] \mathrm{d}x \\
&= (\frac{pl_1}{p-1})^{p-1} \sigma^{-p} \Big(a_1 + b_1 M_1\Big(\int_\Omega |\nabla \psi_1|^p \mathrm{d}x\Big)\Big) \int_\Omega |\nabla \phi_{1,p}|^{p-2} \nabla \phi_{1,p} \cdot \nabla(\phi_{1,p} w) \mathrm{d}x - \\
&\quad (\frac{pl_1}{p-1})^{p-1} \sigma^{-p} \Big(a_1 + b_1 M_1\Big(\int_\Omega |\nabla \psi_1|^p \mathrm{d}x\Big)\Big) \int_\Omega |\nabla \phi_{1,p}|^p w \mathrm{d}x - \\
&= (\frac{pl_1}{p-1})^{p-1} \sigma^{-p} (a_1 + b_1 M_1(\int_\Omega |\nabla \psi_1|^p \mathrm{d}x)) \int_\Omega \lambda_{1,p} |\phi_{1,p}|^{p-2} \phi_{1,p} (\phi_{1,p} w) \mathrm{d}x \\
&\quad (\frac{pl_1}{p-1})^{p-1} \sigma^{-p} \Big(a_1 + b_1 M_1\Big(\int_\Omega |\nabla \psi_1|^p \mathrm{d}x\Big)\Big) \int_\Omega |\nabla \phi_{1,p}|^p w \mathrm{d}x \\
&= (\frac{pl_1}{p-1})^{p-1} \sigma^{-p} \Big(a_1 + b_1 M_1\Big(\int_\Omega |\nabla \psi_1|^p \mathrm{d}x\Big)\Big) \int_\Omega [\lambda_{1,p} \phi_{1,p}^p - |\nabla \phi_{1,p}|^p] w \mathrm{d}x \\
&\le (\frac{pl_1}{p-1})^{p-1} \sigma^{-p} (a_1 + b_1 m_1) \int_\Omega [\lambda_{1,p} \phi_{1,p}^p - |\nabla \phi_{1,p}|^p] w \mathrm{d}x.
\end{aligned}$$

Similarly, we also have

$$\Big(a_2 + b_2 M_2\Big(\int_\Omega |\nabla\psi_2|^q \mathrm{d}x\Big)\Big)\int_\Omega |\nabla\psi_2|^{q-2}\nabla\psi_2\cdot\nabla w \mathrm{d}x$$
$$\le (\frac{ql_2}{q-1})^{q-1}\sigma^{-q}(a_2+b_2m_2)\int_\Omega [\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q] w\mathrm{d}x.$$

Now, by (1.4), we have in $\overline{\Omega}_\delta$,

$$\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p \le -m, \quad \lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q \le -m.$$

It follows that in $\overline{\Omega}_\delta$,

$$(\frac{pl_1}{p-1})^{p-1}\sigma^{-p}(a_1+b_1m_1)[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p] \le -(\frac{pl_1}{p-1})^{p-1}\sigma^{-p} m(a_1+b_1m_1),$$

and

$$(\frac{ql_2}{q-1})^{q-1}\sigma^{-q}(a_2+b_2m_2)[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q] \le -(\frac{ql_2}{q-1})^{q-1}\sigma^{-q} m(a_2+b_2m_2).$$

By the monotone properties of f, g and $\psi_1, \psi_2 \ge 0$, if

$$\lambda \le \overline{\lambda}_1 := \frac{m(a_1+b_1m_1)}{\sigma^p|f(0,0)|}(\frac{pl_1}{p-1})^{p-1},$$

and

$$\lambda \le \overline{\lambda}_2 := \frac{m(a_2+b_2m_2)}{\sigma^q|g(0,0)|}(\frac{ql_2}{q-1})^{q-1},$$

we have

$$(\frac{pl_1}{p-1})^{p-1}\sigma^{-p}(a_1+b_1m_1)[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p] \le \lambda f(\psi_1,\psi_2),$$

and

$$(\frac{ql_2}{q-1})^{q-1}\sigma^{-q}(a_2+b_2m_2)[\lambda_{1,q}\phi_{1,q}^q - |\nabla\phi_{1,q}|^q] \le \lambda g(\psi_1,\psi_2).$$

Next, in $\Omega \setminus \overline{\Omega}_\delta$, we have $\phi_{1,p} \ge \sigma > 0$ and $\phi_{1,q} \ge \sigma > 0$. Thus

$$\psi_1 = l_1\sigma^{\frac{p}{1-p}}\phi_{1,p}^{\frac{p}{p-1}} \ge l_1, \quad \psi_2 = l_2\sigma^{\frac{q}{1-q}}\phi_{1,q}^{\frac{q}{q-1}} \ge l_2,$$

and

$$f(\psi_1,\psi_2) \ge f(l_1,l_2) \ge 0, \quad g(\psi_1,\psi_2) \ge g(l_1,l_2) \ge 0.$$

Therefore, if

$$\lambda \ge \lambda_{*1} := \frac{\lambda_{1,p}(a_1+b_1m_1)}{\sigma^p f(l_1,l_2)}(\frac{pl_1}{p-1})^{p-1},$$

and

$$\lambda \ge \lambda_{*2} := \frac{\lambda_{1,q}(a_2+b_2m_2)}{\sigma^q g(l_1,l_2)}(\frac{ql_2}{q-1})^{q-1},$$

we have

$$(\frac{pl_1}{p-1})^{p-1}\sigma^{-p}(a_1+b_1m_1)[\lambda_{1,p}\phi_{1,p}^p - |\nabla\phi_{1,p}|^p]$$
$$\le (\frac{pl_1}{p-1})^{p-1}\sigma^{-p}(a_1+b_1m_1)\lambda_{1,p} \le \lambda f(\psi_1,\psi_2),$$

and

$$\begin{aligned}&(\frac{ql_2}{q-1})^{q-1}\sigma^{-q}(a_2+b_2m_2)[\lambda_{1,q}\phi_{1,q}^q-|\nabla\phi_{1,q}|^q]\\&\le(\frac{ql_2}{q-1})^{q-1}\sigma^{-q}(a_2+b_2m_2)\lambda_{1,q}\le\lambda g(\psi_1,\psi_2).\end{aligned}$$

Moreover, let $\lambda_*=\min\{\lambda_{*1},\lambda_{*2}\}$ and $\overline{\lambda}=\max\{\overline{\lambda}_1,\overline{\lambda}_2\}$. By condition ($H_4$), we have $\lambda_*<\overline{\lambda}$. Hence, (ψ_1,ψ_2) is a subsolution of problem (1.1).

Next, we construct a supersolution of (1.1). Let

$$(\phi_1,\phi_2)=(\frac{l_1}{\|e\|_\infty}e,\frac{l_2}{\|e\|_\infty}e),$$

in which e is defined by (1.5). Since

$$\begin{aligned}&\Big(a_1+b_1M_1\Big(\int_\Omega|\nabla\phi_1|^p\mathrm{d}x\Big)\Big)\int_\Omega|\nabla\phi_1|^{p-2}\nabla\phi_1\cdot\nabla w\mathrm{d}x\\&=(\frac{l_1}{\|e\|_\infty})^{p-1}\Big(a_1+b_1M_1\Big(\int_\Omega|\nabla\phi_1|^p\mathrm{d}x\Big)\Big)\int_\Omega|\nabla e|^{p-2}\nabla e\cdot\nabla w\mathrm{d}x\\&\ge(\frac{l_1}{\|e\|_\infty})^{p-1}(a_1+b_1m_1)\int_\Omega w\mathrm{d}x,\quad\forall w\in W,\end{aligned}$$

and

$$\begin{aligned}&\Big(a_2+b_2M_2\Big(\int_\Omega|\nabla\phi_2|^q\mathrm{d}x\Big)\Big)\int_\Omega|\nabla\phi_2|^{q-2}\nabla\phi_2\cdot\nabla w\mathrm{d}x\\&\ge(\frac{l_2}{\|e\|_\infty})^{q-1}(a_2+b_2m_2)\int_\Omega w\mathrm{d}x,\quad\forall w\in W,\end{aligned}$$

if

$$\lambda\le\hat{\lambda}_1:=(\frac{l_1}{\|e\|_\infty})^{p-1}\frac{a_1+b_1m_1}{f(l_1,l_2)},\quad\lambda\le\hat{\lambda}_2:=(\frac{l_2}{\|e\|_\infty})^{q-1}\frac{a_2+b_2m_2}{g(l_1,l_2)},$$

we have

$$\Big(a_1+b_1M_1\Big(\int_\Omega|\nabla\phi_1|^p\mathrm{d}x\Big)\Big)\int_\Omega|\nabla\phi_1|^{p-2}\nabla\phi_1\cdot\nabla w\mathrm{d}x\ge\lambda\int_\Omega f(l_1,l_2)w\mathrm{d}x,$$

and

$$\Big(a_2+b_2M_2\Big(\int_\Omega|\nabla\phi_2|^q\mathrm{d}x\Big)\Big)\int_\Omega|\nabla\phi_2|^{q-2}\nabla\phi_2\cdot\nabla w\mathrm{d}x\ge\lambda\int_\Omega g(l_1,l_2)w\mathrm{d}x.$$

Moreover, by $\phi_1\le l_1$, $\phi_2\le l_2$, we obtain $f(\phi_1,\phi_2)\le f(l_1,l_2)$, and $g(\phi_1,\phi_2)\le g(l_1,l_2)$. So

$$\Big(a_1+b_1M_1\Big(\int_\Omega|\nabla\phi_1|^p\mathrm{d}x\Big)\Big)\int_\Omega|\nabla\phi_1|^{p-2}\nabla\phi_1\cdot\nabla w\mathrm{d}x\ge\lambda\int_\Omega f(\phi_1,\phi_2)w\mathrm{d}x,$$

and

$$\Big(a_2+b_2M_2\Big(\int_\Omega|\nabla\phi_2|^q\mathrm{d}x\Big)\Big)\int_\Omega|\nabla\phi_2|^{q-2}\nabla\phi_2\cdot\nabla w\mathrm{d}x\ge\lambda\int_\Omega g(\phi_1,\phi_2)w\mathrm{d}x.$$

Let $\hat{\lambda}=\max\{\hat{\lambda}_1,\hat{\lambda}_2\}$, by condition ($H_5$), we have $\lambda_*<\hat{\lambda}$. Therefore, (ϕ_1,ϕ_2) is a supersolution of problem (1.1).

In addition, we have

$$-\triangle_p\psi_1=(\frac{pl_1}{p-1})^{p-1}\sigma^{-p}(\lambda_{1,p}\phi_{1,p}^p-|\nabla\phi_{1,p}|^p)\le\lambda_{1,p}(\frac{pl_1}{p-1})^{p-1}\sigma^{-p},$$

and

$$-\triangle_p \phi_1 = -(\frac{l_1}{\|e\|_\infty})^{p-1}\triangle_p e = (\frac{l_1}{\|e\|_\infty})^{p-1},$$

by condition (H_5) and Lemma 2.1, we have $\psi_1 \le \phi_1$ in Ω. Similarly, we have $\psi_2 \le \phi_2$ in Ω. Set $\lambda^* := \min\{\overline{\lambda}, \hat{\lambda}\}$, by Proposition 2.1, we conclude that problem (1.1) has a positive solution for any $\lambda \in (\lambda_*, \lambda^*)$. □

Acknowledgements We thank the referees for their time and comments.

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