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Reliability of Parallel Stress-Strength Model

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Abstract In this paper, the reliability of a parallel stress-strength model of exponentiated Pareto distribution is discussed. Different point estimations and interval estimations are proposed. The point estimators obtained are maximum likelihood and Bayesian estimators. The interval estimations obtained are approximate, exact, bootstrap-p and bootstrap-t confidence intervals and Bayesian credible interval. Different methods and the corresponding confidence intervals are demonstrated using some simulation studies.

Keywords parallel stress-strength model; exponentiated Pareto distribution; maximum likelihood estimation; Bayesian estimation; interval estimation

MR(2010) Subject Classification 62F10

1. Introduction

Several researchers have considered the statistical inference of the reliability of the stress-strength model. In the early stage, the maximum likelihood estimator (MLE) of R when X and Y are normally distributed has been considered by [1–4]. When X and Y are Weibull random variables, the estimations of R was considered by McCool [5], Raqab et al. [6] considered the problem when X and Y are generalized exponential distributions. An encyclopedical treatment of the different stress-strength models can be found in the monograph of Kotz et al. [7].

This exponentiated Pareto distribution has been extensively used in the analysis of extreme events (Pickands [8] was apparently the first to use this distribution in this context), especially in hydrology [9], as well as in reliability studies when robustness is required against heavier tailed or lighter tailed alternatives to an exponential distribution. It is flexible enough to accommodate both monotonic as well as non-monotonic failure rates even though the baseline failure rate is monotonic. Modeling survival data by non-monotonic failure rates is desirable, for example, when the course of the disease is such that mortality reaches a peak after some finite period and then slowly declines [10]. Some recent applications of exponentiated Pareto distribution function include the estimation of the finite limit of human lifespan [11]. The estimators of the parameters of exponentiated Pareto distribution had been obtained [12] under different estimation procedures

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for complete sample case. The estimation of parameters had also been obtained [13] under type-I and type-II censoring scheme. Singh et al. [14] developed estimators for the parameters of exponentiated Pareto distribution under progressive type-II censoring with binomial removals. Chen and Cheng [15] discussed the reliability of a system when the strength and the stress imposed on the system are independent, non-identical exponentiated Pareto distributed random variables.

Until now, we only see the reliability $R = P(X < Y)$ of the stress-strength models with one component with the strength Y and one stress X. In this paper, we generalize the traditional system to a parallel model. The parallel stress-strength model occurs when a device under consideration is a combination of k usually independent components with the strengths $Y_1, Y_2, \ldots, Y_k$ and each component of the system is subject to a common shock of a random magnitude X. If the system is operating successfully whenever at least one of the $k \in N^+$ components survives, it is termed parallel in the analogy with electric circuits. For more details about the parallel stress-strength model one can refer to the monograph [7]. In this paper, different point estimators of R are derived, namely, MLE and Bayesian estimators with mean squared error loss functions. Based on the MLE, we can obtain the exact confidence interval of R. Also, we obtain the approximate confidence interval for R by using the approximate normal property of the MLE of R. We also recommend two bootstrap confidence intervals of R. In addition, based on the Bayesian estimator, we obtain the Bayesian credible interval of R. Different methods have been demonstrated by using Monte Carlo simulations.

The rest of the paper is organized as follows. In Section 2, the probability density function (pdf) and cumulative distribution function (cdf) of exponentiated Pareto distribution are presented and the explicit expression of R is derived. In Section 3, we discuss point estimations of R, including MLE and Bayesian estimation. In Section 4, different interval estimators of R are presented, including exact, approximate, bootstrap-p and bootstrap-t confidence intervals and Bayesian credible interval. Some numerical experiments and some discussions are presented in Section 5.

2. Exponentiated Pareto distribution and reliability

A random variable X is said to have exponentiated Pareto distribution, if its pdf is given by

$$f(x;\alpha,\lambda) = \alpha\lambda(1-(1+x)^{-\lambda})^{\alpha-1}(1+x)^{-(\lambda+1)}, \tag{2.1}$$

where $\lambda > 0$, $\alpha > 0$ and $x > 0$. Here α and λ are shape parameters. The cdf is given by

$$F(x;\alpha,\lambda) = (1-(1+x)^{-\lambda})^{\alpha}. \tag{2.2}$$

An exponentiated Pareto distribution will be denoted by $EP(\alpha, \lambda)$.

Let X be a common shock of a random magnitude of a system and a combination of k usually independent components with the strengths $Y_1, Y_2, \ldots, Y_k$ acting on the system. Assume that $X \sim EP(\beta, \lambda)$, $Y_i \sim EP(\alpha, \lambda)$, $i = 1, 2, \ldots, k$, and be independent. Therefore, the reliability of the parallel stress-strength models will be

$$
\begin{aligned}
R = P(X < \max(Y_1, Y_2, \ldots, Y_k)) &= 1 - \int_0^\infty (F_Y(x))^k \mathrm{d}F_X(x) \\
&= \int_0^\infty (1-(1+x)^{-\lambda})^{k\alpha} \beta\lambda(1-(1+x)^{-\lambda})^{\beta-1}(1+x)^{-(\lambda+1)}\mathrm{d}x \\
&= 1 - \frac{\beta}{k\alpha+\beta} = \frac{k\alpha}{k\alpha+\beta}. \qquad (2.3)
\end{aligned}
$$

If α and β are known, the R is simply calculated using (2.3). It can be seen that R does not depend on λ. However, α and β are unknown and λ is known, the estimators of α and β depend on λ, and hence so does the estimator of R.

3. Point estimations of R

In this section, suppose $X_1, X_2, \ldots, X_m$ to be a random sample from $EP(\beta, \lambda)$ and Y_{i1}, Y_{i2}, $\ldots$, Y_{in_i}, $i = 1, 2, \ldots, k$ to be a sample from $EP(\alpha, \lambda)$. We will discuss several point estimations for R given the samples.

3.1. Maximum likelihood estimation of R

To compute the MLE of R, we need to compute the MLE of α and β. As a matter of fact, in order to compute the MLE of α and β, we need to compute the MLE of λ also. The likelihood function for the observed sample is

$$L(\alpha, \beta, \lambda) = \prod_{i=1}^{m} f(x_i; \beta, \lambda) \prod_{i=1}^{k} \prod_{j=1}^{n_i} f(y_{ij}; \alpha\lambda).$$

The log likelihood function for the observed sample is

$$
\begin{aligned}
\ln L(\alpha, \beta, \lambda) =& m \ln\beta + n\ln\alpha + (m+n)\ln\lambda + \\
& (\beta-1)\sum_{i=1}^{m}\ln(1-(1+X_i)^{-\lambda}) + (\alpha-1)\sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln(1-(1+Y_{ij})^{-\lambda}) - \\
& (\lambda+1)(\sum_{i=1}^{m}\ln(1+X_i) + \sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln(1+Y_{ij})), \qquad (3.1)
\end{aligned}
$$

where $n = \sum_{i=1}^{k} n_i$.

To obtain the MLEs of α, β, and λ, we can maximize $\ln L(\alpha, \beta, \lambda)$ directly with respect to α, β, and λ. Differentiating (3.1) with respect to α, β, and λ, respectively and equating to zero, we obtain the following equations

$$\frac{\partial \ln L(\alpha, \beta, \lambda)}{\partial \beta} = \frac{m}{\beta} + \sum_{i=1}^{m}\ln(1-(1+X_i)^{-\lambda}) = 0, \qquad (3.2)$$

$$\frac{\partial \ln L(\alpha, \beta, \lambda)}{\partial \alpha} = \frac{n}{\alpha} + \sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln(1-(1+Y_{ij})^{-\lambda}) = 0, \qquad (3.3)$$

$$\begin{aligned}\frac{\partial \ln L(\alpha,\beta,\lambda)}{\partial\lambda} =&(\beta-1)\sum_{i=1}^{m}\frac{\ln(1+Y_i)(1+X_i)^{-\lambda}}{1-(1+X_i)^{-\lambda}}+(\alpha-1)\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\ln(1+Y_{ij})(1+Y_{ij})^{-\lambda}}{1-(1+Y_{ij})^{-\lambda}}+\\ &\frac{m+n}{\lambda}-\Big(\sum_{i=1}^{m}\ln(1+X_i)+\sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln(1+Y_{ij})\Big)=0. \end{aligned} \tag{3.4}$$

From (3.2) and (3.3), we obtain the MLE of α and β as function of λ, say $\hat{\alpha}(\lambda)$ and $\hat{\beta}(\lambda)$, as

$$\hat{\beta}(\lambda)=-\frac{m}{\sum_{i=1}^{m}\ln(1-(1+X_i)^{-\lambda})}, \tag{3.5}$$

$$\hat{\alpha}(\lambda)=-\frac{n}{\sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln(1-(1+Y_{ij})^{-\lambda})}. \tag{3.6}$$

If the scale parameter λ is known, the MLEs of α and β can be obtained from (3.5) and (3.6). If all the parameters are unknown, we can first estimate the scale parameter by maximizing the profile likelihood function $L(\hat{\alpha}(\lambda),\hat{\beta}(\lambda),\lambda)$, with respect to λ or by solving the following nonlinear equation

$$\begin{aligned}\frac{\partial L(\alpha,\beta,\lambda)}{\partial\lambda} =&(\hat{\beta}(\lambda)-1)\sum_{i=1}^{m}\frac{\ln(1+X_i)(1+X_i)^{-\lambda}}{1-(1+X_i)^{-\lambda}}+(\hat{\alpha}(\lambda)-1)\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\ln(1+Y_{ij})(1+Y_{ij})^{-\lambda}}{1-(1+Y_{ij})^{-\lambda}}+\\ &\frac{m+n}{\lambda}-\Big(\sum_{i=1}^{m}\ln(1+X_i)+\sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln(1+Y_{ij})\Big)=0. \end{aligned} \tag{3.7}$$

Consequently, $\hat{\lambda}$ can be obtained by solving the nonlinear equation

$$h(\lambda)=\lambda, \tag{3.8}$$

where

$$\begin{aligned}h(\lambda)=&(m+n)\Big[-(\hat{\beta}(\lambda)-1)\sum_{i=1}^{m}\frac{\ln(1+X_i)(1+X_i)^{-\lambda}}{1-(1+X_i)^{-\lambda}}-\\ &(\hat{\alpha}(\lambda)-1)\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{\ln(1+Y_{ij})(1+Y_{ij})^{-\lambda}}{1-(1+Y_{ij})^{-\lambda}}+\\ &\Big(\sum_{i=1}^{m}\ln(1+X_i)+\sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln(1+Y_{ij})\Big)\Big]^{-1}.\end{aligned}$$

Since $\hat{\lambda}$ is a fixed point solution of the nonlinear equation (3.7), it therefore can be obtained by using a simple iterative scheme as follows

$$h(\lambda_{(k)})=\lambda_{(k+1)}, \tag{3.9}$$

where $\lambda_{(k)}$ is the the kth iterate of λ. Once we obtain $\hat{\lambda}$, $\hat{\alpha}$ and $\hat{\beta}$ can be obtained from (3.5) and (3.6), respectively. Therefore, the MLE of R becomes

$$\hat{R}_{\text{MLE}}=\frac{k\hat{\alpha}}{k\hat{\alpha}+\hat{\beta}}. \tag{3.10}$$

3.2. Bayesian estimation of R

In this section, we obtain the Bayesian estimation of R under the assumption that the parameters α and β are random variables. It is assumed that α and β have independent gamma priors with the pdfs

$$\pi(\beta) = \frac{b_1^{a_1}}{\Gamma(a_1)}\beta^{a_1-1}e^{-b_1\beta}, \quad \pi(\alpha) = \frac{b_2^{a_2}}{\Gamma(a_2)}\alpha^{a_2-1}e^{-b_2\alpha}.$$

Then parameters $\beta \sim \Gamma(a_1, b_1)$ and $\alpha \sim \Gamma(a_2, b_2)$. Assuming that λ is known, then α and β have independent gamma posterior distributions as follows

$$\beta|\text{data} \sim \Gamma(a_1 + m, b_1 + V), \tag{3.11}$$

$$\alpha|\text{data} \sim \Gamma(a_2 + n, b_2 + W), \tag{3.12}$$

where $V = \sum_{i=1}^{m} \ln(1 - (1 + X_i)^{-\lambda})$ and $W = \sum_{i=1}^{k}\sum_{j=1}^{n_i} \ln(1 - (1 + Y_{ij})^{-\lambda})$.

Since a prior of α and β are independent, using (3.11) and (3.12), the pdf of (α, R) is

$$f(\alpha, r) = k^{A_1}\frac{B_1^{A_1}}{\Gamma(A_1)}\frac{B_2^{A_2}}{\Gamma(A_2)}(1-r)^{(A_1-1)}(r)^{-(1+A_1)}\alpha^{A_1+A_2-1}e^{-(B_2+k\frac{1-r}{r}B_1)\alpha},$$

where $A_1 = a_1 + m$, $A_2 = a_2 + n$, $B_1 = b_1 + V$, $B_2 = b_2 + W$.

Then, the posterior pdf of R is

$$f_R(r) = \int_0^\infty f(\alpha, r)\mathrm{d}\alpha = C(B_2 r + kB_1(1-r))^{-(A_1+A_2)}(1-r)^{A_1-1}r^{A_2-1}, \quad \text{for} \quad 0 < r < 1,$$

and 0 otherwise, where

$$C = k^{A_1}\frac{\Gamma(m + n + a_1 + a_2)}{\Gamma(a_1 + m)\Gamma(a_2 + n)}(b_1 + V)^{a_1+m}(b_2 + W)^{a_2+n}.$$

Now, consider the following loss function

$$L(a, b) = (a - b)^2.$$

It is well known that Bayes estimates with respect to the above loss function is the expectation of the posterior distribution [16]. The Bayes estimate of R under squared error loss cannot be computed analytically.

$$\hat{R}_{\text{BAYES}} = E[R|W, V] = \int_0^1 r f_R(r)\mathrm{d}r. \tag{3.13}$$

Notice that the Bayesian estimator in (3.13) depends on the parameters of the prior distributions of β and α. These parameters could be estimated by means of an empirical bayesian procedure [17]. Therefore, it is proposed to choose $(m + 1, V)$ and $(n + 1, W)$ as the values of (a_1, b_1) and (a_2, b_2), respectively.

4. Interval estimation of R

In this section, we first obtain the exact distribution of $\hat{R}_{\text{MLE}}$ when the common scale parameter is known and obtain the exact confidence interval of R. We also obtain the asymptotic distribution of $\hat{R}$. Based on the asymptotic distribution of $\hat{R}$, the asymptotic confidence interval of R is derived. It is clear that the confidence intervals based on the asymptotic results do not perform very well for small sample sizes. For comparison, another two confidence intervals based on bootstrap methods are proposed in this section. Finally, based on the Bayes estimation of R, we obtain the Bayesian credible interval of R.

4.1. Exact confidence interval

In this section, we also assume that λ is known. Based on Section 3, it is clear that the MLE of R is

$$\hat{R}_{\mathrm{MLE}} = \frac{nk\sum_{i=1}^{m}\ln(1-(1+X_i)^{-\lambda})}{m\sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln(1-(1+Y_{ij})^{-\lambda}) + nk\sum_{i=1}^{m}\ln(1-(1+X_i)^{-\lambda})}.$$

It is easy to see that $\ln(1-(1+X_j)^{-\lambda})$ follows an exponential distribution with parameter β for each $j = 1, 2, \dots, m$ and $\ln(1-(1+Y_{ij})^{-\lambda})$ follows an exponential distribution with parameter α for each $i = 1, 2, \dots, k$; $j = 1, 2, \dots, n_i$. Therefore, $2\beta\sum_{i=1}^{m}\ln(1-(1+X_i)^{-\lambda}) \sim \chi^2(2m)$ and $2\alpha\sum_{i=1}^{k}\sum_{j=1}^{n_i}\ln(1-(1+Y_{ij})^{-\lambda}) \sim \chi^2(2n)$. So,

$$\hat{R}_{\mathrm{MLE}} = \frac{1}{1+\frac{1}{k}\frac{\beta}{\alpha}F},$$

that is,

$$\frac{R}{1-R} \times \frac{1-\hat{R}_{\mathrm{MLE}}}{\hat{R}_{\mathrm{MLE}}} \sim F,$$

where the random variable F follows F distribution with $2n$ and $2m$ degrees of freedom. So, the pdf of $\hat{R}$ is as follows

$$f_{\hat{R}}(r) = C(\frac{\alpha}{\beta})^n \times (1-r)^{n-1} r^{-(1+n)}(1+k\frac{n\alpha}{m\beta}\frac{1-r}{r})^{-(n+m)},$$

where $0 < x < 1$, $C = \frac{\Gamma(m+n)}{\Gamma(m)\Gamma(n)}(k\frac{n}{m})^n$.

The $100(1-\gamma)\%$ confidence interval of R, say, $[L_1, U_1]$, can be obtained as

$$[\frac{1}{1+F_{\gamma/2;2m,2n} \times (1/\hat{R}_{\mathrm{MLE}}-1)}, \frac{1}{1+F_{(1-\gamma/2);2m,2n)} \times (1/\hat{R}_{\mathrm{MLE}}-1)}], \tag{4.1}$$

where $F_{(\gamma/2;2m,2n)}$ and $F_{(1-\gamma/2;2n,2m)}$ are the lower and upper $\gamma/2$th percentile points of the F distribution.

4.2. Approximate confidence interval

We denote the expected Fisher information matrix of $\theta = (\alpha, \beta, \lambda)$ as $J(\theta) = E(I(\theta))$, where $I(\theta) = [I_{ij}]_{i,j=1,2,3}$ is the observed information matrix, i.e.,

$$I(\theta) = -\begin{bmatrix} \frac{\partial^2 L}{\partial\alpha^2} & \frac{\partial^2 L}{\partial\alpha\partial\beta} & \frac{\partial^2 L}{\partial\alpha\partial\lambda} \\ \frac{\partial^2 L}{\partial\beta\partial\alpha} & \frac{\partial^2 L}{\partial\beta^2} & \frac{\partial^2 L}{\partial\beta\partial\lambda} \\ \frac{\partial^2 L}{\partial\lambda\partial\alpha} & \frac{\partial^2 L}{\partial\lambda\partial\beta} & \frac{\partial^2 L}{\partial\lambda^2} \end{bmatrix}.$$

It is easy to see that

$$I_{11} = \frac{n}{\alpha^2}, \quad I_{12} = I_{21} = 0,$$

$$I_{13}=I_{31}=-\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(1+y_{ij})^{-\lambda}\ln(1+y_{ij})}{1-(1+y_{ij})^{-\lambda}},\quad I_{22}=\frac{m}{\beta^2},$$

$$I_{23}=I_{32}=-\sum_{j=1}^{m}\frac{(1+x_j)^{-\lambda}\ln(1+x_j)}{1-(1+x_j)^{-\lambda}},$$

$$I_{33}=\frac{m+n}{\lambda^2}+(\alpha-1)\sum_{i=1}^{k}\sum_{j=1}^{n_i}\frac{(\ln(1+y_{ij}))^2(1+y_{ij})^{-\lambda}}{(1-(1+y_{ij})^{-\lambda})^2}+$$
$$(\beta-1)\sum_{j=1}^{n}\frac{(\ln(1+x_j))^2(1+x_j)^{-\lambda}}{(1-(1+x_j)^{-\lambda})^2}.$$

Through simple computation, we can obtain

$$J_{11}=\frac{n}{\alpha^2},\ J_{12}=J_{21}=0,\ J_{13}=J_{31}=-n\alpha\lambda A(\alpha,\lambda),$$

$$J_{22}=\frac{m}{\beta^2},\quad J_{23}=J_{32}=-m\beta\lambda A(\beta,\lambda),$$

$$J_{33}=\frac{m+n}{\lambda^2}+n\lambda\alpha(\alpha-1)B(\alpha,\lambda)+m\lambda\beta(\beta-1)B(\beta,\lambda),$$

where,

$$A(\alpha,\lambda)=\int_0^{\infty}(1+y)^{-2\lambda-1}\ln(1+y)(1-(1+y)^{-\lambda})^{\alpha-2}\mathrm{d}y,$$

$$B(\alpha,\lambda)=\int_0^{\infty}(1+y)^{-2\lambda-1}(\ln(1+y))^2(1-(1+y)^{-\lambda})^{\alpha-3}\mathrm{d}y.$$

Theorem 4.1 *As $n_i\to\infty$, $i=1,2,\ldots,k$; $m\to\infty$ and $\frac{n}{m}\to p$, then*

$$[\sqrt{n}(\hat\alpha-\alpha),\sqrt{m}(\hat\beta-\beta),\sqrt{m}(\hat\lambda-\lambda)]\to N(0,U^{-1}(\alpha,\beta,\lambda)),$$

where

$$U(\alpha,\beta,\lambda)=\begin{bmatrix}u_{11}&0&u_{13}\\0&u_{22}&u_{23}\\u_{31}&u_{32}&u_{33}\end{bmatrix}$$

and

$$u_{11}=\frac{1}{n}J_{11}=\frac{1}{\alpha^2},\quad u_{13}=u_{31}=\frac{1}{n}J_{13}=-\alpha\lambda A(\alpha,\lambda),$$

$$u_{22}=\frac{1}{m}J_{22}=\frac{1}{\beta^2},\quad u_{23}=u_{32}=\frac{\sqrt{p}}{n}J_{23}=-\frac{\beta\lambda}{\sqrt{p}}A(\beta,\lambda),$$

$$u_{33}=\frac{1}{n}J_{33}=\frac{p+1}{p\lambda^2}+\lambda\alpha(\alpha-1)B(\alpha,\lambda)+\frac{\lambda\beta(\beta-1)}{p}B(\beta,\lambda).$$

Proof The proof follows from the asymptotic normality of MLE. □

Theorem 4.2 *As $n_i\to\infty$, $i=1,2,\ldots,k$; $m\to\infty$ and $\frac{n}{m}\to p$, then*

$$\sqrt{m}(\hat R_{\mathrm{MLE}}-R)\to N(0,B^2),$$

where

$$B^2=\frac{1}{K(\alpha+\beta)^4}[\beta^2(u_{22}u_{33}-u_{23}^2)-2\alpha\beta\sqrt{p}u_{23}u_{31}+p\alpha^2(u_{11}u_{33}-u_{13}^2)],$$

$$K=|U|=u_{11}u_{22}u_{33}-u_{11}u_{23}u_{32}-u_{13}u_{22}u_{31}.$$

Proof Using the results of [18], we can complete the proof. □

Using Theorem 4.2, we can obtain asymptotic confidence interval of R, say $[L_2,U_2]$, is

$$[\hat{R}_{\rm MLE}-Z_{(1-\gamma/2)}\frac{\hat{B}}{\sqrt{m}},\hat{R}_{MLE}+Z_{(1-\gamma/2)}\frac{\hat{B}}{\sqrt{m}}], \tag{4.2}$$

where $Z_{(1-\gamma/2)}$ is the $(1-\gamma/2)$th quantile of the standard normal distribution.

Note that if we want to compute the approximate confidence interval of R, the variance B needs to be estimated. To estimate it, the empirical Fisher information matrix and the MLE of α, β, and λ are used, as follows

$$u_{11}=\frac{1}{\hat{\alpha}^2},\quad u_{13}=u_{31}=-\alpha\lambda A(\hat{\alpha},\hat{\lambda}),$$

$$u_{22}=\frac{1}{\hat{\beta}^2},\quad u_{23}=u_{32}=-\frac{\beta\lambda}{\sqrt{p}}A(\hat{\beta},\hat{\lambda}),$$

$$u_{33}=\frac{p+1}{p\hat{\lambda}^2}+\hat{\lambda}\hat{\alpha}(\hat{\alpha}-1)B(\hat{\alpha},\hat{\lambda})+\frac{\hat{\lambda}\hat{\beta}(\hat{\beta}-1)}{p}B(\hat{\beta},\hat{\lambda}).$$

4.3. Bayesian credible interval

In Section 3.2, we see that the poster distributions of α and β are gamma distribution with parameters $(2m+1,2W)$ and $(2n+1,2V)$, respectively. Thus $4\alpha W$ and $4\beta V$ are independent chi-squared random variables with $2(2n+1)$ and $2(2m+1)$ degrees of freedom, respectively. Therefore,

$$\frac{4\alpha W/2(2n+1)}{4\beta V/2(2m+1)}\sim F(4n+2,4m+2),$$

that is,

$$\frac{(2m+1)W}{(2n+1)V}\times\frac{\alpha}{\beta}\sim F(4n+2,4m+2).$$

From (2.3), we see that $\frac{\alpha}{\beta}=\frac{1}{k}\frac{R}{1-R}$. Thus

$$F=\frac{(2m+1)W}{(2n+1)V}\frac{1}{k}\frac{R}{1-R}, \tag{4.3}$$

is an F distributed random variable with $(4n+2,4m+2)$ degrees of freedom.

Using F in (4.3) as a pivotal quantity, we can obtain $(1-\gamma)\%$ Bayesian credible interval for R as $[L_5,U_5]$, where

$$L_5=\big[F_{\gamma/2}(4m+2,4n+2)\times\frac{1}{k}\frac{(2m+1)W}{(2n+1)V}+1\big]^{-1},$$

and

$$U_5=\big[F_{1-\gamma/2}(4m+2,4n+2)\times\frac{1}{k}\frac{(2m+1)W}{(2n+1)V}+1\big]^{-1}.$$

It is observed that the confidence intervals based on the asymptotic result do not perform well for small sample sizes. We propose two confidence intervals mainly for small sample size.

4.4. Bootstrap-p confidence interval

Percentile bootstrap method based on the idea of Efron [19] is proposed in this subsection. We call it as Boot-p method.

Step 1. From sample $\{y_{i1}, y_{i2}, \dots, y_{in_i}\}$, $i = 1, 2, \dots, k$ and $\{x_1, x_2, \dots, x_m\}$, compute $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$.

Step 2. Use $\hat{\alpha}$ and $\hat{\lambda}$ to generate a bootstrap sample $\{y^*_{i1}, y^*_{i2}, \dots, y^*_{in_i}\}$ and similarly use $\hat{\beta}$ and $\hat{\lambda}$ to generate a bootstrap sample $\{x^*_1, x^*_2, \dots, x^*_m\}$. Compute the bootstrap estimate of R using (3.10), say $\hat{R}^*$.

Step 3. Repeat Step 2 N times.

Step 4. Let $G(x) = P(\hat{R}^* < x)$ be the cumulative distribution of $\hat{R}^*$. Define $\hat{R}_{\text{Boot-}p} = G^{-1}(x)$ for a given x. The approximate $100(1-\gamma)\%$ confidence interval of R, $[L_3, U_3]$, is given by

$$\big[\hat{R}_{\text{Boot-}p}(\frac{\gamma}{2}), \hat{R}_{\text{Boot-}p}(1-\frac{\gamma}{2})\big].$$

4.5. Bootstrap-t confidence interval

Bootstrap t method based on the idea of Hall [20] is proposed in this subsection. We call it as Boot-t method.

Step 1. From sample $\{y_{i1}, y_{i2}, \dots, y_{in_i}\}$, $i = 1, 2, \dots, k$ and $\{x_1, x_2, \dots, x_m\}$ compute $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$.

Step 2. Use $\hat{\alpha}$ and $\hat{\lambda}$ to generate a bootstrap sample $\{y^*_{i1}, y^*_{i2}, \dots, y^*_{in_i}\}$ and similarly use $\hat{\beta}$ and $\hat{\lambda}$ to generate a bootstrap sample $\{x^*_1, x^*_2, \dots, x^*_m\}$. Compute the bootstrap estimate of R using (3.10), say $\hat{R}^*$ and following statistic

$$T^* = \frac{\sqrt{m}(\hat{R}^* - \hat{R})}{\sqrt{\text{Var}(\hat{R}^*)}},$$

where $\sqrt{\text{Var}(\hat{R}^*)}$ is obtained using the observed or expected Fisher information matrix.

Step 3. Repeat Step 2 N times.

Step 4. For the T^* values obtained in Step 2, determine the upper and lower bounds of the $100(1-\gamma)\%$ confidence interval of R as follows: let $H(x) = P(T^* \le x)$ be the cumulative distribution function of T^*. For a given x, define

$$\hat{R}_{\text{Boot-}t}(x) = \hat{R} + m^{-1/2}\sqrt{\text{Var}(\hat{R})}H^{-1}(x).$$

Here also, $\text{Var}(\hat{R})$ can be computed as same as the computation of $\text{Var}(\hat{R}^*)$. The approximate $100(1-\gamma)\%$ confidence interval of R, $[L_4, U_4]$, is given by

$$\big[\hat{R}_{\text{Boot-}t}(\frac{\gamma}{2}), \hat{R}_{\text{Boot-}t}(1-\frac{\gamma}{2})\big].$$

5. Simulations and discussions

In this section, we present some results based on Monte Carlo simulations to demonstrate the performance of the different point estimators and interval estimations for different sample sizes and different parameter values. We consider three numerical experiments separately to draw inference on R.

(i) The point estimations for R, that is, the MLE and Bayesian estimation for R.

(ii) The interval estimations for R, that is, the exact, the approximate and the Bayesian credible interval for R.

(iii) The Bootstrap-p and Bootstrap-t confidence intervals for R.

In all cases, we consider the following small sample size, $k = 2, m = 20, 30$, $(n_1, n_2) = (15, 10)$, $(20, 20)$, $(25, 15)$, and we take $\alpha = 2, 2.5, 3$ and $\beta = 2.5, 3, 3.5, 4, 5, 6$, respectively. All the results are listed based on 10000 replications.

Case (i) Since the parameter λ must be known in the Bayesian estimation for R, without loss of generality, we take $\lambda = 1$ in Case (i). From the sample, we estimate α and β using (3.5) and (3.6), respectively. Finally we obtain the MLE of R using (3.10). In addition, we obtain the Bayesian estimation using (3.13) and we report the average biases and mean squared errors (MSEs) in Table 1 over 10000 replications.

m	(n_1, n_2)	(α, β)	R	$\hat{R}_{\text{MLE}}$		$\hat{R}_{\text{BAYES}}$	
				bias	MSE	bias	MSE
20	(25, 15)	(2.0, 2.5)	0.6154	0.0525	0.0079	0.0525	0.0077
		(2.5 ,3.0)	0.6250	0.0733	0.0131	0.0729	0.0128
		(3.0, 6.0)	0.5000	-0.1753	0.0645	-0.1705	0.0605
	(20, 20)	(2.0, 2.5)	0.6154	0.0536	0.0074	0.0539	0.0073
		(2.5, 3.0)	0.6250	0.0737	0.0125	0.0734	0.0123
		(3.0, 6.0)	0.5000	-0.1772	0.0654	-0.1714	0.0607
	(15, 10)	(2.0, 2.5)	0.6154	0.0543	0.0094	0.0553	0.0092
		(2.5, 3.0)	0.6250	0.0746	0.0145	0.0751	0.0143
		(3.0, 6.0)	0.5000	-0.1760	0.0661	-0.1681	0.0597
	(10, 15)	(2.0, 2.5)	0.6154	0.0552	0.0082	0.0562	0.0082
		(2.5, 3.0)	0.6250	0.0757	0.0135	0.0766	0.0135
		(3.0, 6.0)	0.5000	-0.1762	0.0649	-0.1681	0.0584
30	(25, 15)	(2.0, 2.5)	0.6154	0.0551	0.0082	0.0556	0.0082
		(2.5, 3.0)	0.6250	0.0759	0.0136	0.0762	0.0135
		(3.0, 6.0)	0.5000	-0.1724	0.0620	-0.1680	0.0585
	(20, 20)	(2.0, 2.5)	0.6154	0.0553	0.0078	0.0560	0.0078
		(2.5, 3.0)	0.6250	0.0753	0.0129	0.0755	0.0128
		(3.0, 6.0)	0.5000	-0.1729	0.0620	-0.1680	0.0582
	(15,10)	(2.0, 2.5)	0.6154	0.0570	0.0095	0.0582	0.0095
		(2.5, 3.0)	0.6250	0.0763	0.0144	0.0772	0.0144
		(3.0, 6.0)	0.5000	-0.1703	0.0615	-0.1647	0.0571
	(10, 15)	(2.0, 2.5)	0.6154	0.0579	0.0089	0.0591	0.0090
		(2.5, 3.0)	0.6250	0.0781	0.0142	0.0788	0.0142
		(3.0, 6.0)	0.5000	-0.1739	0.0632	-0.1677	0.0583

Table 1 Biases and MSEs of parameter point estimations

Some of the points are quite clear from Table 1.

• Even for small sample sizes, the performance of the MLEs and Bayesian estimations are quite satisfactory in terms of biases and MSEs. For example, when (α, β)=(2.0, 2.5), $m = 20$ and $(n_1, n_2) = (20, 20)$, the biases and MSEs for above estimation of R are 0.0536, 0.0074 and 0.0539, 0.0073, respectively.

• It is observed that when n_1 or n_2 increase, the MSEs decrease. This verifies the consistency property of the MLE and Bayesian estimation of R. For example, when $m = 30$, $(n_1, n_2) = (10, 15)$ and $(\alpha, \beta) = (2.0, 2.5)$, the MSEs of $(\hat{R}_{\text{MLE}}, \hat{R}_{\text{BAYES}})$ are 0.0089, 0.0090. When (n_1, n_2) increase to $(20, 20)$, the corresponding above values decrease to 0.0078, and 0.0078, respectively.

• For fixed n, as m increases, the MSEs decrease. For fixed m, as n increases, the MSEs decrease as expected.

• The MSE for each point estimation is relevant to the value of R. In fact, it decreases as R increases. For example, when $m = 30$, $(n_1, n_2) = (10, 15)$, $R = 0.5000, 0.6154, 0.6250$, the corresponding MSEs of $\hat{R}_{\text{BAYES}}$ are $0.0583, 0.0090, 0.0142$. The same performances of MSEs of $\hat{R}_{\text{MLE}}$ are also seen.

Case (ii) Now let us consider the exact, approximate interval estimation and Bayesian credible interval for R. In this experiment, we also set $\lambda = 1$. We report the 95% confidence intervals of R and coverage probabilities for all sample sizes and parameter values.

(n_1, n_2)	(α, β)	Exact		Approximate		Bayes	
		C.I.(LEN.)	C.P.	C.I.(LEN.)	C.P.	C.I.(LEN.)	C.P.
(25, 15)	(2.0, 2.5)	(0.504, 0.715)0.210	0.974	(0.242, 0.847)0.605	0.957	(0.533, 0.682)0.148	0.897
	(2.5, 3.0)	(0.511, 0.724)0.208	0.975	(0.351, 0.752)0.401	0.999	(0.544, 0.691)0.146	0.894
	(3.0, 6.0)	(0.390, 0.612)0.221	0.974	(0.393, 0.598)0.205	0.967	(0.418, 0.574)0.155	0.897
(20, 20)	(2.0, 2.5)	(0.504, 0.715)0.210	0.987	(0.239, 0.854)0.615	0.995	(0.533, 0.682)0.148	0.948
	(2.5, 3.0)	(0.514, 0.723)0.209	0.989	(0.346, 0.756)0.409	0.999	(0.543, 0.690)0.147	0.950
	(3.0, 6.0)	(0.390, 0.612)0.222	0.989	(0.394, 0.598)0.204	0.987	(0.417, 0.574)0.156	0.951
(15, 10)	(2.0, 2.5)	(0.496, 0.725)0.229	0.968	(0.475, 0.755)0.279	0.990	(0.531, 0.692)0.160	0.874
	(2.5, 3.0)	(0.504, 0.731)0.227	0.966	(0.472, 0.768)0.296	0.994	(0.539, 0.699)0.159	0.878
	(3.0, 6.0)	(0.380, 0.621)0.241	0.968	(0.309, 0.621)0.312	0.986	(0.414, 0.583)0.169	0.877
(25, 15)	(2.0, 2.5)	(0.518 ,0.703)0.185	0.962	(0.494, 0.732)0.237	0.991	(0.545, 0.676)0.130	0.866
	(2.5, 3.0)	(0.527, 0.711)0.184	0.969	(0.488, 0.750)0.262	0.997	(0.555, 0.684)0.129	0.871
	(3.0, 6.0)	(0.402, 0.598)0.195	0.966	(0.355, 0.576)0.220	0.957	(0.429, 0.567)0.137	0.870
(20, 20)	(2.0, 2.5)	(0.517, 0.704)0.186	0.983	(0.494, 0.732)0.238	0.997	(0.545, 0.676)0.132	0.913
	(2.5, 3.0)	(0.527, 0.712)0.184	0.984	(0.486, 0.750)0.264	0.999	(0.555, 0.685)0.130	0.916
	(3.0, 6.0)	(0.402, 0.598)0.196	0.982	(0.356, 0.578)0.222	0.976	(0.429, 0.567)0.138	0.913
(15, 10)	(2.0, 2.5)	(0.504, 0.712)0.207	0.949	(0.515, 0.714)0.198	0.940	(0.539, 0.685)0.145	0.839
	(2.5, 3.0)	(0.513, 0.719)0.206	0.953	(0.523, 0.722)0.199	0.951	(0.548, 0.692)0.144	0.845
	(3.0, 6.0)	(0.390, 0.609)0.218	0.954	(0.410, 0.591)0.181	0.906	(0.424, 0.57)0.153	0.847

Talbe 2 Confidence intervals for exact, approximate methods and Bayesian credible interval

From Table 2, we can see that

• It is observed that the average length of exact confidence interval is shorter than that of approximate confidence interval but it is longer than that of Bayesian credible interval. For example, when $m = 20$, $(n_1, n_2) = (25, 15)$ and $(\alpha, \beta) = (2.0, 2.5)$, the lengths of three confidence intervals are 0.2104, 0.6058 and 0.1483, respectively. But the performance of the coverage probability is opposite of the three methods.

• The average lengths of all intervals decrease as m or n increases. For example, for exact method, when $(\alpha, \beta) = (2.0, 2.5)$, the lengths of the confidence intervals corresponding to following combinations of $(m, n_1, n_2) = (20, 25, 15)$, $(20, 20, 20)$, $(20, 15, 10)$, $(30, 25, 15)$, $(30, 20, 20)$, $(30, 15, 10)$, are 0.2104, 0.2109, 0.2229 and 0.0858, 0.1862 and 0.2027, respectively.

• It is observed that the exact interval (L_1, U_1) has the largest coverage probability which is the anticipated 95%. Bayesian credible interval (L_5, U_5) has the smallest average coverage probability and it is far from 95%.

• Among the different confidence intervals, Bayesian credible interval has the shortest confidence length.

Case (iii) Since the confidence intervals based on the exact and approximate methods for small sample sizes do not perform well, we present an analysis based on bootstrap method. Here, we assume that λ is unknown. From a sample, we compute the estimate of λ using the iterative algorithm (3.9). We have used the initial estimate to be 1 and the iterative process stops when the difference between the two consecutive iterates is less than 10^{-5}. Once we estimate λ, we estimate α and β using (3.5) and (3.6), respectively. Then we can obtain the Bootstrap-p and Bootstrap-t confidence intervals for R using the methods presented in Sections 4.4 and 4.5. We report the confidence intervals and average coverage probabilities of Bootstrap-p and Bootstrap-t methods in Table 3. The performance of the Bootstrap confidence intervals are quite well.

• The coverage probability of Bootstrap methods is not robust, but it is close to the nominal level in most case. In other side, the performance of the Bootstrap-t method is better than the Bootstrap-p method's.

• The performance of the bootstrap confidence intervals with respect to length is comparable with the length of exact and approximate confidence intervals and similarly, the lengths decrease as the sample sizes increase.

From the above simulation results and discussion, we can see that the performance of the maximum likelihood method is better than the Bayesian method of the point estimation of R in terms of both bias and MSE. It is observed that the exact method of constructing confidence intervals always maintains its coverage percentage at the nominal level but there is a difference about their performances with respect to the average length of the confidence intervals of different combination sets of parameters. Generally, the performance of Bayesian method is the best of all the methods in terms of interval length. The Boot-t confidence intervals and Boot-p confidence intervals both work well in terms of length of confidence intervals in most cases but the performance is not well about the coverage percentage. The Boot-t method seems to work better than the Boot-p method in terms of coverage percentage. It is also clear that the exact method of constructing the confidence intervals is more robust than the other methods mentioned. We recommend to use the exact method for constructing the confidence intervals.

Finally, it should be mentioned that the exact method confidence intervals and Bayesian credible interval are computationally expensive especially for the integral equations. Hence, the bootstrap method can be used as an alterative for this case.

(n_1, n_2)	(α, β)	R	Bootstrap-p			Bootstrap-t		
			C.I.	Length	C.P.	C.I.	Length	C.P.
(25, 15)	(2.0, 2.5)	0.615	(0.155, 0.654)	0.498	0.882	(0.478, 0.686)	0.208	0.956
	(2.5, 3.0)	0.625	(0.441, 0.716)	0.275	0.984	(0.499, 0.700)	0.200	0.952
	(3.0, 6.0)	0.500	(0.301, 0.552)	0.251	0.942	(0.357, 0.567)	0.210	0.950
(20, 20)	(2.0, 2.5)	0.615	(0.494, 0.732)	0.238	0.979	(0.502, 0.662)	0.159	0.950
	(2.5, 3.0)	0.625	(0.498, 0.664)	0.166	0.902	(0.513, 0.679)	0.166	0.950
	(3.0, 6.0)	0.500	(0.346, 0.550)	0.204	0.942	(0.368, 0.556)	0.187	0.950
(15, 10)	(2.0, 2.5)	0.615	(0.475, 0.730)	0.255	0.966	(0.476, 0.704)	0.227	0.950
	(2.5, 3.0)	0.625	(0.484, 0.737)	0.253	0.962	(0.494, 0.709)	0.215	0.950
	(3.0, 6.0)	0.500	(0.407, 0.571)	0.163	0.769	(0.393, 0.630)	0.237	0.960
(25, 15)	(2.0, 2.5)	0.615	(0.512, 0.696)	0.184	0.941	(0.591, 0.692)	0.173	0.952
	(2.5, 3.0)	0.625	(0.531, 0.713)	0.181	0.958	(0.493, 0.661)	0.168	0.950
	(3.0, 6.0)	0.500	(0.305, 0.585)	0.280	0.977	(0.386, 0.573)	0.187	0.950
(20, 20)	(2.0, 2.5)	0.615	(0.5260, 0.687)	0.161	0.955	(0.521, 0.682)	0.159	0.950
	(2.5, 3.0)	0.625	(0.576, 0.722)	0.145	0.813	(0.530, 0.687)	0.157	0.953
	(3.0, 6.0)	0.500	(0.362, 0.550)	0.187	0.920	(0.388, 0.567)	0.179	0.950
(15, 10)	(2.0, 2.5)	0.615	(0.483, 0.717)	0.233	0.958	(0.503, 0.703)	0.199	0.951
	(2.5, 3.0)	0.625	(0.512, 0.722)	0.209	0.967	(0.515, 0.738)	0.223	0.950
	(3.0, 6.0)	0.500	(0.367, 0.600)	0.232	0.942	(0.368, 0.600)	0.231	0.950

Table 3 Bootstrap-p and Bootstrap-t confidence intervals

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