Several Identities for Inverse-Conjugate Compositions

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Abstract
In this paper, we first present several identities related to the inverse-conjugate compositions having parts of size \( \leq 3 \), the compositions into parts equal to 1 or 2, the compositions into odd parts and the compositions into parts greater than 1. In addition, we provide a bijective proof of a relation for inverse-conjugate compositions having parts of size \( \leq k \).

Keywords
inverse-conjugate compositions; identity; Fibonacci number; Tribonacci number; bijective proof

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1. Introduction

A composition of a positive integer \( n \) is a representation of \( n \) as a sequence of positive integers called parts which sum to \( n \). For example, the compositions of 4 are: (4), (3, 1), (1, 3), (2, 2), (2, 1, 1), (1, 2, 1), (1, 1, 2), (1, 1, 1, 1). It is known that there are \( 2^{n-1} \) unrestricted compositions of \( n \). MacMahon [1] devised a graphical representation of a composition, called a zig-zag graph, which resembles the partition Ferrers graph except that the first dot of each part is aligned with the last part of its predecessor. For example, the zig-zag graph of the composition (6, 3, 1, 2, 2) is shown in Figure 1.

\[
\bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

Figure 1 zig-zag graph

The conjugate of a composition is obtained by reading its graph by columns from left to right. Figure 1 gives the conjugate of the composition (6, 3, 1, 2, 2) as (1, 1, 1, 1, 2, 1, 3, 2, 1).

Let \( C \) denote a composition of \( n \). A \( k \)-composition is a composition with \( k \) parts, i.e., \( C = (c_1, c_2, \ldots, c_k) \). The conjugate of \( C \) is denoted by \( C' \) and the inverse of \( C \) is the reversal composition \( \overline{C} = (c_k, c_{k-1}, \ldots, c_1) \). \( C \) is called inverse-conjugate if \( C' = \overline{C} \). For example, (2, 1, 3, 1) is an inverse-conjugate composition of 7.

In 1975, Hoggatt-Bicknell [2] studied ordinary compositions with parts \( \leq k \), and obtained the following result [3, p.72]
**Theorem 1.1** ([2]) Let $C_k(N)$ be the number of compositions of a positive integer $N$ using only the parts $1, 2, \ldots, k$. Then

$$C_k(N) = F_{N+1}^{(k)},$$

where $F_r^{(n)}$ is the $n$-step Fibonacci number.

The $n$-step Fibonacci numbers $F_r^{(n)}$ (see [3]) extend the ordinary Fibonacci numbers.

**Definition 1.2** ([3]) The $n$-step Fibonacci numbers are defined for any positive integer $n$ by

$$F_r^{(n)} = \sum_{i=1}^{n} F_{r-i}^{(n)}, \quad r > 2,$$

with $F_r^{(n)} = 0$ for $r \leq 0$, $F_1^{(n)} = 1$, $F_2^{(n)} = 1$.

Note that the case $n = 1$ gives the sequence of ones, $F_r^{(1)} : 1, 1, 1, \ldots$ while the case $n = 2$ gives the Fibonacci numbers, that is $(F_r^{(2)} = F_r)$: $F_1 = F_2 = 1$, $F_r = F_{r-1} + F_{r-2}$, $r > 2$.

Inverse-conjugate compositions have been studied by some researchers [1, 4–6]. It is known that these compositions are defined for only odd weights, and that there are $2^{n-1}$ inverse-conjugate compositions of $2n - 1$.

Recently Guo-Munagi [7] considered inverse-conjugate compositions with parts of size not exceeding a fixed integer $k > 0$, and obtained their enumeration properties as well as connections with other types of compositions, as summarised in the following three theorems:

**Theorem 1.3** ([7]) Let $IC_k(N)$ denote the number of inverse-conjugate compositions of $N$ into parts of size $\leq k$. Then

$$IC_k(2n - 1) = \sum_{j=1}^{k-1} IC_k(2(n - j) - 1), \quad n > k$$

with $IC_k(2t - 1) = 2^{t-1}$, $t = 1, 2, \ldots, k$.

**Theorem 1.4** ([7]) Let $IC_k(N)$ denote the number of inverse-conjugate compositions of $N$ into parts of size $\leq k$. Then

$$IC_k(2n - 1) = 2 F_r^{(k-1)}, \quad n \geq k - 1,$$

where $F_r^{(n)}$ is the $n$-generalized Fibonacci number.

**Theorem 1.5** ([7]) Let $C_k(n)$ be the number of compositions of a positive integer $n$ using only the parts $1, 2, \ldots, k$. Then

$$IC_{k+1}(2n - 1) = 2C_k(n - 1), \quad n > 1.$$
We first cite the following terminologies and lemmas from [4] that will be used in the proofs later.

Let \( A = (a_1, a_2, \ldots, a_i) \) and \( B = (b_1, b_2, \ldots, b_j) \) be compositions. The concatenation of the parts of \( A \) and \( B \) is defined as \( A|B = (a_1, a_2, \ldots, a_i, b_1, b_2, \ldots, b_j) \). In particular for a nonnegative integer \( c \), we have \( A|(c) = (A; c) \) and \( (c)|A = (c, A) \). The join of \( A \) and \( B \) with notation \( A \uplus B := (a_1, a_2, \ldots, a_i + b_1, b_2, \ldots, b_j) \).

Lemma 2.1 ([4]) An inverse-conjugate composition \( C \) (or its inverse) has the form:

\[
C = (1^{b_r-1}, b_1, 1^{b_r-1} - 2, b_2, 1^{b_r-2}, b_3, \ldots, b_{r-1}, 1^{b_1-2}, b_r), b_i \geq 2.
\]

Lemma 2.2 ([4]) If \( C = (c_1, \ldots, c_k) \) is an inverse-conjugate composition of \( n = 2k - 1 > 1 \), or its inverse, then there is an index \( j \) such that \( c_1 + \cdots + c_j = k - 1 \) and \( c_{j+1} + \cdots + c_k = k \) with \( c_{j+1} > 1 \).

Moreover,

\[
(c_1, \ldots, c_j) = (c_{j+1} - 1, c_{j+2}, \ldots, c_k)'.
\]  

Thus \( C \) can be written in the form

\[
C = A|(1) \uplus B \text{ such that } B' = \overline{A},
\]

where \( A \) and \( B \) are generally different compositions of \( k - 1 \).

To begin with we present the following results for the inverse-conjugate compositions of odd numbers into parts not exceeding 3 according to Theorems 1.3–1.5.

**Theorem 2.3** Let \( IC_3(N) \) denote the number of inverse-conjugate compositions of \( N \) into parts of size \( \leq 3 \). Then

\[
IC_3(2n + 1) = IC_3(2n - 1) + IC_3(2n - 3), \quad n > 2,
\]

with \( IC_3(1) = 1, \ IC_3(3) = 2, \ IC_3(5) = 4 \).

**Theorem 2.4** Let \( IC_3(N) \) denote the number of inverse-conjugate compositions of \( N \) into parts of size \( \leq 3 \). Then

\[
IC_3(2n - 1) = 2F_n, \quad n \geq 2,
\]

where \( F_n \) is the Fibonacci numbers.

**Theorem 2.5** Let \( C_2(n) \) be the number of compositions of a positive integer \( n \) using only the parts 1, 2. Then

\[
IC_3(2n + 1) = 2C_2(n), \quad n \geq 1.
\]

From Theorems 2.4 and 2.5, we also observed that the number of compositions of \( n \) into parts of size 1 or 2 is \( F_{n+1} \). And Theorem 2.5 presents an identity between the number of inverse-conjugate compositions having parts of size \( \leq 3 \) and the number of compositions into parts equal to 1 or 2. In this paper, we provide a bijective proof of Theorem 2.5.

Because an inverse-conjugate composition is always paired with its inverse, we give bijective proofs for only inverse-conjugate compositions having parts \( \leq 3 \) in which the first part is 1. The
Proof For an inverse-conjugate composition $C = (c_1, c_2, \ldots, c_n)$ of $2n + 1$ having parts of size $\leq 3$, and the first part is 1. From Lemma 2.2 we know that there is an index $j$ such that $c_1 + c_2 + \cdots + c_j = n$ and $c_{j+1} + \cdots + c_n = n + 1$ with $c_{j+1} > 1$, or $c_1 + c_2 + \cdots + c_j = n + 1$ and $c_{j+1} + \cdots + c_n = n$ with $c_j > 1$. And from Lemma 2.1, we know that the number of 1’s to the right of the part 3 is at most 1 in an inverse-conjugate composition. So we consider the following two cases.

Case 1 When $c_1 + c_2 + \cdots + c_j = n$, using $C$ we first obtain a composition $B = (c_1, c_2, \ldots, c_j)$. Next, for each part of size 3 in composition $B$, we do the following operation: if 3 is followed by 1, we replace 3 by “2, 1”; otherwise replace 3 by “1, 1, 1”. In this way, we obtain a composition of $n$ with parts of size $\leq 2$ and the first part is 1. But this case does not include compositions with three parts being “1, 1, 1” on the left end.

Case 2 When $c_1 + c_2 + \cdots + c_j = n + 1$ with $c_j > 1$, we first get a composition $A = (1, c_{j+1}, \ldots, c_n)'$, where the last part of $A$ is 1 because of $c_n > 1$, and the first part of $A$ is $> 1$. Using $A$ we obtain a composition $D$ of $n$ by deleting the last part 1 of $A$. Similarly, for composition $D$, we replace 3 by “2, 1” if the part on the right side of it is 1, otherwise replace 3 by “1, 1, 1”. In this way, we get a composition of $n$ having parts of size $\leq 2$ and the first part is $> 1$. And this case includes compositions having three parts are “1, 1, 1” on the left end.

Conversely, for a composition $K$ into parts of size $\leq 2$ of $n$, we consider the following three cases.

Case a When there are at most two 1’s on the left end of $K$, we first do the following operation: replace “2, 1” with 3 if there are parts “2, 1, 1”, or replace “1, 1, 1” with 3 if there are parts “1, 1, \ldots, 1” from right to left in composition $K$. So we get a composition $M$. Next, we get a composition $R = M|\{(1)|M\}'$. Thereupon the composition $R$ is an inverse-conjugate composition of $2n + 1$ with parts $\leq 3$, and the first part is 1. Here the composition $R$ satisfies $c_1 + c_2 + \cdots + c_j = n$ and $c_{j+1} + \cdots + c_n = n + 1$ with $c_{j+1} > 1$.

For example, the composition $(1, 1, 2, 2, 1, 1, 1, 2)$ of 11 into 1’s and 2’s produces the inverse-conjugate composition $(1, 1, 2, 2, 3, 2, 2, 1, 2, 2, 3)$ of 23 as follows:

$$(1, 1, 2, 2, 1, 1, 1, 2) \rightarrow (1, 1, 2, 2, 3, 2) \rightarrow (1, 1, 2, 2, 3, 2, 2, 1, 2, 2, 3).$$

Case b When there are three parts “1, 1, 1” on the left end of $K$, we first replace “1, 1, 1” by 3, and then replace “2, 1” with 3 if there are parts “2, 1, 1”, or replace “1, 1, 1” with 3 if there are parts “1, 1, \ldots, 1” from right to left in $K$. In this way, we have a composition $N$. Next, we obtain a composition $H = N|\{(1)\}$. Using $H$ we get a composition $F$ by replacing the first part $\lambda$ of $H$ with $\lambda - 1$. Finally, we obtain a composition $G = F \uplus H'$. Hence the composition $G$ is an inverse-conjugate composition of $2n + 1$ with parts $\leq 3$, and the first part is 1. Here the
composition $G$ satisfies $c_1 + c_2 + \cdots + c_j = n + 1$ and $c_{j+1} + \cdots + c_n = n$ with $c_j > 1$.

**Case c** When the first part of $K$ is 2, we obtain a composition $P$ using the same steps in Case b except that the first part 2 remains the same. Hence the composition $P$ is an inverse-conjugate composition of $2n + 1$ with parts $\leq 3$, and the first part is 1. Here the composition $P$ satisfies $c_1 + c_2 + \cdots + c_j = n + 1$ and $c_{j+1} + \cdots + c_n = n$ with $c_j > 1$.

Thus we complete the proof. $\square$

We cite an example to illustrate Theorem 2.5.

**Example 2.6** Let $n = 5$. The corresponding relations between the inverse-conjugate compositions of 11 into parts of size $\leq 3$ and the compositions of 5 into 1’s and 2’s are as follows.

(1, 1, 3, 2, 1, 3) $\leftrightarrow$ (1, 1, 1, 1, 1) $\leftrightarrow$ (3, 1, 2, 3, 1, 1),

(1, 2, 1, 2, 3, 2) $\leftrightarrow$ (2, 1, 2) $\leftrightarrow$ (2, 3, 2, 1, 2, 1),

(1, 3, 2, 2, 1, 2) $\leftrightarrow$ (2, 1, 1, 2) $\leftrightarrow$ (2, 1, 2, 3, 1, 1),

(1, 2, 2, 2, 2, 2) $\leftrightarrow$ (1, 2, 2) $\leftrightarrow$ (2, 2, 2, 2, 1, 1),

(1, 1, 2, 1, 3, 3) $\leftrightarrow$ (1, 1, 2, 1) $\leftrightarrow$ (3, 3, 1, 2, 1, 1),

(1, 3, 1, 3, 1, 2) $\leftrightarrow$ (1, 2, 1, 1) $\leftrightarrow$ (2, 1, 3, 1, 3, 1),

(1, 2, 3, 1, 2, 2) $\leftrightarrow$ (1, 1, 1, 2) $\leftrightarrow$ (2, 2, 1, 3, 2, 1),

(1, 1, 2, 2, 2, 3) $\leftrightarrow$ (2, 2, 1) $\leftrightarrow$ (3, 2, 2, 2, 1, 1).

It is known that the number of compositions of $n$ into odd parts is $F_n$, and the number of compositions of $n$ into parts greater than 1 is $F_{n-1}$. And then combined with Theorem 2.4, the following identities are also obtained.

**Theorem 2.7** Let $C_{\text{odd}}(n)$ be the number of compositions of a positive integer $n$ into odd parts. Then

$$IC_3(2n − 1) = 2C_{\text{odd}}(n), \quad n > 1. \quad (2.6)$$

**Proof** For an inverse-conjugate composition $C$ of $2n − 1$ with parts $\leq 3$, and the first part is 1, using proof of Theorem 2.5 we obtain a composition $B$ of $n − 1$ with parts of size 1, 2. Next append 1 to the left end of $B$ to obtain a composition $H$, then adjoin 1 and all adjacent 2’s on the right of it to produce new parts from left to right in $H$. Hence we obtain a composition of $n$ into odd parts.

Clearly this correspondence is one-to-one, and vice versa. We complete the proof. $\square$

**Theorem 2.8** Let $C_{>1}(n)$ be the number of compositions of $n$ into parts greater than 1. Then

$$IC_3(2n − 1) = 2C_{>1}(n + 1), \quad n > 1. \quad (2.7)$$

**Proof** For an inverse-conjugate composition $C$ of $2n − 1$ with parts $\leq 3$, and the first part is 1, firstly, we obtain a composition $B$ of $n − 1$ with parts of size 1, 2 using proof of Theorem 2.5. Next a composition $D$ is obtained by appending 1 in both the first end and the last end of $B$. Finally, we derive the conjugate $D'$ of $D$. Since the parts of $D$ are 1’s or 2’s and both ends are 1’s, so the $D'$ is a composition of $n + 1$ with the parts greater than 1.
For example, the inverse-conjugate composition \( (1, 2, 1, 2, 3, 2) \) and its inverse composition 
\( (2, 3, 2, 1, 2, 1) \) of 11 into parts \( \leq 3 \) produce the composition \( (2, 3, 2) \) of 7 with parts greater than 1 as follows:

\[
(1, 2, 1, 2, 3, 2) \rightarrow (1, 3, 2) \rightarrow (2, 1, 2, 1) \rightarrow (2, 1, 2) \rightarrow (1, 2, 1, 2, 1) \rightarrow (2, 3, 2); \\
(2, 3, 2, 1, 2, 1) \rightarrow (1, 2, 1, 2, 3, 2) \rightarrow (1, 3, 2) \rightarrow (2, 1, 2, 1) \rightarrow (2, 1, 2) \\
\rightarrow (1, 2, 1, 2, 1) \rightarrow (2, 3, 2).
\]

Obviously, this correspondence is one-to-one, and vice versa. We complete the proof. □

We cite an example to illustrate Theorem 2.8.

**Example 2.9** Let \( n = 6 \). The corresponding relations between the inverse-conjugate compositions of 11 into parts of size \( \leq 3 \) and the compositions of 7 into parts greater than 1 are as follows.

\[
(1, 1, 3, 2, 1, 3) \leftrightarrow (7) \leftrightarrow (3, 1, 2, 3, 1, 1), \\
(1, 2, 1, 2, 3, 2) \leftrightarrow (2, 3, 2) \leftrightarrow (2, 3, 2, 1, 2, 1), \\
(1, 3, 2, 2, 1, 2) \leftrightarrow (2, 5) \leftrightarrow (2, 1, 2, 3, 1), \\
(1, 2, 2, 2, 2, 2) \leftrightarrow (3, 2, 2) \leftrightarrow (2, 2, 2, 2, 2, 1), \\
(1, 1, 2, 1, 3, 3) \leftrightarrow (4, 3) \leftrightarrow (3, 3, 1, 2, 1, 1), \\
(1, 3, 1, 3, 1, 2) \leftrightarrow (3, 4) \leftrightarrow (2, 1, 3, 1, 3, 1), \\
(1, 2, 3, 1, 2, 2) \leftrightarrow (5, 2) \leftrightarrow (2, 2, 1, 3, 2, 1), \\
(1, 1, 2, 2, 2, 3) \leftrightarrow (2, 2, 3) \leftrightarrow (3, 2, 2, 2, 1, 1).
\]

3. A bijective proof of Theorem 1.5

Theorem 1.5 is the generalization of Theorem 2.5, so it has an important theoretical meaning to provide a bijective proof of Theorem 1.5. Although the proof is similar to that of Theorem 2.5, we still give a bijective proof of Theorem 1.5 in this section.

**Proof** For an inverse-conjugate composition \( C = (c_1, c_2, \ldots, c_n) \) of \( 2n - 1 \) with parts of size \( \leq k \), and the first part is 1. From Lemma 2.2 we know that there is an index \( j \) such that \( c_1 + c_2 + \cdots + c_j = n - 1 \) and \( c_{j+1} + \cdots + c_n = n \) with \( c_{j+1} > 1 \), or \( c_1 + c_2 + \cdots + c_j = n \) and \( c_{j+1} + \cdots + c_n = n - 1 \) with \( c_{j+1} > 1 \). Using Lemma 2.1 we know that the number of 1’s on the right of \( k \) is at most \( k - 2 \) in an inverse-conjugate composition. Thus we consider the following two cases.

**Case 1** When \( c_1 + c_2 + \cdots + c_j = n - 1 \), we first obtain a composition \( B = (c_1, c_2, \ldots, c_j) \). Next, for \( B \) we do the following transform: If there are no 1’s on the right of the part \( k \), we replace \( k \) by \( \underbrace{1, 1, \ldots, 1}_k \). If \( k \) is followed by \( d \) 1’s, we replace \( k \) with \( \underbrace{d + 1, 1, 1, \ldots, 1}_{k+d-1} \), where \( 1 \leq d \leq k - 2 \). In this way, we obtain a composition of \( n - 1 \) into parts of size \( \leq k - 1 \) and the first part is 1. But this case does not include the compositions with \( k \) parts being \( \underbrace{1, 1, \ldots, 1}_k \) on the left end.
Case 2 When \( c_1 + c_2 + \cdots + c_j = n \) with \( c_j > 1 \), we first obtain a composition \( A = (1, c_{j+1}, \ldots, c_n) \), where the last part of \( A \) is 1 because of \( c_n > 1 \), and the first part of \( A \) is > 1. Next, a composition \( D \) of \( n - 1 \) is got by deleting the last part 1 of \( A \). Similarly, we replace \( k \) by \( "d + 1, 1, 1, \ldots, 1" \) when there are \( d \)'s on the right of the part \( k \), where \( 0 \leq d \leq k - 2 \). In this way, we obtain a composition of \( n - 1 \) with parts of size \( \leq k - 1 \) and the first part is > 1. And this case includes compositions with \( k \) parts being \( "1, 1, \ldots, 1" \) on the left end.

Conversely, for a composition \( S \) with parts of size \( \leq k - 1 \) of \( n - 1 \), we consider the following three cases.

Case a When the first part of \( S \) is 1 and there are not \( k \) parts "1,1,..,1" on the left end, we do the following operation: replace "1,1,1,..,1" with \( k \) if there are parts "1,1,1,..,1", where \( 2 \leq l \leq k - 1 \), or replace "1,1,1,..,1" with \( k \) if there are \( t \) parts 1,1,..,1 from right to left in composition \( S \). So we obtain a composition \( T \). Next, we have a composition \( R = T|((1)|T)' \). Thereupon the composition \( R \) is an inverse-conjugate composition of \( 2n - 1 \) with parts \( \leq k \), and the first part is 1. Here the composition \( R \) satisfies \( c_1 + c_2 + \cdots + c_j = n - 1 \) and \( c_{j+1} + \cdots + c_n = n \) with \( c_{j+1} > 1 \).

Case b When there are \( k \) parts "1,1,..,1" on the left end of \( S \), we first replace "1,1,..,1" by \( k \), and then replace "1,1,1,..,1" with \( k \) if there are parts "1,1,1,..,1", where \( 2 \leq l \leq k - 1 \), or replacing "1,1,1,..,1" with \( k \) if there are \( t \) parts "1,1,1,..,1" from right to left in \( S \). In this way, we have a composition \( U \). Next, we have a composition \( H = U'|\{1\} \), and then we obtain a composition \( F \) by replacing the first part \( \lambda \) of \( H \) with \( \lambda - 1 \). Finally, we obtain a composition \( G = \bar{F} \cup H' \). Hence the composition \( G \) is an inverse-conjugate composition of \( 2n - 1 \) with parts \( \leq k \), and the first part is 1. Here the composition \( G \) satisfies satisfies \( c_1 + c_2 + \cdots + c_j = n \) and \( c_{j+1} + \cdots + c_n = n - 1 \) with \( c_j > 1 \).

Case c When the first part of \( S \) is \( h \), where, 1 < \( h < k \), we obtain a composition \( P \) using the same steps in Case 2 except that the first part \( h \) remains the same. Hence the composition \( P \) is an inverse-conjugate composition of \( 2n - 1 \) with parts \( \leq k \), and the first part is 1. Here the composition \( P \) satisfies \( c_1 + c_2 + \cdots + c_j = n \) and \( c_{j+1} + \cdots + c_n = n - 1 \) with \( c_j > 1 \).

We complete the proof. \( \square \)

In particular, we give the following interesting relations for the inverse-conjugate compositions into parts of size \( \leq 4 \).

**Corollary 3.1** Let \( IC_4(N) \) denote the number of inverse-conjugate compositions of \( N \) into
parts of size $\leq 4$. Then

$$IC_4(2n + 1) = IC_4(2n - 1) + IC_4(2n - 3) + IC_4(2n - 5), \quad n > 3, \quad (3.1)$$

with $IC_4(1) = 1, IC_4(3) = 2 IC_4(5) = 4, IC_4(7) = 8$.

**Corollary 3.2** Let $IC_4(N)$ denote the number of inverse-conjugate compositions of $N$ into parts of size $\leq 4$. Then

$$IC_4(2n + 1) = 2C_3(n), \quad n \geq 1. \quad (3.2)$$

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**References**