Unique Results for a New Fourth-Order Boundary Value Problem

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Abstract In this paper, we investigate the existence and uniqueness of solutions for a new fourth-order differential equation boundary value problem:

$$\begin{cases} u^{(4)}(t) = f(t, u(t)) - b, \ 0 < t < 1, \\ u(0) = u'(0) = u'(1) = u^{(3)}(1) = 0, \end{cases}$$

where $f \in C([0,1] \times (-\infty, +\infty), (-\infty, +\infty))$, $b \geq 0$ is a constant. The novelty of this paper is that the boundary value problem is a new type and the method is a new fixed point theorem of φ-(h, e)-concave operators.

Keywords   fourth-order boundary value problem; unique solution; φ-(h, e)-concave operator; existence and uniqueness

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1. Introduction

In [1], Yang gave some priori estimates for the positive solutions of the following fourth-order differential equation

$$u^{(4)}(t) = g(t)f(t, u(t)), \ \ 0 < t < 1,$$

under two-point boundary conditions

$$u(0) = u'(0) = u'(1) = u^{(3)}(1) = 0,$$

and obtained the existence and nonexistence of positive solutions for the boundary value problem, where $f \in C([0,1] \times [0, +\infty), [0, +\infty))$, $g \in C([0,1], [0, +\infty))$ with $\int_0^1 g(t)\mathrm{d}t > 0$. The method is the Krasnoselskii fixed point theorem.

Based upon [1], Zhang [2] established the existence and iteration of monotone positive solutions for the following fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = q(t)f(t, u(t), u'(t)), \ 0 < t < 1, \\ u(0) = u'(0) = u'(1) = u^{(3)}(1) = 0, \end{cases}$$

where $f \in C([0,1] \times [0, +\infty) \times [0, +\infty), [0, +\infty))$, $q \in C((0,1), [0, +\infty))$. The method is monotone iterative technique. That is, the author gave the existence of monotone positive solutions for the

problem by making an iterative scheme whose starting point is simple quadratic function or the zero function.

Recently, there are many papers studying the existence or multiplicity of positive solutions for fourth-order boundary value problems, see [1–20] for example. Also, the uniqueness of positive solutions for the fourth-order boundary value problems has been studied by some researchers [8, 9, 15–20]. In [9], by using two fixed point theorems for mixed monotone operators with perturbation, we gave some existence and uniqueness results for monotone positive solutions to the following elastic beam equation with boundary conditions

$$\begin{cases} u^{(4)}(t) = f(t, u(t), u'(t)), \ 0 < t < 1, \\ u(0) = u'(0) = 0, \\ u''(1) = 0, \ u^{(3)}(1) = g(u(1)), \end{cases}$$

where $f \in C([0,1] \times \mathbf{R} \times \mathbf{R})$ and $g \in C(\mathbf{R})$ are real functions. Recently, we also studied the approximations of the monotone positive solutions for the following fourth-order boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)), \ 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0, \\ u^{(3)}(1) + g(u(1)) = 0, \end{cases}$$

where $f \in C([0,1] \times [0, +\infty), [0, +\infty))$, $g \in C([0, +\infty), [0, +\infty))$. The methods used are two fixed point theorems of a sum operator in partial ordering Banach space [20].

Different from these papers mentioned above, we will consider a new fourth-order differential equation boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)) - b, \ 0 < t < 1, \\ u(0) = u'(0) = u'(1) = u^{(3)}(1) = 0, \end{cases} \tag{1.1}$$

where $f \in C([0,1] \times (-\infty, +\infty), (-\infty, +\infty))$, $b > 0$ is a constant. To my knowledge, there are no papers considering problem (1.1). Generally, to get the unique results of solutions, the Banach fixed point theorem is needed. Here the method is different. We will use a novel fixed point theorem to obtain the unique results for problem (1.1). We will present the existence and uniqueness of solutions for problem (1.1). Our analysis relies on a new fixed point theorem of φ-(h, e)-concave operators. It should be pointed out that our results are interesting and our methods are novel.

2. Auxiliary results

In order to establish our main results, we list some lemmas and concepts.

Lemma 2.1 ([1]) *If f is continuous, then problem (1.1) has an integral formulation given by*

$$u(t) = \int_0^1 G(t, s)[f(s, u(s)) - b]\mathrm{d}s, \tag{2.1}$$

where

$$G(t,s) = \frac{1}{12} \begin{cases} s^2(6t - 3t^2 - 2s), \ 0 \leq s \leq t \leq 1, \\ t^2(6s - 3s^2 - 2t), \ 0 \leq t \leq s \leq 1. \end{cases} \tag{2.2}$$

Lemma 2.2 ([3]) *The function $G(t,s)$ satisfies the following inequality*

$$\frac{1}{12}t^2 s^2 \leq G(t,s) \leq \frac{1}{2}t^2 s, \quad \text{for any } t, s \in [0,1].$$

Next we introduce some basic facts and fixed point theorems in abstract spaces [16,18,21–23], which play a crucial role in next proofs of main results.

Definition 2.3 *Let $(E, \|\cdot\|)$ be a real Banach space, θ be the zero element of E. A set $P \subset E$ is a cone. E is partially ordered by P, i.e., $x \leq y$ if and only if $y - x \in P$. P is normal if there is $N > 0$ such that, if $x, y \in E, \theta \leq x \leq y$, then $\|x\| \leq N\|y\|$. The infimum of such constants N is called the normality constant of P.*

For $x, y \in E$, the notation $x \sim y$ denotes that there exist $\lambda > 0$ and $\mu > 0$ such that $\lambda x \leq y \leq \mu x$. It is clear that $\sim$ is an equivalence relation. For fixed $h > \theta$ (i.e., $h \geq \theta$ and $h \neq \theta$), define $P_h = \{x \in E | x \sim h\}$. It is easy to see that $P_h \subset P$.

Definition 2.4 *An operator $A : E \to E$ is increasing if $x \leq y$ implies $Ax \leq Ay$.*

Definition 2.5 *Let $e \in P$ with $\theta \leq e \leq h$. Define a new set*

$$P_{h,e} = \{x \in E | x + e \in P_h\}.$$

Then we can see that $h \in P_{h,e}$ and

$$P_{h,e} = \{x \in E | \text{ there exist } \mu = \mu(h,e,x) > 0, \nu = \nu(h,e,x) > 0 \text{ such that } \mu h \leq x + e \leq \nu h\}.$$

Let $A : P_{h,e} \to E$ be a given operator. For $x \in P_{h,e}$ and $\lambda \in (0,1)$, there exists $\varphi(\lambda) > \lambda$ such that

$$A(\lambda x + (\lambda - 1)e) \geq \varphi(\lambda)Ax + (\varphi(\lambda) - 1)e.$$

Then A is called a φ-(h,e)-concave operator.

Theorem 2.6 ([22]) *Suppose that P is normal and A is an increasing φ-(h,e)-concave operator with $Ah \in P_{h,e}$. Then A has a unique fixed point x^* in $P_{h,e}$. Moreover, for any given points $w_0 \in P_{h,e}$, putting the sequence $w_n = Aw_{n-1}$, $n = 1, 2, \ldots$, we get $\|w_n - x^*\| \to 0$ as $n \to \infty$.*

If $e = \theta$, then we have

Theorem 2.7 ([23]) *Suppose that A is an increasing φ-(h,θ)-concave operator and P is normal, $Ah \in P_h$. Then A has a unique fixed point x^* in P_h. Further, for any point $v_0 \in P_h$, constructing the sequence $v_n = Av_{n-1}$, $n = 1, 2, \ldots$, we have $\|v_n - x^*\| \to 0$ as $n \to \infty$.*

3. Main results

In this section, we discuss the existence and uniqueness of solutions for problem (1.1). The tools are Theorems 2.6 and 2.7 which are relatively new, and so the method is new to the

fourth-order boundary value problems.

Problem (1.1) is considered in the function space $C[0,1] = \{x : [0,1] \to R$ is continuous$\}$, which is a Banach space and the norm is $\|x\| = \sup\{|x(t)| : t \in [0,1]\}$. Set $P = \{x \in C[0,1] | x(t) \geq 0, t \in [0,1]\}$, a normal cone in $C[0,1]$. It is well known that this space can be equipped with a partial order given by $x, y \in C[0,1]$, $x \leq y \Leftrightarrow x(t) \leq y(t)$, $t \in [0,1]$. Let

$$e(t) = \frac{1}{24}bt^2(t-2)^2, \quad t \in [0,1].$$

Theorem 3.1 *Suppose that*

(H_1) $f : [0,1] \times [-\frac{b}{24}, +\infty) \to (-\infty, +\infty)$ *is increasing with respect to the second variable;*

(H_2) *For any $\lambda \in (0,1)$, there is $\varphi(\lambda) > \lambda$ such that*

$$f(t, \lambda x + (\lambda - 1)y) \geq \varphi(\lambda)f(t, x), \ \forall \ t \in [0,1], x \in (-\infty, +\infty), y \in [0, \frac{b}{24}];$$

(H_3) $f(t,0) \geq 0$ *with $f(t,0) \not\equiv 0$ for $t \in [0,1]$.*

Then problem (1.1) has a unique solution u^* in $P_{h,e}$, where $h(t) = Lt^2, t \in [0,1]$ with $L \geq \frac{1}{6}b$. Moreover, for any given $w_0 \in P_{h,e}$, making a sequence

$$w_n(t) = \int_0^1 G(t,s)f(s, w_{n-1}(s))ds - \frac{1}{24}bt^2(t-2)^2, \quad n = 1, 2, \ldots,$$

we have $w_n(t) \to u^*(t)$ as $n \to \infty$.

Proof Firstly, for $t \in [0,1]$,

$$e(t) = \frac{1}{24}bt^2(t-2)^2 \geq 0 \text{ and } e(t) \leq \frac{b}{24}. \tag{3.1}$$

That is, $e \in P$. Further, for $t \in [0,1]$,

$$e(t) = \frac{1}{24}bt^2(t-2)^2 \leq \frac{1}{6}bt^2 \leq Lt^2 = h(t).$$

Hence, $0 \leq e(t) \leq h(t)$. In addition, $P_{h,e} = \{u \in C[0,1] | u + e \in P_h\}$. From Lemma 2.1, problem (1.1) has an integral formulation given by

$$u(t) = \int_0^1 G(t,s)f(s, u(s))ds - b\int_0^1 G(t,s)ds$$

$$= \int_0^1 G(t,s)f(s, u(s))ds - \frac{b}{24}t^2(t^2 + 4 - 4t)$$

$$= \int_0^1 G(t,s)f(s, u(s))ds - e(t).$$

For any $u \in P_{h,e}$, we consider the following operator of the form

$$Au(t) = \int_0^1 G(t,s)f(s, u(s))ds - e(t), \quad t \in [0,1].$$

So $u(t)$ is the solution of problem (1.1) if and only if $u(t) = Au(t)$. Next we divide several steps to show that A satisfies the conditions of Theorem 2.6.

Step 1. We prove that $A : P_{h,e} \to E$ is a φ-(h,e)-concave operator. For $u \in P_{h,e}, \lambda \in (0,1)$,

from (3.1) and (H_2) we have

$$A(\lambda u + (\lambda - 1)e)(t) = \int_0^1 G(t,s)f(s,\lambda u(s) + (\lambda - 1)e(s))\mathrm{d}s - e(t)$$

$$\geq \varphi(\lambda)\int_0^1 G(t,s)f(s,u(s))\mathrm{d}s - e(t)$$

$$= \varphi(\lambda)[\int_0^1 G(t,s)f(s,u(s))\mathrm{d}s - e(t)] + [\varphi(\lambda) - 1]e(t)$$

$$= \varphi(\lambda)Au(t) + [\varphi(\lambda) - 1]e(t).$$

So we get

$$A(\lambda u + (\lambda - 1)e) \geq \varphi(\lambda)Au + [\varphi(\lambda) - 1]e, \quad u \in P_{h,e}, \lambda \in (0,1).$$

This implies that A is φ-(h,e)-concave operator.

Step 2. We show that $A : P_{h,e} \to E$ is an increasing operator. For $u \in P_{h,e}$, we know $u + e \in P_h$. This indicates that there exists $\mu > 0$ such that $u(t) + e(t) \geq \mu h(t), t \in [0,1]$. And thus

$$u(t) \geq \mu h(t) - e(t) \geq -e(t) \geq -\frac{b}{24}.$$

Note that from condition (H_1), we easily know that $A : P_{h,e} \to E$ is increasing.

Step 3. We prove that $Ah \in P_{h,e}$. So we need to prove $Ah + e \in P_h$. Considering Lemma 2.2 with (H_1), (H_3),

$$Ah(t) + e(t) = \int_0^1 G(t,s)f(s,h(s))\mathrm{d}s = \int_0^1 G(t,s)f(s,Ls^2)\mathrm{d}s$$

$$\leq \int_0^1 \frac{1}{2}st^2 f(s,L)\mathrm{d}s \leq \frac{1}{2}\int_0^1 sf(s,L)\mathrm{d}s \cdot t^2$$

$$= \frac{1}{2L}\int_0^1 sf(s,L)\mathrm{d}s \cdot h(t)$$

$$Ah(t) + e(t) = \int_0^1 G(t,s)f(s,Ls^2)\mathrm{d}s$$

$$\geq \int_0^1 \frac{1}{12}s^2t^2 f(s,0)\mathrm{d}s = \frac{1}{12}\int_0^1 s^2 f(s,0)\mathrm{d}s \cdot t^2$$

$$= \frac{1}{12L}\int_0^1 s^2 f(s,0)\mathrm{d}s \cdot h(t).$$

Let

$$l_1 = \frac{1}{12L}\int_0^1 s^2 f(s,0)\mathrm{d}s, \quad l_2 = \frac{1}{2L}\int_0^1 sf(s,L)\mathrm{d}s.$$

Because $L > 0$ and from (H_1), (H_3),

$$\int_0^1 sf(s,L)\mathrm{d}s \geq \int_0^1 s^2 f(s,0)\mathrm{d}s > 0$$

and thus $l_2 \geq l_1 > 0$. So this shows that $Ah + e \in P_h$.

Consequently, by using Theorem 2.6, the operator A has a unique fixed point u^* in $P_{h,e}$ and thus

$$u^*(t) = \int_0^1 G(t,s)f(s,u^*(s))\mathrm{d}s - e(t), \quad t \in [0,1].$$

Evidently, $u^*(t)$ is a solution. Moreover, for any $w_0 \in P_{h,e}$, the sequence $w_n = Aw_{n-1}$, $n = 1, 2, \ldots$ satisfies $w_n \to u^*$ as $n \to \infty$. That is,

$$w_n(t) = \int_0^1 G(t,s)f(s,w_{n-1}(s))\mathrm{d}s - \frac{1}{24}bt^2(t-2)^2, \quad n = 1, 2, \ldots,$$

and $w_n(t) \to u^*(t)$ as $n \to \infty$. $\square$

Corollary 3.2 *Let all the conditions of Theorem 3.1 be satisfied. Suppose that there exists $t_0 \in [0,1]$ such that*

$$\int_0^1 G(t_0,s)[f(s,0) - b]\mathrm{d}s \not\equiv 0.$$

Then problem (1.1) has a unique nontrivial solution u^* in $P_{h,e}$, where $h(t) = Lt^2, t \in [0,1]$ with $L \geq \frac{1}{6}b$. Moreover, for any given $w_0 \in P_{h,e}$, making a sequence

$$w_n(t) = \int_0^1 G(t,s)f(s,w_{n-1}(s))\mathrm{d}s - \frac{1}{24}bt^2(t-2)^2, \quad n = 1, 2, \ldots,$$

we have $w_n(t) \to u^*(t)$ as $n \to \infty$.

Proof From the proof of Theorem 3.1, we get the unique solution u^* of (1.1) by

$$u^*(t) = \int_0^1 G(t,s)f(s,u^*(s))\mathrm{d}s - e(t) = \int_0^1 G(t,s)[f(s,u^*(s)) - b]\mathrm{d}s, \quad t \in [0,1].$$

If $u^*(t) \equiv 0$, then $\int_0^1 G(t,s)[f(s,0-b]\mathrm{d}s \equiv 0, t \in [0,1]$. This contradicts the condition. So u^* is the nontrivial solution. $\square$

When $b = 0$, we can obtain the uniqueness and existence of positive solutions for problem (1.1) by using Theorem 2.7.

Theorem 3.3 *Suppose that*
(H_4) *$f: [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous with $f(t,0) \not\equiv 0$;*
(H_5) *For any $t \in [0,1]$, $f(t,x)$ is increasing with respect to the second variable;*
(H_6) *For any $\lambda \in (0,1)$, there exists $\varphi(\lambda) > \lambda$ such that*

$$f(t,\lambda x) \geq \varphi(\lambda)f(t,x), \quad \forall \, t \in [0,1], \; x \in [0,+\infty).$$

Then problem (1.1) has a unique positive solution u^* in P_h, where $h(t) = t^2, t \in [0,1]$. Moreover, for any given $w_0 \in P_h$, making a sequence

$$w_n(t) = \int_0^1 G(t,s)f(s,w_{n-1}(s))\mathrm{d}s, \quad n = 1, 2, \ldots,$$

we have $w_n(t) \to u^*(t)$ as $n \to \infty$.

Remark 3.4 If $b > 0$, we cannot obtain Theorem 3.1 and Corollary 3.2 by using previous

methods used in $[2, 3, 8, 9, 11, 12, 17, 20]$. Comparing Theorem 3.3 with the main results in $[1, 2]$, we present some alternative approaches to study the similar type of problems under different conditions.

4. Examples

To illustrate our main results, we give two examples.

Example 4.1 Consider the following fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = t^{\frac{2}{5}}(u(t) + \frac{1}{24})^{\frac{1}{5}}(t-2)^{\frac{2}{5}} - 1, \ 0 < t < 1, \\ u(0) = u'(0) = u'(1) = u^{(3)}(1) = 0. \end{cases} \quad (4.1)$$

Evidently, problem (4.1) fits the framework of problem (1.1). In this example, let $b = 1$ and

$$f(t, x) = t^{\frac{2}{5}}(x + \frac{1}{24})^{\frac{1}{5}}(t-2)^{\frac{2}{5}}.$$

Take $e(t) = \frac{1}{24}t^2(t-2)^2$, $h(t) = \frac{1}{6}t^2$, $t \in [0, 1]$. Then $e(t) \leq \frac{1}{24}$, $e(t) \leq h(t)$, $t \in [0, 1]$. In addition,

$$f(t, x) = [t^2(x + \frac{1}{24})(t-2)^2]^{\frac{1}{5}} = [24e(t)x + e(t)]^{\frac{1}{5}}.$$

Obviously, $f : [0, 1] \times [-\frac{1}{24}, +\infty) \to (-\infty, +\infty)$ is continuous, increasing with respect to the second variable. Moreover, for any $\lambda \in (0, 1), x \in (-\infty, +\infty), y \in [0, \frac{1}{24}]$, we have

$$\begin{aligned} f(t, \lambda x + (\lambda - 1)y) &= \{24e(t)[\lambda x + (\lambda - 1)y] + e(t)\}^{\frac{1}{5}} \\ &= \lambda^{\frac{1}{5}}\{24e(t)[x + (1 - \frac{1}{\lambda})y] + \frac{1}{\lambda}e(t)\}^{\frac{1}{5}} \\ &= \lambda^{\frac{1}{5}}[24e(t)x + (1 - \frac{1}{\lambda})24e(t)y + \frac{1}{\lambda}e(t)]^{\frac{1}{5}} \\ &\geq \lambda^{\frac{1}{5}}[24e(t)x + (1 - \frac{1}{\lambda})e(t) + \frac{1}{\lambda}e(t)]^{\frac{1}{5}} \\ &= \lambda^{\frac{1}{5}}[24e(t)x + e(t)]^{\frac{1}{5}}. \end{aligned}$$

Let $\varphi(\lambda) = \lambda^{\frac{1}{5}}$. Then $\varphi(\lambda) > \lambda$, $\lambda \in (0, 1)$ and

$$f(t, \lambda x + (\lambda - 1)y) \geq \varphi(\lambda)f(t, x), \quad \forall t \in [0, 1], x \in (-\infty, +\infty), y \in [0, \frac{1}{24}].$$

Further, $f(t, 0) = [e(t)]^{\frac{1}{5}} \geq 0$ with $f(t, 0) \not\equiv 0$ for $t \in [0, 1]$. Hence, all the conditions of Theorem 3.1 are satisfied. Therefore, problem (4.1) has a unique solution u^* in P_h, where $h(t) = \frac{1}{6}t^2$, $t \in [0, 1]$. And, for any initial value $u_0 \in P_h$, making the sequence

$$u_{n+1}(t) = \int_0^1 G(t, s)s^{\frac{2}{5}}(u_n(s) + \frac{1}{24})^{\frac{1}{5}}(s-2)^{\frac{2}{5}}\mathrm{d}s - \frac{1}{24}t^2(t-2)^2, \quad n = 0, 1, 2, \dots,$$

one has $u_n(t) \to u^*(t)$ as $n \to \infty$. $\square$

Example 4.2 Consider the following fourth-order boundary value problem:

$$\begin{cases} u^{(4)}(t) = [u(t)]^\alpha + \sin t, \ 0 < t < 1, \\ u(0) = u'(0) = u'(1) = u^{(3)}(1) = 0, \end{cases} \quad (4.2)$$

where $\alpha \in (0,1)$. Evidently, problem (4.2) fits the framework of problem (1.1) with $b = 0$. In this example, let $f(t,x) = x^\alpha + \sin t$. Obviously, $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous. And it is easy to see that $f(t,x)$ is increasing in $x \in [0,+\infty)$ for fixed $t \in [0,1]$. Moreover, $f(t,0) = \sin t \not\equiv 0$. Set $\varphi(\tau) = \tau^\alpha$, $\tau \in (0,1)$. Then $\varphi(\tau) \in (\tau,1)$ and

$$f(t,\tau x) = \tau^\alpha x^\alpha + \sin t \geq \tau^\alpha(x^\alpha + \sin t) = \varphi(\tau)f(t,x),$$

for $t \in [0,1]$, $x \geq 0$. Hence, all the conditions of Theorem 3.3 are satisfied, then problem (4.2) has a unique positive solution u^* in P_h, where $h(t) = t^2$, $t \in [0,1]$. And, for any initial values $u_0 \in P_h$, making the sequence

$$u_{n+1}(t) = \int_0^1 G(t,s)[u_n^\alpha(s) + \sin s]\mathrm{d}s, \quad n = 0,1,2,\ldots,$$

we have $u_n(t) \to u^*(t)$ as $n \to \infty$, where $G(t,s)$ is given as (2.2).

References

[1] Bo YANG. *Positive solutions for a fourth order boundary value problem*. Electron. J. Qual. Theory Differ. Equ., 2005, **3**: 1–17.

[2] Xiaoping ZHANG. *Existence and iteration of monotone positive solutions for an elastic beam with a corner*. Nonlinear Anal. Real World Appl., 2009, **10**(4): 2097–2103.

[3] E. ALVRS, T. MA, M. PELICER. *Monotone positive solutions for a fourth order equation with nonlinear boundary conditions*. Nonlinear Anal., 2009, **71**(9): 3834–3841.

[4] Zhanbing BAI. *The upper and lower solution method for some fourth-order boundary value problems*. Nonlinear Anal., 2007, **67**(6): 1704–1709.

[5] A. CABADA, S. TERSIAN. *Multiplicity of solutions of a two point boundary value problem for a fourth-order equation*. Appl. Math. Comput., 2013, **219**(10): 5261–5267.

[6] J. GRAEF, Lingju KONG, Qingkai KONG, et al. *Positive solutions to a fourth order boundary value problem*. Results Math., 2011, **59**(1-2): 141–155.

[7] T. JANKOESKI, R. JANKOWSKI. *Multiple solutions of boundary-value problems for fourth-order differential equations with deviating arguments*. J. Optim. Theory Appl., 2010, **146**(1): 105–115.

[8] Shunyong LI, Xiaoqin ZHANG. *Existence and uniqueness of monotone positive solutions for an elastic beam equation with nonlinear boundary conditions*. Comput. Math. Appl., 2012, **63**(9): 1355–1360.

[9] Shunyong LI, Chengbo ZHAI. *New existence and uniqueness results for an elastic beam equation with nonlinear boundary conditions*. Bound. Value Probl., 2015, **104**: 1–12.

[10] Yongxiang LI. *Existence of positive solutions for the cantilever beam equations with fully nonlinear terms*. Nonlinear Anal. Real World Appl., 2016, **27**: 221–237.

[11] Xuezhe LV, Libo WANG, Minghe PEI. *Monotone positive solution of a fourth-order BVP with integral boundary conditions*. Bound. Value Probl., 2015, **172**: 1–12.

[12] Minghe PEI, Sungkag CHANG. *Monotone iterative technique and symmetric positive solutions for a fourth-order boundary value problem*. Math. Comput. Modelling, 2010, **51**(9-10): 1260–1267.

[13] Wenxia WANG, Yanping ZHENG, Hui YANG, et al. *Positive solutions for elastic beam equations with nonlinear boundary conditions and a parameter*. Bound. Value Probl., 2014, **80**: 1–17.

[14] J. R. L. WEBB, G. INFANTE, D. FRANCO. *Positive solutions of nonlinear fourth-order boundary-value problems with local and non-local boundary conditions*. Proc. Roy. Soc. Edinburgh Sect. A, 2008, **138**(2): 427–446.

[15] Chengbo ZHAI, Ruiping SONG, Qianqian HAN. *The existence and the uniqueness of symmetric positive solutions for a fourth-order boundary value problem*. Comput. Math. Appl., 2011, **62**(6): 2639–2647.

[16] Chengbo ZHAI, Lingling ZHANG. *New fixed point theorems for mixed monotone operators and local existence-uniqueness of positive solutions for nonlinear boundary value problems*. J. Math. Anal. Appl., 2011, **382**(2): 594–614.

[17] Chengbo ZHAI, Chunrong JIANG. *Existence and uniqueness of convex monotone positive solutions for boundary value problems of an elastic beam equation with a parameter.* Electron. J. Qual. Theory Differ. Equ., 2015, **81**: 1–11.

[18] Chengbo ZHAI, D. R. ANDERSON. *A sum operator equation and applications to nonlinear elastic beam equations and Lane-Emden-Fowler equations.* J. Math. Anal. Appl., 2011, **375**(2): 388–400.

[19] Chengbo ZHAI, Chunrong JIANG. *Existence of nontrivial solutions for a nonlinear fourth-order boundary value problem via iterative method.* J. Nonlinear Sci. Appl., 2016, **9**(6): 4295–4304.

[20] Chengbo ZHAI, Chunrong JIANG, Shunyong LI. *Approximating monotone positive solutions of a nonlinear fourth-order boundary value problem via sum operator method.* Mediterr. J. Math., 2017, **14**(2): 1–12.

[21] Dajun GUO, V. LAKSHMIKANTHAM. *Nonlinear Problems in Abstract Cones.* Academic Press Inc, Boston and New York, 1988.

[22] Chengbo ZHAI, Li WANG. *φ-(h, e)-concave operators and applications.* J. Math. Anal. Appl., 2017, **454**(2): 571–584.

[23] Chengbo ZHAI, Fang WANG. *Properties of positive solutions for the operator equation $Ax = \lambda x$ and applications to fractional differential equations with integral boundary conditions.* Adv. Difference Equ., 2015, **366**: 1–10.