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Multi-Wavelet Bessel Sequences in Sobolev Spaces

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Abstract Bessel sequence plays an important role in the study of frames for a Hilbert space with the convergence of a frame series, which has been widely studied in the literature. This paper addresses multi-wavelet Bessel sequences in Sobolev spaces setting, the result obtained is useful for the study of multi-wavelet frames in these spaces.

Keywords multi-wavelet; Bessel sequence; frame; Sobolev spaces

MR(2010) Subject Classification 42C40; 42C15

1. Introduction

In view of the great design freedom and the efficient application in practice, such as image restoration, signal denoising and the numerical solution of operator equations, wavelet frames have been extensively investigated by many researchers [1–8]. The Bessel sequence is very important in the study of frames for a Hilbert space with the convergence of a frame series [9–12], which has been widely studied in the literature.

Recently, Han and Shen [10] gave a sufficient condition for a $2I_d$ wavelet sequence to be Bessel sequence in Sobolev space $H^s(\mathbb{R}^d)$, $s>0$. Li, Yang and Yuan [13] generalized this result to Bessel M-multiwavelet sequences with M being an isotropic expansive matrix. In this paper, we further generalize the refinable function of [10, Theorem 2.3] and [13, Theorem 2.1] to a vector and the wavelet function to a finite number of vectors, and address multi-wavelet Bessel sequences in Sobolev spaces setting.

We first give some necessary notations and notions. We denote by $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}_0$ the set of integers, the set of positive integers, and the set of nonnegative integers, respectively. Let $d\in\mathbb{N}$, we denote by $\mathbb{T}^d=[0,1)^d$ the d-dimensional torus, and, for a Lebesgue measurable set E in $\mathbb{R}^d$, by $|E|$ its Lebesgue measure and χ_E the characteristic function of E, respectively. We write δ as the Dirac sequences such that $\delta_{0,0}=1$, and $\delta_{0,k}=0$ for $0\neq k\in\mathbb{Z}^d$. For a function f in $L^1(\mathbb{R}^d)$, its Fourier transform $\hat{f}$ is defined by $\hat{f}(\cdot)=\int_{\mathbb{R}^d} f(x)e^{-2\pi i\langle x,\,\cdot\rangle}\mathrm{d}x$, and is naturally extended to tempered distributions, where $\langle\cdot,\,\cdot\rangle$ denotes the Euclidean inner product on $\mathbb{R}^d$.

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For $s \in \mathbb{R}$, the Sobolev space $H^s(\mathbb{R}^d)$ consists of all distributions f such that

$$\|f\|^2_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 (1 + \|\xi\|_2^2)^s \mathrm{d}\xi < \infty,$$

where $\|\cdot\|_2$ denotes the Euclidean norm on $\mathbb{R}^d$. It is noted that, $H^s(\mathbb{R}^d)$ is a separable Hilbert space under the definition of the inner product:

$$\langle f,\, g\rangle_{H^s(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}(1 + \|\xi\|_2^2)^s \mathrm{d}\xi, \quad f, g \in H^s(\mathbb{R}^d).$$

Obviously, $H^0(\mathbb{R}^d) = L^2(\mathbb{R}^d)$, and $H^{s_1}(\mathbb{R}^d) \subseteq H^{s_2}(\mathbb{R}^d)$ iff $s_1 \geq s_2$. Furthermore, for every $g \in H^{-s}(\mathbb{R}^d)$,

$$\langle f,\, g\rangle = \int_{\mathbb{R}^d} \hat{f}(\xi)\overline{\hat{g}(\xi)}\mathrm{d}\xi, \quad f \in H^s(\mathbb{R}^d)$$

gives a continuous functional on $H^s(\mathbb{R}^d)$.

For $f, g : \mathbb{R}^d \mapsto \mathbb{C}$, we define

$$[f,\, g]_t(\cdot) = \sum_{k \in \mathbb{Z}^d} f(\cdot + k)\overline{g(\cdot + k)}(1 + \|\cdot + k\|_2^2)^t, \quad t \in \mathbb{R}.$$

We denote by M^* its conjugate transpose for a $d \times d$ order matrix M, by Γ_{M^*} a full set of ${M^*}^{-1}\mathbb{Z}^d/\mathbb{Z}^d$, i.e., a set of representatives of distinct cosets of ${M^*}^{-1}\mathbb{Z}^d/\mathbb{Z}^d$. It is called a dilation matrix if M is an integer matrix, and its eigenvalues are all greater than one in modulus. In this paper, we always assume that M is isotropic, i.e., M is similar to a diagonal matrix $\mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_d)$ satisfying $|\lambda_1| = |\lambda_2| = \cdots = |\lambda_d| = |\det M|^{\frac{1}{d}}$. Moreover, for convenient narration, we write $m = |\det M|^{\frac{1}{d}}$ and write

$$f_{j,k}(\cdot) = m^{\frac{jd}{2}} f(M^j \cdot - k) \quad \text{and} \quad f^s_{j,k}(\cdot) = m^{-js} f_{j,k}(\cdot) = m^{j(\frac{d}{2} - s)} f(M^j \cdot - k)$$

for a distribution f, $j \in \mathbb{Z}$, $k \in \mathbb{Z}^d$ and $s \in \mathbb{R}$.

Given $r \in \mathbb{N}$, let $\phi = (\phi_1, \phi_2, \ldots, \phi_r)^T \in (H^s(\mathbb{R}^d))^r$ be an M-refinable function vector satisfying the refinement equation, i.e., there exists an $r \times r$ order matrix $\hat{a}$, called refinement mask symbol such that

$$\hat{\phi}(M^* \cdot) = \hat{a}(\cdot)\hat{\phi}(\cdot) \ \text{ a.e. on } \ \mathbb{R}^d. \tag{1.1}$$

Given $L \in \mathbb{N}$. Wavelet function vectors $\psi_l = (\psi_1^l, \psi_2^l, \ldots, \psi_r^l)^T$ with $l = 1, 2, \ldots, L$ are defined by

$$\hat{\psi}_l(M^* \cdot) = \hat{b}^l(\cdot)\hat{\phi}(\cdot), \quad l = 1, 2, \ldots, L, \tag{1.2}$$

where $\hat{b}^l(\cdot) = (\hat{b}^l_{n,m}(\cdot))^r_{n,m=1}$ with $l = 1, 2, \ldots, L$ being a sequence of $r \times r$ order matrices of $\mathbb{Z}^d$-periodic measurable functions on $\mathbb{R}^d$, called wavelet masks symbol. Define a multi-wavelet system

$$\begin{aligned} X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L) = &\{\phi_{n;0,k} : n = 1, 2, \ldots, r; k \in \mathbb{Z}^d\} \cup \\ &\{\psi^{l,s}_{n;j,k} : n = 1, 2, \ldots, r; j \in \mathbb{N}_0, k \in \mathbb{Z}^d, \quad l = 1, 2, \ldots, L\}. \end{aligned} \tag{1.3}$$

$X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L)$ is called a multi-wavelet Bessel sequence (MWBS) in $H^s(\mathbb{R}^d)$ if there exists $B > 0$ such that

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{Z}^d}|\langle f,\,\phi_{n;0,k}\rangle_{H^s(\mathbb{R}^d)}|^2+\sum_{n=1}^{r}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}|\langle f,\,\psi^{l,s}_{n;j,k}\rangle_{H^s(\mathbb{R}^d)}|^2\le B\|f\|^2_{H^s(\mathbb{R}^d)},\quad \forall f\in H^s(\mathbb{R}^d),$$

where B is called a Bessel bound; it is called a multi-wavelet frame (MWF) in $H^s(\mathbb{R}^d)$ if there exist $0 < A \le B < \infty$ such that

$$\begin{aligned}A\|f\|^2_{H^s(\mathbb{R}^d)}\le&\sum_{n=1}^{r}\sum_{k\in\mathbb{Z}^d}|\langle f,\,\phi_{n;0,k}\rangle_{H^s(\mathbb{R}^d)}|^2+\sum_{n=1}^{r}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}|\langle f,\,\psi^{l,s}_{n;j,k}\rangle_{H^s(\mathbb{R}^d)}|^2\\ \le& B\|f\|^2_{H^s(\mathbb{R}^d)},\quad \forall f\in H^s(\mathbb{R}^d),\end{aligned}$$

where A and B are called frame bounds.

2. Some necessary lemmas

In this section, we provide some necessary lemmas which are used for later.

By a standard argument, we have

Lemma 2.1 *Let $s \in \mathbb{R}$. Define λ by*

$$\widehat{\lambda f}(\cdot)=(1+\|\cdot\|_2^2)^{\frac{s}{2}}\hat{f}(\cdot) \tag{2.1}$$

for $f \in H^s(\mathbb{R}^d)$ or $L^2(\mathbb{R}^d)$. Then λ is a unitary operator both from $H^s(\mathbb{R}^d)$ onto $L^2(\mathbb{R}^d)$ and from $L^2(\mathbb{R}^d)$ onto $H^{-s}(\mathbb{R}^d)$.

Lemma 2.2 *Let $s \in \mathbb{R}$, and $X^s(\phi;\psi_1,\psi_2,\dots,\psi_L)$ be a multi-wavelet system in $H^s(\mathbb{R}^d)$. Then $X^s(\phi;\psi_1,\psi_2,\dots,\psi_L)$ is a MWBS in $H^s(\mathbb{R}^d)$ with Bessel bound B if and only if*

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{Z}^d}|\langle f,\,\phi_{n;0,k}\rangle|^2+\sum_{n=1}^{r}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}|\langle f,\,\psi^{l,s}_{n;j,k}\rangle|^2\le B\|f\|^2_{H^{-s}(\mathbb{R}^d)}\ \ \textit{for}\ \ f\in H^{-s}(\mathbb{R}^d). \tag{2.2}$$

Proof By Lemma 2.1, we know $X^s(\phi;\psi_1,\psi_2,\dots,\psi_L)$ is a MWBS in $H^s(\mathbb{R}^d)$ with Bessel bound B if and only if

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{Z}^d}|\langle f,\,\lambda\phi_{n;0,k}\rangle|^2+\sum_{n=1}^{r}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}|\langle f,\,\lambda\psi^{l,s}_{n;j,k}\rangle|^2\le B\|f\|^2_{H^{-s}(\mathbb{R}^d)}\ \ \text{for}\ \ f\in L^2(\mathbb{R}^d). \tag{2.3}$$

Since λ is a unitary operator, we have

$$\langle f,\,\lambda\phi_{n;0,k}\rangle=\langle\lambda f,\,\phi_{n;0,k}\rangle \text{ and } \langle f,\,\lambda\psi^{l,s}_{n;j,k}\rangle=\langle\lambda f,\,\psi^{l,s}_{n;j,k}\rangle,$$

and

$$\|f\|^2_{L^2(\mathbb{R}^d)}=\|\lambda f\|^2_{H^{-s}(\mathbb{R}^d)}.$$

It follows that (2.3) is equivalent to

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{Z}^d}|\langle\lambda f,\,\phi^{n}_{0,k}\rangle|^2+\sum_{n=1}^{r}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}|\langle\lambda f,\,\psi^{n,s}_{l,j,k}\rangle|^2\le B\|\lambda f\|^2_{H^{-s}(\mathbb{R}^d)}\ \ \text{for}\ \ f\in L^2(\mathbb{R}^d). \tag{2.4}$$

This leads to the lemma since λ is a unitary operator from $L^2(\mathbb{R}^d)$ onto $H^{-s}(\mathbb{R}^d)$ by Lemma 2.1. $\square$

Lemma 2.3 ([13, Lemma 1.1]) *Let M be a $d\times d$ order isotropic dilation matrix. Then there exists a norm $\|\cdot\|$ on $\mathbb{R}^d$ such that $\|M^*\cdot\|=m\|\cdot\|$. Furthermore, there exist positive constants ϱ_1 and ϱ_2 such that $\varrho_2\|\cdot\|\le\|\cdot\|_2\le\varrho_1\|\cdot\|$.*

Lemma 2.4 ([13, Lemma 2.1]) *For $\eta>\zeta>0$, define*

$$B_{\zeta,\eta}(\xi)=\sum_{j=0}^{\infty}m^{-2j\zeta}(1+\varrho_1^2\|\xi\|^2)^{\zeta}(1+m^{-2j-2}\varrho_2^2\|\xi\|^2)^{-\eta},\ \xi\in\mathbb{R}^d, \tag{2.5}$$

where ϱ_1,ϱ_2 and $\|\cdot\|$ are as in Lemma 2.3. Then there exists a positive constant C such that $B_{\zeta,\eta}(\xi)<C$, $\forall\xi\in\mathbb{R}^d$.

Lemma 2.5 *Let $0\ne s\in\mathbb{R}$ and $\phi=(\phi_1,\phi_2,\dots,\phi_r)^T\in(H^s(\mathbb{R}^d))^r$. If $[\hat\phi_n,\,\hat\phi_n]_t\in L^\infty(\mathbb{R}^d)$ for some $t>s$ with $n=1,2,\dots,r$, then*

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{Z}^d}|\langle g,\,\phi_{n;0,k}\rangle|^2\le\sum_{n=1}^{r}\|[\hat\phi_n,\,\hat\phi_n]_s\|_{L^\infty(\mathbb{R}^d)}\|g\|^2_{H^{-s}(\mathbb{R}^d)} \tag{2.6}$$

for $g\in H^{-s}(\mathbb{R}^d)$.

Proof Since for any $n\in\{1,2,\dots,r\}$, $\phi_n\in H^s(\mathbb{R}^d)$ and $g\in H^{-s}(\mathbb{R}^d)$, we have $\hat g\overline{\hat\phi_n}\in L^1(\mathbb{R}^d)$. Applying the Plancherel theorem and the Parseval identity, by a simple computation we have

$$\begin{aligned}\sum_{k\in\mathbb{Z}^d}|\langle g,\,\phi_n(\cdot-k)\rangle|^2&=\sum_{k\in\mathbb{Z}^d}\Big|\int_{\mathbb{R}^d}\hat g(\xi)\overline{\hat\phi_n(\xi)}e^{2\pi i\langle k,\,\xi\rangle}\mathrm{d}\xi\Big|^2\\&=\sum_{k\in\mathbb{Z}^d}\Big|\sum_{k'\in\mathbb{Z}^d}\int_{\mathbb{T}^d}\hat g(\xi+k')\overline{\hat\phi_n(\xi+k')}e^{2\pi i\langle k,\,\xi\rangle}\mathrm{d}\xi\Big|^2\\&=\int_{\mathbb{T}^d}\Big|\sum_{k'\in\mathbb{Z}^d}\hat g(\xi+k')\overline{\hat\phi_n(\xi+k')}\Big|^2\mathrm{d}\xi\\&=\int_{\mathbb{T}^d}|[\hat g,\,\hat\phi_n]_0(\xi)|^2\mathrm{d}\xi.\end{aligned}\tag{2.7}$$

By the Cauchy Schwarz's inequality, we have $|[\hat g,\,\hat\phi_n]_0(\xi)|^2\le[\hat g,\,\hat g]_{-s}(\xi)[\hat\phi_n,\,\hat\phi_n]_s(\xi)$ for almost every $\xi\in\mathbb{R}^d$. Since $t>s$ and $[\hat\phi_n,\,\hat\phi_n]_t\in L^\infty(\mathbb{R}^d)$, it follows that

$$[\hat\phi_n,\,\hat\phi_n]_s(\xi)\le[\hat\phi_n,\,\hat\phi_n]_t(\xi).$$

Therefore, $[\hat\phi_n,\,\hat\phi_n]_s\in L^\infty(\mathbb{R}^d)$, and thus we deduce from (2.7) that

$$\begin{aligned}\sum_{n=1}^{r}\sum_{k\in\mathbb{Z}^d}|\langle g,\,\phi_n(\cdot-k)\rangle|^2&\le\sum_{n=1}^{r}\int_{\mathbb{T}^d}[\hat g,\,\hat g]_{-s}(\xi)[\hat\phi_n,\,\hat\phi_n]_s(\xi)\mathrm{d}\xi\\&\le\sum_{n=1}^{r}\|[\hat\phi_n,\,\hat\phi_n]_s\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{T}^d}[\hat g,\,\hat g]_{-s}(\xi)\mathrm{d}\xi\\&=\sum_{n=1}^{r}\|[\hat\phi_n,\,\hat\phi_n]_s\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}|\hat g(\xi)|^2(1+\|\xi\|_2^2)^{-s}\mathrm{d}\xi\end{aligned}$$

$$= \sum_{n=1}^{r} \|[\hat{\phi}_n,\, \hat{\phi}_n]_s\|_{L^\infty(\mathbb{R}^d)} \|g\|^2_{H^{-s}(\mathbb{R}^d)}. \quad \square \tag{2.8}$$

Lemma 2.6 *Let $0 \neq s < t$, and $\hat{b}^l(\cdot) = (\hat{b}^l_{n,m}(\cdot))^r_{n,m=1}$, $l = 1, 2, \ldots, L$ be a sequence of $r \times r$ order matrices of $\mathbb{Z}^d$-periodic measurable functions on $\mathbb{R}^d$, define*

$$\Delta_{s,t}(\xi) = \sum_{j=0}^{\infty} m^{-2js}(1 + \|\xi\|_2^2)^s \sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\hat{b}^l_{n,m}(M^{*^{-j-1}}\xi)|^2 (1 + \|M^{*^{-j-1}}\xi\|_2^2)^{-t}, \quad \xi \in \mathbb{R}^d.$$

If there exists a nonnegative number $\alpha > -s$ and a positive constant C such that

$$\sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\hat{b}^l_{n,m}(\cdot)|^2 \leq C \min(1,\, \|\cdot\|_2^{2\alpha}), \text{ a.e. on } \mathbb{R}^d, \tag{2.9}$$

then $\Delta_{s,t} \in L^\infty(\mathbb{R}^d)$.

Proof Let us consider the two cases $s > 0$ and $s < 0$ separately.

Suppose $s > 0$. Since $t > s$, by Lemma 2.3, we have

$$\Delta_{s,t}(\xi) \leq \sum_{j=0}^{\infty} m^{-2js}(1 + \varrho_1^2\|\xi\|^2)^s \sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\hat{b}^l_{n,m}(M^{*^{-j-1}}\xi)|^2 (1 + m^{-2j-2}\varrho_2^2\|\xi\|^2)^{-t}. \tag{2.10}$$

By Lemma 2.4, there exists a poitive constant C' such that

$$B_{s,t}(\xi) = \sum_{j=0}^{\infty} m^{-2js}(1 + \varrho_1^2\|\xi\|^2)^s (1 + m^{-2j-2}\varrho_2^2\|\xi\|^2)^{-t} \leq C', \quad \forall \xi \in \mathbb{R}^d. \tag{2.11}$$

This implies that $\Delta_{s,t}(\xi) \leq C'C$, $\forall \xi \in \mathbb{R}^d$, i.e., $\Delta_{s,t} \in L^\infty(\mathbb{R}^d)$.

Suppose $s < 0$. Without loss of generality, we assume that $s < t < 0$. By Lemma 2.3, we have

$$\begin{aligned} \Delta_{s,t}(\xi) \leq & \sum_{j=0}^{\infty} m^{-2js}(1 + \varrho_2^2\|\xi\|^2)^s \sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\hat{b}^l_{n,m}(M^{*^{-j-1}}\xi)|^2 (1 + m^{-2j-2}\varrho_1^2\|\xi\|^2)^{-t} \\ =: & \,\Theta_{s,t}(\xi). \end{aligned} \tag{2.12}$$

For $\varrho_1\|\xi\| \leq 1$ and $j \geq 0$, we have

$$(1 + m^{-2j-2}\varrho_1^2\|\xi\|^2)^{-t} \leq 2^{-t} \text{ and } (1 + \varrho_2^2\|\xi\|^2)^s \leq 1.$$

Since $\alpha \geq 0$, $\alpha + s > 0$, by Lemma 2.3 and Eq. (2.9), we have the following estimate

$$\begin{aligned} \Theta_{s,t}(\xi) \leq & 2^{-t}\sum_{j=0}^{\infty} m^{-2js} \sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\hat{b}^l_{n,m}(M^{*^{-j-1}}\xi)|^2 \\ \leq & 2^{-t}C\sum_{j=0}^{\infty} m^{-2js}\|M^{*^{-j-1}}\xi\|_2^{2\alpha} \\ \leq & 2^{-t}Cm^{-2\alpha}\sum_{j=0}^{\infty} m^{-2j(\alpha+s)}(\varrho_1\|\xi\|)^{2\alpha} \\ \leq & 2^{-t}Cm^{-2\alpha}\sum_{j=0}^{\infty} m^{-2j(\alpha+s)} = \frac{2^{-t}Cm^{-2\alpha}}{1 - m^{-2(\alpha+s)}} < \infty. \end{aligned} \tag{2.13}$$

For $\varrho_1\|\xi\|>1$, there exists $J\in\mathbb{N}_0$ such that $m^J\le\varrho_1\|\xi\|<m^{J+1}$. Then for $j=0,1,\ldots,J$, we have

$$(1+m^{-2j-2}\varrho_1^2\|\xi\|^2)^{-t}\le(1+m^{2(J-j)})^{-t}=m^{-2(J-j)t}(m^{-2(J-j)}+1)^{-t}\le 2^{-t}m^{-2(J-j)t}$$

and

$$(1+\varrho_2^2\|\xi\|^2)^s\le(1+\varrho_2^2\varrho_1^{-2}m^{2J})^s\le\varrho_2^{2s}\varrho_1^{-2s}m^{2Js}.$$

Write $\Theta_{s,t}(\xi)=\Theta^1_{s,t}(\xi)+\Theta^2_{s,t}(\xi)$, where

$$\Theta^1_{s,t}(\xi)=\sum_{j=0}^{J}m^{-2js}(1+\varrho_2^2\|\xi\|^2)^s\sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r}|\hat{b}^l_{n,m}(M^{*^{-j-1}}\xi)|^2(1+m^{-2j-2}\varrho_1^2\|\xi\|^2)^{-t},$$

$$\Theta^2_{s,t}(\xi)=\sum_{j=J+1}^{\infty}m^{-2js}(1+\varrho_2^2\|\xi\|^2)^s\sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r}|\hat{b}^l_{n,m}(M^{*^{-j-1}}\xi)|^2(1+m^{-2j-2}\varrho_1^2\|\xi\|^2)^{-t}.$$

Then by $m^J\le\varrho_1\|\xi\|<m^{J+1}$ and $J\in\mathbb{N}_0$, it follows from $s<t<0$ that

$$\begin{aligned}\Theta^1_{s,t}(\xi)&=\sum_{j=0}^{J}m^{-2js}(1+\varrho_2^2\|\xi\|^2)^s\sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r}|\hat{b}^l_{n,m}(M^{*^{-j-1}}\xi)|^2(1+m^{-2j-2}\varrho_1^2\|\xi\|^2)^{-t}\\&\le C\varrho_2^{2s}\varrho_1^{-2s}2^{-t}\sum_{j=0}^{J}m^{-2(J-j)(t-s)}\le C\varrho_2^{2s}\varrho_1^{-2s}2^{-t}\sum_{j=0}^{\infty}m^{-2j(t-s)}\\&=\varrho_2^{2s}\varrho_1^{-2s}2^{-t}\frac{1}{1-m^{-2(t-s)}}<\infty.\end{aligned}\tag{2.14}$$

Since $m^J\le\varrho_1\|\xi\|<m^{J+1}$, we have for $j\ge J+1$

$$(1+m^{-2j-2}\varrho_1^2\|\xi\|^2)^{-t}\le(1+m^{2(J-j)})^{-t}\le 2^{-t}$$

and

$$(1+\varrho_2^2\|\xi\|^2)^s\le(1+\varrho_2^2\varrho_1^{-2}m^{2J})^s\le\varrho_2^{2s}\varrho_1^{-2s}m^{2Js}.$$

Since $\alpha\ge0,\alpha+s>0$, by Lemma 2.3 and Eq. (2.9), we have

$$\begin{aligned}\Theta^2_{s,t}(\xi)&=\sum_{j=J+1}^{\infty}m^{-2js}(1+\varrho_2^2\|\xi\|^2)^s\sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r}|\hat{b}^l_{n,m}(M^{*^{-j-1}}\xi)|^2(1+m^{-2j-2}\varrho_1^2\|\xi\|^2)^{-t}\\&\le2^{-t}\varrho_2^{2s}\varrho_1^{-2s}\sum_{j=J+1}^{\infty}m^{-2(j-J)s}\sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r}|\hat{b}^l_{n,m}(M^{*^{-j-1}}\xi)|^2\\&\le2^{-t}\varrho_2^{2s}\varrho_1^{-2s}C\sum_{j=J+1}^{\infty}m^{-2(j-J)s}\|M^{*^{-j-1}}\xi\|_2^{2\alpha}\\&\le2^{-t}\varrho_2^{2s}\varrho_1^{-2s}C\sum_{j=J+1}^{\infty}m^{-2(j-J)s}m^{-2\alpha(j+1)}(\varrho_1\|\xi\|)^{2\alpha}\\&\le2^{-t}\varrho_2^{2s}\varrho_1^{-2s}C\sum_{j=J+1}^{\infty}m^{-2(j-J)(\alpha+s)}=2^{-t}\varrho_2^{2s}\varrho_1^{-2s}C\sum_{j=1}^{\infty}m^{-2j(\alpha+s)}\\&=2^{-t}\varrho_2^{2s}\varrho_1^{-2(\alpha+s)}C\frac{m^{-2(\alpha+s)}}{1-m^{-2s}}<\infty.\end{aligned}\tag{2.15}$$

Therefore, for the case $s < 0$, we conclude that $\Delta_{s,t} \in L^\infty(\mathbb{R}^d)$. $\square$

3. Multi-wavelet Bessel sequences

In this section, we will study multi-wavelet Bessel sequences in Sobolev spaces setting.

Theorem 3.1 *Given $s \in \mathbb{R}$, let $\phi = (\phi_1, \phi_2, \ldots, \phi_r)^T \in (H^s(\mathbb{R}^d))^r$ be an M-refinable function vector satisfying the refinable Eq. (1.1), and let $\hat{b}^l(\cdot) = (\hat{b}^l_{n,m}(\cdot))_{n,m=1}^r, l = 1, 2, \ldots, L$ be a sequence of $r \times r$ order matrices of $\mathbb{Z}^d$-periodic measurable functions on $\mathbb{R}^d$, $\psi_l = (\psi^l_1, \psi^l_2, \ldots, \psi^l_r)^T$, $l = 1, 2, \ldots, L$, be the wavelet function vectors defined by (1.2), and $X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L)$ be the multi-wavelet systems defined by (1.3). Assume that*

(i) $[\hat{\phi}_n, \hat{\phi}_n]_t \in L^\infty(\mathbb{R}^d)$ for some $t > s$ with $n = 1, 2, \ldots, r$;

(ii) There exists a nonnegative number $\alpha > -s$ and a positive constant C such that

$$\sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r} |\hat{b}^l_{n,m}(\cdot)|^2 \le C \min(1, \|\cdot\|_2^{2\alpha}), \text{ a.e. on } \mathbb{R}^d.$$

Then $X^s(\phi; \psi_1, \psi_2, \ldots, \psi_L)$ is a MWBS in $H^s(\mathbb{R}^d)$.

Proof For the case $s = 0$, we take $0 < s_0 < \min\{t, \alpha\}$, then the conditions (i) and (ii) hold for $s = s_0$. Therefore, the conclusion holds for $s = 0$ if it holds for $s = s_0$. So, in order to finish the proof, we need to prove the conclusion holds for $s \neq 0$. By Lemma 2.2, it is enough to prove that there exists a positive constant B such that

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{Z}^d} |\langle g,\, \phi_{n;0,k}\rangle|^2 + \sum_{n=1}^{r}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d} |\langle g,\, \psi^{l,s}_{n;j,k}\rangle|^2 \le B\|g\|^2_{H^{-s}(\mathbb{R}^d)} \text{ for } g \in H^{-s}(\mathbb{R}^d). \quad (3.1)$$

For the first part, by Lemma 2.5, we have

$$\sum_{n=1}^{r}\sum_{k\in\mathbb{Z}^d} |\langle g,\, \phi_{n;0,k}\rangle|^2 \le \sum_{n=1}^{r} \|[\hat{\phi}_n,\, \hat{\phi}_n]_s\|_{L^\infty(\mathbb{R}^d)} \|g\|^2_{H^{-s}(\mathbb{R}^d)} \text{ for } g \in H^{-s}(\mathbb{R}^d). \quad (3.2)$$

Next, we check the second part. For $g \in H^{-s}(\mathbb{R}^d)$, compute

$$\begin{aligned}
\sum_{k\in\mathbb{Z}^d} |\langle g,\, \psi^{l,s}_{n;j,k}\rangle|^2 &= m^{-j(d+2s)} \sum_{k\in\mathbb{Z}^d} \Big| \int_{\mathbb{R}^d} \hat{g}(\xi)\overline{\hat{\psi}^l_n(M^{*-j}\xi)} e^{2\pi i\langle k,\, M^{*^{-j}}\xi\rangle} \mathrm{d}\xi \Big|^2 \\
&= m^{j(d-2s)} \sum_{k\in\mathbb{Z}^d} \Big| \sum_{k'\in\mathbb{Z}^d} \int_{\mathbb{T}^d} \hat{g}(M^{*^j}(\xi+k'))\overline{\hat{\psi}^l_n(\xi+k')} e^{2\pi i\langle k,\, \xi\rangle} \mathrm{d}\xi \Big|^2 \\
&= m^{j(d-2s)} \int_{\mathbb{T}^d} \Big| \sum_{k'\in\mathbb{Z}^d} \hat{g}(M^{*^j}(\xi+k'))\overline{\hat{\psi}^l_n(\xi+k')} \Big|^2 \mathrm{d}\xi \\
&= m^{j(d-2s)} \int_{\mathbb{T}^d} |[\hat{g}(M^{*^j}\cdot),\, \hat{\psi}^l_n(\cdot)]_0(\xi)|^2 \mathrm{d}\xi. \qquad (3.3)
\end{aligned}$$

By the definition in (1.2), we can get each component of $\hat{\psi}_l$

$$\hat{\psi}^l_n(\cdot) = \sum_{m=1}^{r} \hat{b}^l_{n,m}(M^{*^{-1}}\cdot)\hat{\phi}_n(M^{*^{-1}}\cdot) \text{ for } n = 1, 2, \ldots, r \text{ and } l = 1, 2, \ldots, L,$$

and it follows from (3.3) that

$$\begin{aligned}
&\sum_{k\in\mathbb{Z}^d}|\langle g,\,\psi^{l,s}_{n;j,k}\rangle|^2\\
&= m^{j(d-2s)}\int_{\mathbb{T}^d}\Big|\sum_{k\in\mathbb{Z}^d}\sum_{m=1}^{r}\hat g(M^{*^j}(\xi+k))\overline{\hat b^l_{n,m}(M^{*^{-1}}(\xi+k))\hat\phi_n(M^{*^{-1}}(\xi+k))}\Big|^2\mathrm{d}\xi\\
&= m^{j(d-2s)}\int_{\mathbb{T}^d}\Big|\sum_{\gamma\in\Gamma_{M^*}}\sum_{m=1}^{r}\overline{\hat b^l_{n,m}(M^{*^{-1}}\xi+\gamma)}[\hat g(M^{*^{j+1}}\cdot),\,\hat\phi_n]_0(M^{*^{-1}}\xi+\gamma)\Big|^2\mathrm{d}\xi\\
&\le m^{(j+1)d-2js}\sum_{\gamma\in\Gamma_{M^*}}\int_{\mathbb{T}^d}\Big|\sum_{m=1}^{r}\hat b^l_{n,m}(M^{*^{-1}}\xi+\gamma)[\hat g(M^{*^{j+1}}\cdot),\,\hat\phi_n]_0(M^{*^{-1}}\xi+\gamma)\Big|^2\mathrm{d}\xi\\
&\le m^{(j+2)d-2js}\int_{\mathbb{T}^d}\sum_{m=1}^{r}|\hat b^l_{n,m}(\xi)|^2[\hat g(M^{*^{j+1}}\cdot),\,\hat g(M^{*^{j+1}}\cdot)]_{-t}(\xi)[\hat\phi_n,\,\hat\phi_n]_t(\xi)\mathrm{d}\xi\\
&\le m^{(j+2)d-2js}\max_{1\le n\le r}\{\|[\hat\phi_n,\,\hat\phi_n]_t\|_{L^\infty(\mathbb{R}^d)}\}\int_{\mathbb{T}^d}\sum_{m=1}^{r}|\hat b^l_{n,m}(\xi)|^2[\hat g(M^{*^{j+1}}\cdot),\,\hat g(M^{*^{j+1}}\cdot)]_{-t}(\xi)\mathrm{d}\xi\\
&= m^{(j+2)d-2js}\max_{1\le n\le r}\{\|[\hat\phi_n,\,\hat\phi_n]_t\|_{L^\infty(\mathbb{R}^d)}\}\int_{\mathbb{R}^d}\sum_{m=1}^{r}|\hat b^l_{n,m}(\xi)|^2|\hat g(M^{*^{j+1}}\xi)|^2(1+\|\xi\|_2^2)^{-t}\mathrm{d}\xi\\
&= m^{d-2js}\max_{1\le n\le r}\{\|[\hat\phi^n,\,\hat\phi^n]_t\|_{L^\infty(\mathbb{R}^d)}\}\times\\
&\quad\int_{\mathbb{R}^d}\sum_{m=1}^{r}|\hat b^l_{n,m}(M^{*^{-j-1}}\xi)|^2|\hat g(\xi)|^2(1+\|M^{*^{-j-1}}\xi\|_2^2)^{-t}\mathrm{d}\xi. \qquad (3.4)
\end{aligned}$$

Hence, we conclude that

$$\begin{aligned}
\sum_{n=1}^{r}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}|\langle g,\,\psi^{l,s}_{n;j,k}\rangle|^2 &\le m^d\max_{1\le n\le r}\{\|[\hat\phi_n,\,\hat\phi_n]_t\|_{L^\infty(\mathbb{R}^d)}\}\int_{\mathbb{R}^d}|\hat g(\xi)|^2(1+\|\xi\|_2^2)^{-s}\times\\
&\quad\sum_{j=0}^{\infty}m^{-2js}(1+\|\xi\|_2^2)^{s}\sum_{l=1}^{L}\sum_{n=1}^{r}\sum_{m=1}^{r}|\hat b^l_{n,m}(M^{*^{-j-1}}\xi)|^2(1+\|M^{*^{-j-1}}\xi\|_2^2)^{-t}\mathrm{d}\xi. \qquad (3.5)
\end{aligned}$$

By Lemma 2.6, we get from (3.5) that

$$\begin{aligned}
&\sum_{n=1}^{r}\sum_{l=1}^{L}\sum_{j=0}^{\infty}\sum_{k\in\mathbb{Z}^d}|\langle g,\,\psi^{l,s}_{n;j,k}\rangle|^2\\
&\le m^d\max_{1\le n\le r}\{\|[\hat\phi_n,\,\hat\phi_n]_t\|_{L^\infty(\mathbb{R}^d)}\}\|\Delta_{s,t}\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}|\hat g(\xi)|^2(1+\|\xi\|_2^2)^{-s}\\
&= m^d\max_{1\le n\le r}\{\|[\hat\phi_n,\,\hat\phi_n]_t\|_{L^\infty(\mathbb{R}^d)}\}\|\Delta_{s,t}\|_{L^\infty(\mathbb{R}^d)}\|g\|^2_{H^{-s}(\mathbb{R}^d)}.
\end{aligned}$$

Consequently, (3.1) holds with

$$B=\sum_{n=1}^{r}\|[\hat\phi_n,\,\hat\phi_n]_s\|_{L^\infty(\mathbb{R}^d)}+m^d\max_{1\le n\le r}\{\|[\hat\phi_n,\,\hat\phi_n]_t\|_{L^\infty(\mathbb{R}^d)}\}\|\Delta_{s,t}\|_{L^\infty(\mathbb{R}^d)}. \qquad (3.6)$$

The proof is completed. □

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References

[1] Chenglong BAO, Bin DONG, Likun HOU, et al. *Image restoration by minimizing zero norm of wavelet frame coefficients*. Inverse Problems, 2016, **32**(11): 1–27.

[2] Dirong CHEN. *Regularized generalized inverse accelerating linearized alternating minimization algorithm for frame-based poissonian image deblurring*. SIAM J. Imaging Sci., 2014, **7**(2): 716–739.

[3] Jianfeng CAI, Bin DONG, Zuowei SHEN. *Image restoration: A wavelet frame based model for piecewise smooth functions and beyond*. Appl. Comput. Harmon. Anal., 2016, **41**(1): 94–138.

[4] Bin DONG. *Sparse representation on graphs by tight wavelet frames and applications*. Appl. Comput. Harmon. Anal., 2017, **42**(3): 452–479.

[5] Bin DONG, Qingtang JIANG, Zuowei SHEN. *Image restoration: Wavelet frame shrinkage, nonlinear evolution pdes, and beyond*. Multiscale Model. Simul., 2017, **15**(1): 606–660.

[6] S. DIDAS, J. WEICKERT. *Integrodifferential equations for continuous multiscale wavelet shrinkage*. Inverse Probl. Imaging, 2007, **1**(1): 47–62.

[7] R. STEVENSON. *Adaptive solution of operator equations using wavelet frames*. SIAM. J. Numer. Anal., 2004, **41**(3): 1074–1100.

[8] Haimiao ZHANG, Yichuan DONG, Qibin FAN. *Wavelet frame based Poisson noise removal and image deblurring*. Signal Process, 2017, **137**: 363–372.

[9] Bin HAN, Qun MO. *Multiwavelet frames from refinable function vectors*. Adv. Comput. Math., 2003, **18**(2): 211–245.

[10] Bin HAN, Zuowei SHEN. *Dual wavelet frames and Riesz bases in Sobolev spaces*. Constr. Approx., 2009, **29**(3): 369–406.

[11] Bin HAN, Zuowei SHEN. *Characterization of Sobolev spaces of arbitrary smoothness using nonstationary tight wavelet frames*. Isr. J. Math., 2009, **172**(1): 371–398.

[12] Dayong LU, Dengfeng LI. *A characterization of orthonormal wavelet families in Sobolev spaces*. Acta Math. Sci., 2011, **31**(4): 1475–1488.

[13] Youfa LI, Shouzhi YANG, Dehui YUAN. *Bessel multiwavelet sequences and dual multiframelets in Sobolev spaces*. Adv. Comput. Math., 2013, **38**(3): 491–529.