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Weak Hopf Algebras Corresponding to the Non-Standard Deformation of Type B_n

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Abstract We introduce a non-standard quantum group $X_q(B_n)$ of type B_n which is a Hopf algebra. Then we replace the group of grouplike elements of $X_q(B_n)$, and obtain a weak Hopf algebra $\mathfrak{w}X_q(B_n)$. Finally, we describe the Ext-quiver of $\mathfrak{w}X_q(B_n)$ as a coalgebra.

Keywords non-standard quantum group; weak Hopf algebra; Ext quiver

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1. Introduction

All algebras in this paper are considered over the complex field $\mathbb{C}$, the nonzero parameter $q \in \mathbb{C}$ is not a root of unity.

Ge et al. [1] introduced a new quantum group and in [2] Jing et al. derived all finite dimensional irreducible representations of this new quantum group. Aghamohammadi et al. [3, 4] introduced mulliparametric generalizations of type A_{n-1} and type B_n. On the other hand, Li and Duplij [5, 6] defined weak Hopf algebras, which are bialgebras with weak antipodes, but not Hopf algebras. Yang [7] constructed weak Hopf algebras corresponding to Cartan matrices, determined their PBW bases according to the definition, then some properties related to the weak Hopf algebra $wsl_q(2)$ were studied in [8, 9]. Cheng [10] researched weak Hopf algebras corresponding to $U_q(sl_n)$, and gave their Ext quivers. In 2016, Cheng and Yang [11] constructed a weak Hopf algebra $\mathfrak{w}X_q(A_1)$ corresponding to the non-standard quantum group $X_q(A_1)$, and described the PBW basis of $\mathfrak{w}X_q(A_1)$. Following the non-standard quantum group $X_q(B_n)$ in [4], in this short note we construct a non-standard quantum group $\mathfrak{w}X_q(B_n)$ by weakening the grouplike elements, then study the Ext quiver of its coalgebra.

The paper is arranged as follows. In Section 2, we give some definitions and relations of $X_q(B_n)$. Then we establish a weak quantum algebra $\mathfrak{w}X_q(B_n)$ by weakening the antipode, and prove that $\mathfrak{w}X_q(B_n)$ is a weak Hopf algebra. In Section 3, we study the weak Hopf algebra structure of $\mathfrak{w}X_q(B_n)$, and give the Ext quiver of its coalgebra.

2. The weak Hopf algebra $\mathfrak{w}X_q(B_n)$

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Definition 2.1 *$X_q(B_n)$ is the associative algebra over the field $\mathbb{C}$ with 1, generated by $K_1^{\pm 1}, K_2^{\pm 1}, \ldots, K_n^{\pm 1}, E_1, \ldots, E_n, F_1, \ldots, F_n$ with the following relations*

$$K_iK_i^{-1} = K_i^{-1}K_i = 1,\ K_iK_j = K_jK_i,$$

$$K_iE_j = E_jK_i,\ K_iF_j = F_jK_i,\quad i \neq j, j+1,$$

$$K_iE_i = q_i^{-1}E_iK_i,\ K_iF_i = q_iF_iK_i,$$

$$K_{i+1}E_i = q_{i+1}E_iK_{i+1},\ K_{i+1}F_i = q_{i+1}^{-1}F_iK_{i+1},$$

$$(q_i - q_{i+1})E_i^2 = (q_i - q_{i+1})F_i^2 = 0,\quad i \neq n,$$

$$E_iE_j = E_jE_i,\ F_iF_j = F_jF_i,\quad |i-j| \geq 2,$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_i^{-1}K_{i+1} - K_iK_{i+1}^{-1}}{q - q^{-1}},\quad i \neq n,$$

$$E_nF_j - F_jE_n = \delta_{n,j}\frac{K_n^{-1} - K_n}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}},$$

$$q_iE_i^2E_{i\pm 1} - (1 + q_iq_{i+1})E_iE_{i\pm 1}E_i + q_{i+1}E_{i\pm 1}E_i^2 = 0,\quad i \neq n,$$

$$q_iF_i^2F_{i\pm 1} - (1 + q_iq_{i+1})F_iF_{i\pm 1}F_i + q_{i+1}F_{i\pm 1}F_i^2 = 0,\quad i \neq n,$$

$$E_n^3E_{n-1} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{q_n^{\frac{1}{2}}} E_n^2E_{n-1}E_n + \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{q_n^{\frac{1}{2}}} E_nE_{n-1}E_n^2 - E_{n-1}E_n^3 = 0,$$

$$F_n^3F_{n-1} - \begin{bmatrix} 3 \\ 1 \end{bmatrix}_{q_n^{\frac{1}{2}}} F_n^2F_{n-1}F_n + \begin{bmatrix} 3 \\ 2 \end{bmatrix}_{q_n^{\frac{1}{2}}} F_nF_{n-1}F_n^2 - F_{n-1}F_n^3 = 0,$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, $[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}}$, $q_i = q$ *or* $-q^{-1}$, $1 \leq i \leq n$.

If all $q_i = q$, then Serre relations of $X_q(B_n)$ are the same with $U_q(B_n)$. If $q_i \neq q_{i+1}$ $(1 \leq i \leq n)$, then $E_i^2 = F_i^2 = 0$, and we call it the non-standard quantum groups corresponding to B_n, denoted by $X_q(B_n)$, where $q_i = q$ or $-q^{-1}$ $(1 \leq i \leq n)$.

Proposition 2.2 *$X_q(B_n)$ is a Hopf algebra with comultiplication Δ, counit ε and antipode S defined as follows*

$$\Delta : X_q(B_n) \to X_q(B_n) \otimes X_q(B_n),$$

$$\Delta(K_i) = K_i \otimes K_i,\ \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$

$$\Delta(E_i) = K_iK_{i+1}^{-1} \otimes E_i + E_i \otimes 1, i \neq n, \Delta(E_n) = K_n \otimes E_n + E_n \otimes 1,$$

$$\Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i^{-1}K_{i+1}, i \neq n, \Delta(F_n) = 1 \otimes F_n + F_n \otimes K_n^{-1},$$

$$\varepsilon : X_q(B_n) \to \mathbb{C},$$

$$\varepsilon(K_i) = 1,\ \varepsilon(E_i) = \varepsilon(F_i) = 0,$$

$$S : X_q(B_n) \to X_q(B_n),$$

$$S(K_i) = K_i^{-1},\ S(E_i) = -K_i^{-1}K_{i+1}E_i,\ S(F_i) = -F_iK_iK_{i+1}^{-1}.$$

Proof Indeed, Δ defines a morphism of algebra from $X_q(B_n)$ into $X_q(B_n) \otimes X_q(B_n)$. It is easy to prove that Δ keeps relations similar to the proof of [12, Proposition VII.1.1] and [13, Appendix in Chap. 4].

Similiarly, one can easily prove that ε defines an algebra morphism from $X_q(B_n)$ into $\mathbb{C}$ and S defines an antipode of $X_q(B_n)$. Therefore, $X_q(B_n)$ is a Hopf algebra with the comultiplication Δ, the counit ε and the antipode S. □

In the following, we construct a weak Hopf algebra $\mathfrak{w}X_q(B_n)$ corresponding to $X_q(B_n)$. Firstly, we replace $K_i^{\pm 1}$ by $K_i, \overline{K}_i$, and add a new generator J such that $K_i\overline{K}_i = \overline{K}_iK_i = J$ for all $i = 1, 2, \ldots, n$.

If E_i (resp., F_i) satisfies

$$K_iE_j = E_jK_i \text{ (resp., } K_iF_j = F_jK_i), \quad i \neq j, j+1,$$

$$K_iE_i = q_i^{-1}E_iK_i \text{ (resp., } K_iF_i = q_iF_iK_i),$$

$$K_{i+1}E_i = q_{i+1}E_iK_{i+1} \text{ (resp., } K_{i+1}F_i = q_{i+1}^{-1}F_iK_{i+1}),$$

$$\overline{K}_iE_j = E_j\overline{K}_i \text{ (resp., } \overline{K}_iF_j = F_j\overline{K}_i), \quad i \neq j, j+1,$$

$$\overline{K}_iE_i = q_iE_i\overline{K}_i \text{ (resp., } \overline{K}_iF_i = q_i^{-1}F_i\overline{K}_i),$$

$$\overline{K}_{i+1}E_i = q_{i+1}^{-1}E_i\overline{K}_{i+1} \text{ (resp., } \overline{K}_{i+1}F_i = q_{i+1}F_i\overline{K}_{i+1}),$$

then E_i (resp., F_i) is said to be of type I.

If E_i (resp., F_i) satisfies

$$K_iE_j\overline{K}_i = E_j \text{ (resp., } K_iF_j\overline{K}_i = F_j), \quad i \neq j, j+1,$$

$$K_iE_i\overline{K}_i = q_i^{-1}E_i \text{ (resp., } K_iF_i\overline{K}_i = q_iF_i),$$

$$K_{i+1}E_i\overline{K}_{i+1} = q_{i+1}E_j \text{ (resp., } K_{i+1}F_i\overline{K}_{i+1} = q_{i+1}^{-1}F_i),$$

then E_i (resp., F_i) is said to be of type II.

Remark 2.3 We define the notation $d_i = (k_i|\overline{k}_i), k_i, \overline{k}_i = 0$ or 1 to represent the type of E_i and F_i. The information before | is related to E_i and the information after | is related to F_i. The notation $d_i = (k_i \mid \overline{k}_i)$ indicates that if k_i or $\overline{k}_i = 1$, then the corresponding generator E_i or F_i is of type I; if k_i or $\overline{k}_i = 0$, then the corresponding generator E_i or F_i is of type II. We say that E_i and F_i are of type d_i if E_i and F_i are of type I or type II according to d_i.

Now we give the definition of the algebra $\mathfrak{w}X_q(B_n)$.

Definition 2.4 *The algebra $\mathfrak{w}X_q(B_n)$ is an associative algebra over $\mathbb{C}$ with 1 generated by $J, K_1, K_2, \ldots, K_n$, $\overline{K}_1, \overline{K}_2, \ldots, \overline{K}_n$, $E_1, \ldots, E_n, F_1, \ldots, F_n$, satisfying the following relations:*

$$K_iK_j = K_jK_i,\ \overline{K}_i\overline{K}_j = \overline{K}_j\overline{K}_i,\ K_i\overline{K}_j = \overline{K}_jK_i, \tag{2.1}$$

$$K_i\overline{K}_i = J = \overline{K}_iK_i,\ K_iJ = JK_i = K_i,\ \overline{K}_iJ = J\overline{K}_i = \overline{K}_i, \tag{2.2}$$

$$E_i, F_i \text{ are of type } d_i, \tag{2.3}$$

$$(q_i - q_{i+1})E_i^2 = (q_i - q_{i+1})F_i^2 = 0, \quad i \neq n, \tag{2.4}$$

$$E_iE_j = E_jE_i,\ F_iF_j = F_jF_i, |i-j| \geq 2, \tag{2.5}$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{\overline{K}_iK_{i+1} - K_i\overline{K}_{i+1}}{q-q^{-1}}, \quad i \neq n, \tag{2.6}$$

$$E_nF_j - F_jE_n = \delta_{n,j}\frac{\overline{K}_n - K_n}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}}, \tag{2.7}$$

$$q_iE_i^2E_{i\pm1} - (1+q_iq_{i+1})E_iE_{i\pm1}E_i + q_{i+1}E_{i\pm1}E_i^2 = 0, \quad i \neq n, \tag{2.8}$$

$$q_iF_i^2F_{i\pm1} - (1+q_iq_{i+1})F_iF_{i\pm1}F_i + q_{i+1}F_{i\pm1}F_i^2 = 0, \quad i \neq n, \tag{2.9}$$

$$E_n^3E_{n-1} - \begin{bmatrix}3\\1\end{bmatrix}_{q_n^{\frac{1}{2}}} E_n^2E_{n-1}E_n + \begin{bmatrix}3\\2\end{bmatrix}_{q_n^{\frac{1}{2}}} E_nE_{n-1}E_n^2 - E_{n-1}E_n^3 = 0, \tag{2.10}$$

$$F_n^3F_{n-1} - \begin{bmatrix}3\\1\end{bmatrix}_{q_n^{\frac{1}{2}}} F_n^2F_{n-1}F_n + \begin{bmatrix}3\\2\end{bmatrix}_{q_n^{\frac{1}{2}}} F_nF_{n-1}F_n^2 - F_{n-1}F_n^3 = 0, \tag{2.11}$$

where $\begin{bmatrix}n\\k\end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$, $[n]_q = \frac{q^n - q^{-n}}{q-q^{-1}}$, $q_i = q$ *or* $-q^{-1}$, $1 \leq i \leq n$.

Remark 2.5 We use the notation $d = (d_1, d_2, \ldots, d_n)$ to denote the type of E_i, F_i in $\mathfrak{w}X_q(B_n)$. The algebra $\mathfrak{w}X_q(B_n)$ is said to be of type d.

Now, we define the comultiplication and counit of $\mathfrak{w}X_q(B_n)$ as follows,

$$\Delta(K_i) = K_i \otimes K_i,\ \Delta(\overline{K}_i) = \overline{K}_i \otimes \overline{K}_i,\ \Delta(J) = J \otimes J,$$

if E_i (resp., F_i) is of type I,

$$\Delta(E_i) = K_i\overline{K}_{i+1} \otimes E_i + E_i \otimes 1\, (i \neq n), \Delta(E_n) = K_n \otimes E_n + E_n \otimes 1$$

$$(\text{resp., } \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_{i+1}\overline{K}_i\, (i \neq n), \Delta(F_n) = 1 \otimes F_n + F_n \otimes \overline{K}_n);$$

if E_i (resp., F_i) is of type II,

$$\Delta(E_i) = K_i\overline{K}_{i+1} \otimes E_i + E_i \otimes J\, (i \neq n), \Delta(E_n) = K_n \otimes E_n + E_n \otimes J;$$

$$(\text{resp., } \Delta(F_i) = J \otimes F_i + F_i \otimes \overline{K}_iK_{i+1}\, (i \neq n), \Delta(F_n) = J \otimes F_n + F_n \otimes \overline{K}_n).$$

$$\varepsilon(K_i) = \varepsilon(\overline{K}_i) = \varepsilon(J) = 1,\ \varepsilon(E_i) = \varepsilon(F_i) = 0.$$

The maps Δ, ε can be extended to $\mathfrak{w}X_q(B_n)$ naturally such that $\mathfrak{w}X_q(B_n)$ is a bialgebra.

Theorem 2.6 *Keeping notations as above and assume that $J \neq 1$, then $\mathfrak{w}X_q(B_n)$ is a weak Hopf algebra with the weak antipode*

$$T(K_i) = \overline{K}_i,\ T(\overline{K}_i) = K_i, T(J) = J,$$

$$T(E_i) = -\overline{K}_iK_{i+1}E_i,\ T(F_i) = -F_iK_i\overline{K}_{i+1}.$$

Moreover, it is not a Hopf algebra.

Proof It is sufficient to prove that T can define a weak antipode of $\mathfrak{w}X_q(B_n)$.

It is easy to see that T is an algebra isomorphism from $\mathfrak{w}X_q(B_n)$ to $\mathfrak{w}X_q(B_n)^{op}$ as algebra. The proof is more or less similar to [7, Theorem 3.1].

Now we need to show that T is a weak antipode, that is,

$$(\mathrm{id} * T * \mathrm{id})(x) = \mathrm{id}(x),\ (T * \mathrm{id} * T)(x) = T(x)$$

for all $x \in \mathfrak{w}X_q(B_n)$.

Let

$$\Delta_2 = (\mathrm{id} \otimes \Delta) \circ \Delta.$$

If E_i is of type I, then

$$\Delta_2(E_i) = K_i\overline{K}_{i+1} \otimes \Delta(E_i) + E_i \otimes \Delta(1) = K_i\overline{K}_{i+1} \otimes K_i\overline{K}_{i+1} \otimes E_i + K_i\overline{K}_{i+1} \otimes E_i \otimes 1 + E_i \otimes 1 \otimes 1.$$

In this case,

$$(\mathrm{id} * T * \mathrm{id})(E_i) = K_i\overline{K}_{i+1}T(K_i\overline{K}_{i+1})E_i + K_i\overline{K}_{i+1}T(E_i) + E_i = E_i = \mathrm{id}(E_i),$$

$$(T * \mathrm{id} * T)(E_i) = T(K_i\overline{K}_{i+1})K_i\overline{K}_{i+1}T(E_i) + T(K_i\overline{K}_{i+1})E_iT(1) + T(E_i) = J^2T(E_i) = T(E_i).$$

If E_i is of type II, then

$$\begin{aligned}\Delta_2(E_i) &= K_i\overline{K}_{i+1} \otimes \Delta(E_i) + E_i \otimes \Delta(J)\\ &= K_i\overline{K}_{i+1} \otimes K_i\overline{K}_{i+1} \otimes E_i + K_i\overline{K}_{i+1} \otimes E_i \otimes 1 + E_i \otimes J \otimes J.\end{aligned}$$

In this case,

$$\begin{aligned}(\mathrm{id} * T * \mathrm{id})(E_i) &= K_i\overline{K}_{i+1}T(K_i\overline{K}_{i+1})E_i + K_i\overline{K}_{i+1}T(E_i) + E_iT(J)J\\ &= E_iJ = \mathrm{id}(E_i)\ (\text{note that } E_iJ = E_i),\end{aligned}$$

$$\begin{aligned}(T * \mathrm{id} * T)(E_i) &= T(K_i\overline{K}_{i+1})K_i\overline{K}_{i+1}T(E_i) + T(K_i\overline{K}_{i+1})E_iT(J) + T(E_i)JT(J)\\ &= J^2T(E_i) = T(E_i).\end{aligned}$$

The other relations can be checked similarly.

On the other hand, similar to [7], one can show that

(a) The comultiplication of generators is a linear expression of generators;

(b) $(T * \mathrm{id})(x)$ or $(\mathrm{id} * T)(x)$ (for x being the generators $K_i, \overline{K}_i, E_i, F_i$) is a central element of $\mathfrak{w}X_q(B_n)$.

Now we assume that

$$\mathrm{id} * T * \mathrm{id}(x) = x,\ T * \mathrm{id} * T(x) = T(x)$$

and

$$\mathrm{id} * T * \mathrm{id}(y) = y,\ T * \mathrm{id} * T(y) = T(y)$$

for $x, y \in \mathfrak{w}X_q(B_n)$, we can prove

$$(\mathrm{id} * T * \mathrm{id})(xy) = xy,\ (T * \mathrm{id} * T)(xy) = T(xy)$$

by induction. This means that T indeed is a weak antipode. Hence $\mathfrak{w}X_q(B_n)$ is a weak Hopf algebra.

Moreover, T is not an antipode. If $T:\mathfrak{w}X_q(B_n)\longrightarrow \mathfrak{w}X_q(B_n)$ is an antipode, then

$$(T*\mathrm{id})(J)=u\varepsilon(J)=(\mathrm{id}*T)(J).$$

We get

$$T(J)J=1=JT(J),$$

and J is invertible. However, $J(1-J)=0$ and $J\neq 1$. That is a contradiction. Hence $\mathfrak{w}X_q(B_n)$ is not a Hopf algebra. □

3. Algebraic structure of $\mathfrak{w}X_q(B_n)$

It is easy to see that the elements J and $1-J$ are a pair of orthogonal central idempotent elements. Let $\omega_q=\mathfrak{w}X_q(B_n)J$, $\overline{\omega}_q=\mathfrak{w}X_q(B_n)(1-J)$. We have

Proposition 3.1 *If the weak Hopf algebra $\mathfrak{w}X_q(B_n)$ is of type d, then $\mathfrak{w}X_q(B_n)=\omega_q\oplus\overline{\omega}_q$. Moreover, ω_q is isomorphic to $X_q(B_n)$ as Hopf algebra.*

Proof Since J is a central idempotent element, we get $\mathfrak{w}X_q(B_n)=\omega_q\oplus\overline{\omega}_q$. The subalgebra ω_q can be viewed as an algebra generated by $E_1J,\dots,E_nJ,F_1J,\dots,F_nJ,K_1,\dots,K_{n+1},\overline{K}_1,\dots,\overline{K}_{n+1}$ with the induced relations. It is easy to see that ω_q is a Hopf subalgebra of $\mathfrak{w}X_q(B_n)$.

Set $\rho: X_q(B_n)\longrightarrow\omega_q$,

$$\rho(K_i')=K_i,\ \rho(K_i'^{-1})=\overline{K}_i,\ \rho(E_j')=E_jJ,\ \rho(F_j')=F_jJ,$$

where $K_i',K_i'^{-1}$ $(1\le i\le n+1)$, E_j',F_j' $(1\le j\le n)$ are generators of $X_q(B_n)$. It is easy to see that ρ is a well-defined surjective algebra homomorphism.

Let $\phi:\mathfrak{w}X_q(B_n)\longrightarrow X_q(B_n)$ be a map given by

$$\phi(1)=1,\phi(J)=1,\phi(E_j)=E_j',\phi(F_j)=F_j',\phi(K_i)=K_i',\phi(\overline{K}_i)=K_i'^{-1}.$$

We can check that ϕ is a well-defined algebra homomorphism. If we consider the restricted homomorphism $\phi|_{\omega_q}$ of ϕ, then we have $\phi|_{\omega_q}\circ\rho=\mathrm{id}_{X_q(B_n)}$. So ρ is injective. Hence, $\omega_q\cong X_q(B_n)$. □

Remark 3.2 As Hopf algebra, we have

$$\mathfrak{w}X_q(B_n)/\langle J-1\rangle\cong X_q(B_n)$$

where $\langle J-1\rangle$ is a two-sides ideal generated by $J-1$.

Recall that the notation $d=(d_1,d_2,\dots,d_n)$, where $d_i=(k_i|\overline{k}_i)$, $k_i,\overline{k}_i=0$ or 1. Let $\sigma=\{i|k_i\neq 0\}$, $\overline{\sigma}=\{i|\overline{k_i}\neq 0\}$ and $X_i=E_i(1-J)$, $Y_j=F_j(1-J)$, where $i\in\sigma,j\in\overline{\sigma}$. If E_i (resp., F_i) is of type I, then $X_i\neq 0$ (resp., $Y_i\neq 0$); If E_i (resp., F_i) is of type II, then $X_i=0$ (resp., $Y_i=0$). In fact, $\overline{\omega}_q$ can be viewed as an algebra generated by $\{X_i,Y_j|i\in\sigma,j\in\overline{\sigma}\}\cup\{1-J\}$ with the relation

$$X_iY_j=Y_jX_i \text{ for all } i\in\sigma,j\in\overline{\sigma}.$$

In the following, we consider the coalgebra structure of $\mathfrak{w}X_q(B_n)$ by Ext quivers. Let C be a graded coalgebra, and

$$C = gr_{\mathcal{F}}(C) = \bigoplus_{n\geq 0}(C_n/C_{n-1}),$$

where $C_n = \Delta^{-1}(C_{n-1}\otimes C + C\otimes C_0)$ and $\mathcal{F} = \{C_n\}_{n\geq 0}$ is a filtration of coradical of C. Let $C(n) = C_n/C_{n-1}$. We have

$$C = \bigoplus_{n\geq 0} C(n),\ C(0) = \mathbb{C}G(C),\ G(C) = \{g\in C|\Delta(g) = g\otimes g, \varepsilon(g) = 1\}.$$

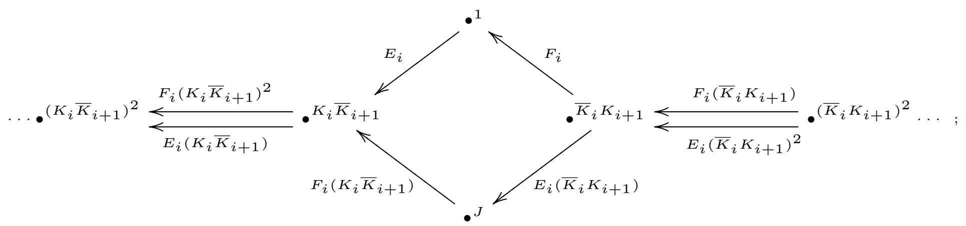
The elements in $G(C)$ are called group-like elements.

$$C(1) = \bigoplus_{g,h\in G(C)} C(1)^{hg},\ C(1)^{hg} = \{c\in C(1)|\Delta(c) = h\otimes c + c\otimes g\},$$

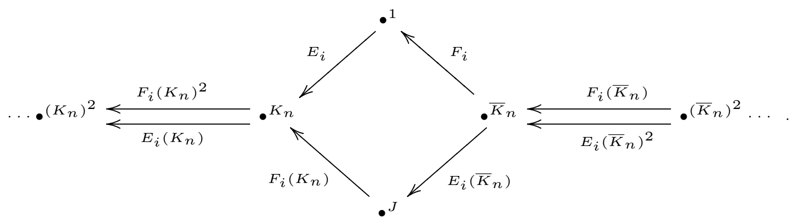
where the elements in $C(1)^{hg}$ are called $(h:g)$-skew primitive elements. Define the quiver $Q = Q(C) = (Q_0, Q_1)$ with the vertex set $Q_0 = G(C)$ and for $g, h\in G(C)$, there are t_{gh} arrows from g to h, where $t_{gh} = \dim_{\mathbb{C}} C(1)^{hg}$. One can refer to [10, 14–17]. In the following quivers, the labelling x of the arrow $\bullet^h \xleftarrow{x} \bullet^g$ means that $0\neq x\in C(1)^{hg}$ consisting of a basis of $C(1)^{hg}$.

Theorem 3.3 *For the weak Hopf algebra $\mathfrak{w}X_q(B_n)$ of type d, the Ext quiver of its coalgebra is a union of some of the following quivers.*

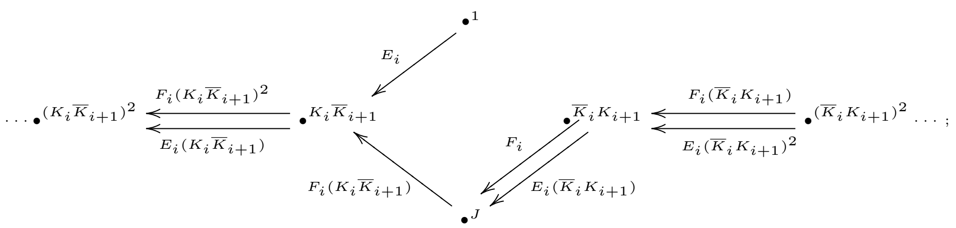
(1) Assume that $d_i = (1|1)$. If $i\neq n$, the corresponding quiver is



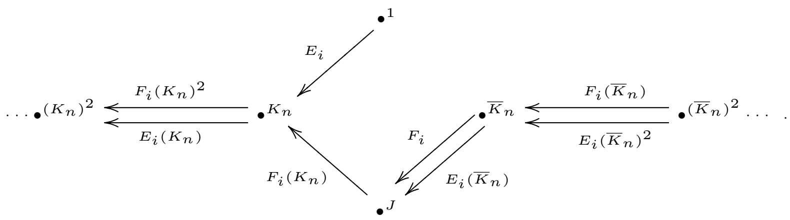
if $i = n$, the corresponding quiver is



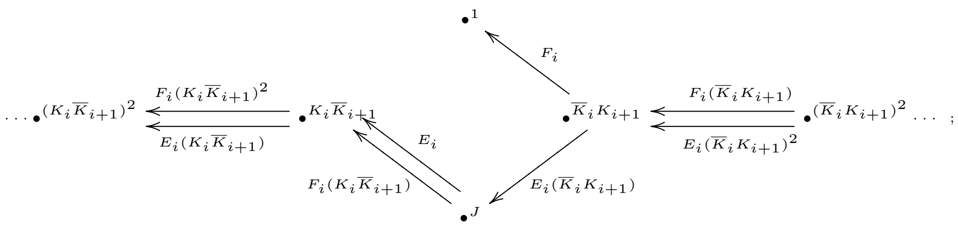
(2) Assume that $d_i = (1|0)$. If $i\neq n$, the corresponding quiver is



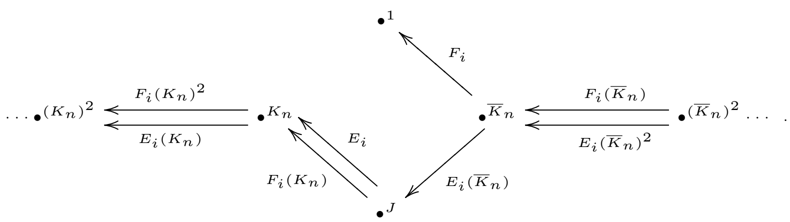
If $i = n$, the corresponding quiver is



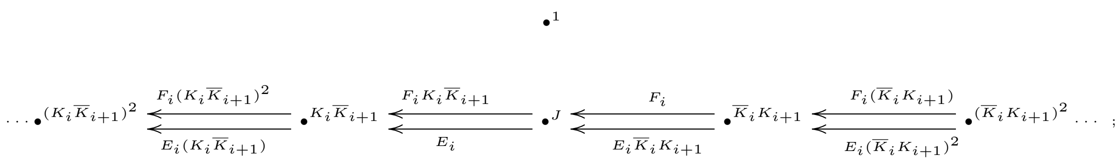
(3) *Assume that $d_i = (0|1)$. If $i \neq n$, the corresponding quiver is*



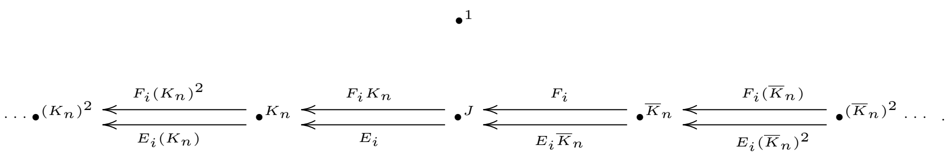
if $i = n$, the corresponding quiver is



(4) *Assume that $d_i = (0|0)$. If $i \neq n$, the corresponding quiver is*



if $i = n$, the corresponding quiver is



Proof (1) In first, it is easy to see that $\mathfrak{w}X_q(B_n)$ is a graded coalgebra, and we get $\mathfrak{w}X_q(B_n) = \bigoplus_{n\geq 0} C(n)$, where $C(n)$ is defined as above. Let h, g be the group-like elements of $\mathfrak{w}X_q(B_n)$, which are generated by $K_i, \overline{K}_i$ $(1 \leq i \leq n)$. Similar to the method in [10] and [15], we can obtain that $((K_i\overline{K}_{i+1})^{m+1} : (K_i\overline{K}_{i+1})^m)$-skew primitive elements, $((K_n)^{m+1} : (K_n)^m)$-skew primitive elements, $((K_{i+1}\overline{K}_i)^m : (K_{i+1}\overline{K}_i)^{m+1})$-skew primitive elements, $((\overline{K}_n)^m : (\overline{K}_n)^{m+1})$-skew primitive elements in $C(1)$ as follows form the arrow set

$$E_i(K_i\overline{K}_{i+1})^m,\ F_i(K_i\overline{K}_{i+1})^{m+1},\ E_i(K_n)^m,\ F_i(K_n)^{m+1},$$

$$F_i(K_{i+1}\overline{K}_i)^m,\ E_i(K_{i+1}\overline{K}_i)^{m+1},\ F_i(\overline{K}_n)^m,\ E_i(\overline{K}_n)^{m+1}.$$

Moreover, we calculate the comultiplication of these skew primitive elements.

$$\Delta(E_i(K_i\overline{K}_{i+1})^m) = (K_i\overline{K}_{i+1})^{m+1}\otimes E_i(K_i\overline{K}_{i+1})^m + E_i(K_i\overline{K}_{i+1})^m\otimes(K_i\overline{K}_{i+1})^m,\quad m\geq 1, i\neq n,$$

$$\Delta(E_i(K_n)^m) = (K_n)^{m+1}\otimes E_i(K_n)^m + E_i(K_n)^m\otimes(K_n)^m,\quad m\geq 1, i=n,$$

$$\Delta(F_i(K_{i+1}\overline{K}_i)^m) = (K_{i+1}\overline{K}_i)^m\otimes F_i(K_{i+1}\overline{K}_i)^m + F_i(K_{i+1}\overline{K}_i)^m\otimes(K_{i+1}\overline{K}_i)^{m+1},\quad m\geq 1, i\neq n,$$

$$\Delta(F_i(\overline{K}_n)^m) = (\overline{K}_n)^m\otimes F_i(\overline{K}_n)^m + F_i(\overline{K}_n)^m\otimes(\overline{K}_n)^{m+1},\quad m\geq 1, i=n,$$

$$\Delta(E_i(K_{i+1}\overline{K}_i)^m) = (K_{i+1}\overline{K}_i)^{m-1}\otimes E_i(K_{i+1}\overline{K}_i)^m + E_i(K_{i+1}\overline{K}_i)^m\otimes(K_{i+1}\overline{K}_i)^m,\quad m\geq 2, i\neq n,$$

$$\Delta(E_i(\overline{K}_n)^m) = (\overline{K}_n)^{m-1}\otimes E_i(\overline{K}_n)^m + E_i(\overline{K}_n)^m\otimes(\overline{K}_n)^m,\quad m\geq 2, i=n,$$

$$\Delta(F_i(K_i\overline{K}_{i+1})^m) = (K_i\overline{K}_{i+1})^m\otimes F_i(K_i\overline{K}_{i+1})^m + F_i(K_i\overline{K}_{i+1})^m\otimes(K_i\overline{K}_{i+1})^{m-1},\quad m\geq 2, i\neq n,$$

$$\Delta(F_i(K_n)^m) = (K_n)^m\otimes F_i(K_n)^m + F_i(K_n)^m\otimes(K_n)^{m-1},\quad m\geq 2, i=n,$$

and

$$\Delta(E_iK_{i+1}\overline{K}_i) = J\otimes E_iK_{i+1}\overline{K}_i + E_iK_{i+1}\overline{K}_i\otimes K_{i+1}\overline{K}_i,\quad i\neq n,$$

$$\Delta(E_i\overline{K}_n) = J\otimes E_i\overline{K}_n + E_i\overline{K}_n\otimes\overline{K}_n,\quad i=n,$$

$$\Delta(F_iK_i\overline{K}_{i+1}) = K_i\overline{K}_{i+1}\otimes F_iK_i\overline{K}_{i+1} + F_iK_i\overline{K}_{i+1}\otimes J,\quad i\neq n,$$

$$\Delta(F_iK_n) = K_n\otimes F_iK_n + F_iK_n\otimes J,\quad i=n.$$

We also know that if E_i is of type I, then

$$\Delta(E_i) = K_i\overline{K}_{i+1}\otimes E_i + E_i\otimes 1\,(i\neq n),\ \Delta(E_n) = K_n\otimes E_n + E_n\otimes 1;$$

if F_i is of type I, then

$$\Delta(F_i) = 1\otimes F_i + F_i\otimes K_{i+1}\overline{K}_i\,(i\neq n),\ \Delta(F_n) = 1\otimes F_n + F_n\otimes\overline{K}_n;$$

if E_i is of type II, then

$$\Delta(E_i) = K_i\overline{K}_{i+1}\otimes E_i + E_i\otimes J\,(i\neq n),\ \Delta(E_n) = K_n\otimes E_n + E_n\otimes J;$$

if F_i is of type II, then

$$\Delta(F_i) = J\otimes F_i + F_i\otimes\overline{K}_iK_{i+1}\,(i\neq n),\ \Delta(F_n) = J\otimes F_n + F_n\otimes\overline{K}_n.$$

Therefore, according to the value of d_i which reflects the type of E_i and F_i, the statements (1)–(4) are obtained directly. □

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