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(α,β)-Soft Ideals of Weak-BCI-Algebras

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Abstract In this paper, we firstly introduce the concept of (α,β)-soft ideals of Weak-BCI-algebras after endowing a parameter set as a Weak-BCI-algebra. When $U=[0,1]$, $\alpha=U$, $\beta=\emptyset$, it becomes the hesitant fuzzy ideals of Weak-BCI-algebras. Then important properties of (α,β)-soft ideals of Weak-BCI-algebras are given. Finally, we investigate the properties of the homomorphism image and inverse image of (α,β)-soft ideals of Weak-BCI-algebras.

Keywords soft set; weak-BCI-algebra; (α,β)-soft ideal; homomorphism

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1. Introduction

The soft set theory was firstly proposed by Molodtsov in 1999 (see [1]), which offers a general mathematical tool for dealing with uncertainties. With the rapid development of the soft set theory, it has been applied to many fields, such as game theory, operations research, decision making problem, information science and so on [2–6].

In 2003, Maji et al. [7] defined the operations of two soft sets and proved some propositions on soft set operations. In 2007, Aktas and Cagman [8] proposed the notion of soft groups and studied the algebraic properties. In 2008, Feng et al. [9] introduced the notion of soft semirings and the notion of soft ideals of soft semirings. Jun [10] introduced the notion of soft BCK/BCI-algebras. Jun and Park [11] discussed the applications of soft sets in ideal theory of BCK/BCI-algebras.

In 2008, Wen [12] proposed the new concept of soft subgroups and normal soft subgroups, endowed the parameter set as algebraic structure of group, and obtained some important conclusions. Inspired by this idea, Liao's team studied a series of new soft set algebras [13–17]. In recent years, the studies of hesitant fuzzy algebras [18–22] are a special case of this soft set algebras (The case of the universe set $U=[0,1]$).

In 2005, Chen and Pu [23] proposed the notion of Weak-BCI-algebras and studied some properties. In this paper, we introduce the notion of (α,β)-soft ideals of Weak-BCI-algebras after endowing the parameter set as a Weak-BCI-algebra. Then the notion of hesitant fuzzy

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ideals of Weak-BCI-algebras is the special case when $U=[0,1]$, $\alpha=U$, $\beta=\emptyset$. We give an example to show that an (α,β)-soft ideal of a Weak-BCI-algebra exists, and it is different from the standard soft ideal of Weak-BCI-algebra and the new type of soft ideal of Weak-BCI-algebra. It is a new algebraic structure. We discuss the properties of (α,β)-soft ideals of Weak-BCI-algebras in various soft set operations. Furthermore, we give the equivalent characterization of (α,β)-soft ideals of Weak-BCI-algebras by using dual soft sets. We give the sufficient condition for (α,β)-soft ideals of Weak-BCI-algebras by using the level set of soft sets. It is shown that the dual soft sets and the level set of soft sets are two different algebraic structures. Finally, the properties of the homomorphism image and inverse image of (α,β)-soft ideals of Weak-BCI-algebras are discussed.

2. Preliminaries

In this section, we give some definitions and theorems which are useful for the later use.

Definition 2.1 ([23]) *An algebra $(X;*,0)$ of type (2,0) is called a Weak-BCI-algebra if the following conditions hold for all $x,y,z\in X$:*

(1) $x*0=x$,

(2) $x*x=0$,

(3) $(x*y)*z=(x*z)*y$.

Definition 2.2 ([22]) *Let X be a reference set. A hesitant fuzzy set on X is defined in terms of a function that when applied to X returns a subset of [0,1], which can be viewed as the following mathematical representation:*

$$H_X:=\{(x,h_X(x)|x\in X\},\quad \text{where}\ \ h_X:X\to P([0,1]).$$

Definition 2.3 ([23]) *A nonempty subset I of a Weak-BCI-algebra X is called an ideal of X if it satisfies the following axioms::*

(1) $0\in I$,

(2) $(\forall x\in X)(\forall y\in I)(x*y\in I\Rightarrow x\in I)$.

Obviously, 0 and X are both ideals of X.

Theorem 2.4 ([16]) *Let $\{A_i\}_{i\in I}$ be subalgebras of a Weak-BCI-algebra X. Then $\cap_{i\in I}A_i$ is a subalgebra of X.*

Definition 2.5 *Let X and Y be Weak-BCI-algebras. A mapping $f:X\to Y$ is called a homomorphism from X to Y if $f(x*y)=f(x)*f(y)$, for all $x,y\in X$. If f is a surjection, then f is called a surjective homomorphism.*

Definition 2.6 ([1]) *A pair (F,A) is called a soft set over U, where F is a mapping given by $F:A\to P(U)$.*

Definition 2.7 ([7]) *For two soft sets (F,A) and (G,B) over a common universe U, we say that*

(F,A) is a soft subset of (G,B) if

(1) $A \subseteq B$;

(2) $\forall x \in A, F(x) = G(x)$.

We write $(F,A) \widetilde{\subseteq} (G,B)$.

Theorem 2.8 ([16]) *Let X_1, X_2 be Weak-BCI-algebras. If $(x_1,x_2)*(y_1,y_2) = (x_1*y_1, x_2*y_2)$, for all $(x_1,x_2),(y_1,y_2) \in X_1 \times X_2$, then $(X_1 \times X_2; *, (0,0))$ is a Weak-BCI-algebra.*

Definition 2.9 ([7]) *If (F,A) and (G,B) are two soft sets over a common universe U, then (F,A) AND (G,B) denoted by $(F,A)\wedge(G,B)$ is defined by $(F,A)\wedge(G,B) = (H, A\times B)$, where $H(x,y) = F(x)\cap G(y)$, for all $(x,y)\in A\times B$.*

Definition 2.10 ([24]) *If (F,A) and (G,B) are two soft sets over a common universe U, then restricted intersection of (F,A) and (G,B) is defined to be the soft set (H,C) satisfying the conditions:*

(1) $C = A\cap B$;

(2) $\forall x \in C$, $H(x) = F(x)\cap G(x)$.

In this case, we write $(H,C) = (F,A)\cap_R (G,B)$.

Definition 2.11 ([12]) *Let $H : X \to P(U), g \mapsto H(g)$ be a soft set. Then $A_H : U \to P(X)$, $x \mapsto A_H(x) = \{g | x \in H(g)\}$ is called a dual soft set of H. Let $A : U \to P(X)$ be a soft set. Then $H_A : X \to P(U)$, $g \mapsto H_A(g) = \{x | g \in A(x)\}$ is called a dual soft set of A.*

Definition 2.12 ([13]) *Let X_1, X_2 be Weak-BCI-algebras, U be an initial universe set, $P(U)$ be the power set of U, $f : X_1 \to X_2$ be a mapping, and $H_1 : X_1 \to P(U)$, $H_2 : X_2 \to P(U)$ be soft sets. Define:*

$$f(H_1)(x_2) = \begin{cases} \bigcup\limits_{f(x_1)=x_2} H_1(x_1), & f^{-1}(x_2) \neq \emptyset, \\ \emptyset, & f^{-1}(x_2) = \emptyset, \end{cases}$$

and $f^{-1}(H_2)(x_1) = H_2(f(x_1))$. Then $f(H_1)$ and $f^{-1}(H_2)$ are soft sets of X_2 and X_1, respectively. $f(H_1)$ is called the image of H_1 by f, and $f^{-1}(H_2)$ is called the preimage of H_2 by f.

3. (α,β)-soft ideal of Weak-BCI-algebras

In this section, the definition of (α,β)-soft ideal of Weak-BCI-algebras is introduced and related properties are investigated.

Definition 3.1 *Let X be a Weak-BCI-algebra, and (F,A) be a soft set on X. If $F(x)$ is an ideal of X for all $x\in A$, then (F,A) is called a standard soft ideal of X.*

This standard soft ideal of X has not been studied.

Definition 3.2 *Let X be a Weak-BCI-algebra, $H : X \to P(U)$ be a soft set. H is called a new type of soft ideal of X, denoted by (H,X), if the following conditions hold for all $x,y\in X$:*

(1) $H(0) \supseteq H(x)$;

(2) $H(x) \supseteq H(x * y) \cap H(y)$.

Definition 3.3 *Let U be an initial universe set, $\alpha, \beta \subseteq U, \beta \subset \alpha$, X be a Weak-BCI-algebra, $H : X \to P(U)$ be a soft set. H is called an (α, β)-soft ideal of X, denoted by $(H_{(\alpha,\beta)}, X)$, if the following conditions hold for all $x, y \in X$:*

(1) $H(0) \cup \beta \supseteq H(x) \cap \alpha$;

(2) $H(x) \cup \beta \supseteq H(x * y) \cap H(y) \cap \alpha$.

Without causing confusion, it can be abbreviated as (H, X).

(α, β)-soft ideal of X is nontrivial generalization of a new type of soft ideal of X.

When $U = [0, 1]$, $\alpha = U$, $\beta = \emptyset$, accordingly, we get the definition of a hesitant fuzzy ideal of Weak-BCI-algebra as follows:

Definition 3.4 *Let X be a Weak-BCI-algebra, $H : X \to P([0, 1])$ be a hesitant fuzzy set. H is called a hesitant fuzzy ideal of X, if the following conditions hold for all $x, y \in X$:*

(1) $H(0) \supseteq H(x)$;

(2) $H(x) \supseteq H(x * y) \cap H(y)$.

The following example shows the existence of (α, β)-soft ideal of X, and it is different from the usual soft ideal and the new type of soft ideal of Weak-BCI-algebra. It is a new soft algebraic structure.

Example 3.5 Let $U = X = \{0, 1, 2, 3\}$, $\alpha = \{0, 1\}$, $\beta = \{1\}$, and the operation "$*$" be given by Table 1. It can be verified that $(X; *, 0)$ is a Weak-BCI-algebra. Let $H : X \to P(U)$, $H(0) = \{0, 1\}$, $H(1) = \{1, 2, 3\}$, $H(2) = \{1, 2\}$, $H(3) = \{0, 3\}$. Then H is an (α, β)-soft ideal of X by definition 3.3. Since $H(0) = \{0, 1\}$ is not an ideal of X, it follows from definition 3.1 that H is not a standard soft ideal of X. And since $H(0) \supseteq H(1)$ is invalid, it follows from definition 3.2 that H is not a new type of soft ideal of X.

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	1
2	2	0	0	1
3	3	3	3	0

Table 1 The operation table of "$*$"

Theorem 3.6 *Let X_1, X_2 be Weak-BCI-algebras, X_1 be a subalgebra of X_2, (H_1, X_1), (H_2, X_2) be two soft sets, and H_2 be an (α, β)-soft ideal of X_2. If $(H_1, X_1) \widetilde{\subseteq} (H_2, X_2)$, then H_1 is an (α, β)-soft ideal of X_1.*

Proof Obviously, $0 \in X_1$. If $(H_1, X_1) \widetilde{\subseteq} (H_2, X_2)$, then $H_1(0) = H_2(0)$. Since H_2 is an (α, β)-soft ideal of X_2, for all $x \in X_1 \subseteq X_2$, we have

$$H_1(0) \cup \beta = H_2(0) \cup \beta \supseteq H_2(x) \cap \alpha = H_1(x) \cap \alpha. \quad (3.1)$$

And for all $x,y\in X_1$, since X_1 is a subalgebra of X_2, we have $x*y\in X_1$. If $(H_1,X_1)\widetilde{\subseteq}(H_2,X_2)$, then $H_1(x)=H_2(x)$, $H_1(x*y)=H_2(x*y), H_1(y)=H_2(y)$, hence

$$\begin{aligned} H_1(x)\cup\beta =&H_2(x)\cup\beta\supseteq H_2(x*y)\cap H_2(y)\cap\alpha\\ =&H_1(x*y)\cap H_1(y)\cap\alpha. \end{aligned}\tag{3.2}$$

Therefore, H_1 is an (α,β)-soft ideal of X_1. □

Theorem 3.7 *Let X_1, X_2 be Weak-BCI-algebras, H_1 and H_2 be (α,β)-soft ideal of X_1 and X_2, respectively. If $(H, X_1\times X_2)=(H_1,X_1)\wedge(H_2,X_2)$, then H is an (α,β)-soft ideal of $X_1\times X_2$.*

Proof By Theorem 2.8, $X_1\times X_2$ is a Weak-BCI-algebra. For all $x\in X_1\times X_2$, $x=(x_1,x_2)$, $x_i\in X_i$ $(i=1,2)$, we have

$$H(x)\cap\alpha=(H_1(x_1)\cap H_2(x_2))\cap\alpha\subseteq(H_1(0)\cup\beta)\cap(H_2(0)\cup\beta)=H(0)\cup\beta.\tag{3.3}$$

And for all $x,y\in X_1\times X_2$, $x=(x_1,x_2)$, $y=(y_1,y_2)$, $x_i,y_i\in X_i$ $(i=1,2)$, we have

$$\begin{aligned} H(x)\cup\beta =&(H_1(x_1)\cap H_2(x_2))\cup\beta\\ \supseteq&(H_1(x_1*y_1)\cap H_1(y_1)\cap\alpha)\cap(H_2(x_2*y_2)\cap H_2(y_2)\cap\alpha)\\ =&H((x_1*y_1,x_2*y_2))\cap H((y_1,y_2))\cap\alpha\\ =&H(x*y)\cap H(y)\cap\alpha. \end{aligned}\tag{3.4}$$

Therefore, H is an (α,β)-soft ideal of $X_1\times X_2$. □

Theorem 3.8 *Let X be a Weak-BCI-algebra, X_1, X_2 be subalgebras of X. Let H_1 and H_2 be (α,β)-soft ideal of X_1 and X_2, respectively. If $(H,X_1\cap X_2)=(H_1,X_1)\cap_R(H_2,X_2)$, then H is an (α,β)-soft ideal of $X_1\cap X_2$.*

Proof It follows from Theorem 2.4 that $X_1\cap X_2$ is a subalgebra of X. For all $x\in X_1\cap X_2$, we have

$$H(x)\cap\alpha =H_1(x)\cap H_2(x)\cap\alpha\subseteq(H_1(0)\cup\beta)\cap(H_2(0)\cup\beta)=H(0)\cup\beta.\tag{3.5}$$

And for all $x,y\in X_1\cap X_2$, we have

$$\begin{aligned} H(x)\cup\beta =&(H_1(x)\cap H_2(x))\cup\beta\\ \supseteq&(H_1(x*y)\cap H_1(y)\cap\alpha)\cap(H_2(x*y)\cap H_2(y)\cap\alpha)\\ =&H(x*y)\cap H(y)\cap\alpha. \end{aligned}\tag{3.6}$$

Therefore, H is an (α,β)-soft ideal of $X_1\cap X_2$. □

Theorem 3.9 *Let X be a Weak-BCI-algebra, and $\alpha-\beta\neq\emptyset$. Then the following statements hold:*

(1) $H: X\to P(U)$ is an (α,β)-soft ideal of X if and only if for all $u\in\alpha-\beta$, $A_H(u)\neq\emptyset$ is an ideal of X.

(2) Let $A: U\to P(X)$ be a soft set. For all $u\in\alpha-\beta$, $A(u)\neq\emptyset$ is an ideal of X if and only

if H_A is an (α,β)-soft ideal of X.

Proof (1) Suppose that H is an (α,β)-soft ideal of X. For all $u\in\alpha-\beta$, if $A_H(u)=\{x|u\in H(x)\}\neq\emptyset$, then for all $x\in A_H(u)$, we have $u\in H(x)\cap\alpha\subseteq H(0)\cup\beta$, thus $u\in H(0)$, that is, $0\in A_H(u)$. If $x*y\in A_H(u)$ and $y\in A_H(u)$, then $u\in H(x*y)\cap H(y)\cap\alpha\subseteq H(x)\cup\beta$, thus $u\in H(x)$, that is $x\in A_H(u)$. Therefore, $A_H(u)$ is an ideal of X.

Conversely, suppose that for all $u\in\alpha-\beta, A_H(u)\neq\emptyset$ is an ideal of X. For all $x\in X$, if $H(x)\cap\alpha=\emptyset$, then $H(x)\cap\alpha\subseteq H(0)\cup\beta$; if $H(x)\cap\alpha\neq\emptyset$, then for all $u\in H(x)\cap\alpha$, since $0\in A_H(u)$, we have $u\in H(0)\subseteq H(0)\cup\beta$, hence, $H(x)\cap\alpha\subseteq H(0)\cup\beta$. And for all $x,y\in X$, if $H(x*y)\cap H(y)\cap\alpha=\emptyset$, then $H(x*y)\cap H(y)\cap\alpha\subseteq H(x)\cup\beta$; if $H(x*y)\cap H(y)\cap\alpha\neq\emptyset$, then for all $u\in H(x*y)\cap H(y)\cap\alpha$, we have $x*y\in A_H(u)$ and $y\in A_H(u)$ which imply that $x\in A_H(u)$, thus $u\in H(x)$, therefore $H(x*y)\cap H(y)\cap\alpha\subseteq H(x)\cup\beta$.

(2) Can be proved by a similar way. □

Definition 3.10 *Let X be a Weak-BCI-algebra, $H:X\to P(U)$ be a soft set. The set $H_\gamma=\{x|H(x)\supseteq\gamma,\gamma\in P(U)\}$ is called a γ-level set of H.*

Theorem 3.11 *Let X be a Weak-BCI-algebra, $H:X\to P(U)$ be a soft set. Then H is an (α,β)-soft ideal of X if $H_\gamma\neq\emptyset$ is an ideal of X, for all $\gamma\in P(U)$.*

Proof For all $x\in X$, if $H(x)\cap\alpha=\emptyset$, then $H(x)\cap\alpha\subseteq H(0)\cup\beta$. If $H(x)\cap\alpha\neq\emptyset$, let $H(x)\cap\alpha=\gamma$, since $H_\gamma\neq\emptyset$ is an ideal of X, we have $0\in H_\gamma$, that is $H(0)\supseteq\gamma$, hence $H(0)\cup\beta\supseteq\gamma=H(x)\cap\alpha$. For all $x,y\in X$, if $H(x*y)\cap H(y)\cap\alpha=\emptyset$, then $H(x*y)\cap H(y)\cap\alpha\subseteq H(x)\cup\beta$. If $H(x*y)\cap H(y)\cap\alpha\neq\emptyset$, let $H(x*y)\cap H(y)\cap\alpha=\gamma$. We have $H(x*y)\supseteq\gamma$ and $H(y)\supseteq\gamma$. Then $x*y\in H_\gamma$ and $y\in H_\gamma$. Since $H_\gamma\neq\emptyset$ is an ideal of X, we have $x\in H_\gamma$, that is $H(x)\supseteq\gamma$, hence $H(x)\cup\beta\supseteq H(x*y)\cap H(y)\cap\alpha$. Therefore, H is an (α,β)-soft ideal of X. □

The converse of Theorem 3.11 may not be true in general (see Example 3.12).

Example 3.12 Let U, X, α, β,H be the same as Example 3.5. Assume that $\gamma=\{1,2\}$. Then $H_\gamma=\{1,2\}$ is not an ideal of X.

From Theorems 3.9 and 3.11, we know that the dual soft sets and the level set of soft sets are two different algebraic structures.

4. Image and preimage of (α,β)-soft ideal of Weak-BCI-algebras

In this section, the properties of image and preimage of (α,β)-soft ideal of Weak-BCI-algebras are investigated.

Theorem 4.1 *Let X_1,X_2 be Weak-BCI-algebras, U be an initial universe set, $f:X_1\to X_2$ be a homomorphism from X_1 to X_2, $H_1:X_1\to P(U)$, $H_2:X_2\to P(U)$ be soft sets. If H_2 is an (α,β)-soft ideal of X_2, then $f^{-1}(H_2)$ is an (α,β)-soft ideal of X_1.*

Proof If H_2 is an (α,β)-soft ideal of X_2, then for all $x_1 \in X_1$, let $f(x_1) = y_1 \in X_2$. We have

$$\begin{aligned} f^{-1}(H_2)(x_1) \cap \alpha = H_2(f(x_1)) \cap \alpha = H_2(y_1) \cap \alpha \subseteq H_2(0) \cup \beta \\ = H_2(f(0)) \cup \beta = f^{-1}(H_2)(0) \cup \beta. \end{aligned} \tag{4.1}$$

For all x_1, $x_2 \in X_1$, let $f(x_1) = y_1, f(x_2) = y_2 \in X_2$. We have

$$\begin{aligned} & f^{-1}(H_2)(x_1 * x_2) \cap f^{-1}(H_2)(x_2) \cap \alpha = H_2(f(x_1 * x_2)) \cap H_2(f(x_2)) \cap \alpha \\ & = H_2(f(x_1) * f(x_2)) \cap H_2(f(x_2)) \cap \alpha = H_2(y_1 * y_2) \cap H_2(y_2) \cap \alpha \\ & \subseteq H_2(y_1) \cup \beta = H_2(f(x_1)) \cup \beta = f^{-1}(H_2)(x_1) \cup \beta. \end{aligned} \tag{4.2}$$

Therefore, $f^{-1}(H_2)$ is an (α,β)-soft ideal of X_1. $\square$

Theorem 4.2 *Let X_1, X_2 be Weak-BCI-algebras, U be an initial universe set. $f : X_1 \to X_2$ be a surjective homomorphism from X_1 to X_2, $H_1 : X_1 \to P(U)$, $H_2 : X_2 \to P(U)$ be soft sets. Then H_2 is an (α,β)-soft ideal of X_2 if and only if $f^{-1}(H_2)$ is an (α,β)-soft ideal of X_1.*

Proof Necessity is given by Theorem 4.1.

Conversely, assume that $f^{-1}(H_2)$ is an (α,β)-soft ideal of X_1. For all $y_1 \in X_2$, since f is a surjective homomorphism from X_1 to X_2, there exists $x_1 \in X_1$, such that $f(x_1) = y_1$, hence

$$\begin{aligned} H_2(y_1) \cap \alpha = H_2(f(x_1)) \cap \alpha = f^{-1}(H_2)(x_1) \cap \alpha \\ \subseteq f^{-1}(H_2)(0) \cup \beta = H_2(f(0)) \cup \beta = H_2(0) \cup \beta. \end{aligned} \tag{4.3}$$

For all y_1, $y_2 \in X_2$, since f is a surjective homomorphism from X_1 to X_2, there exists x_1, $x_2 \in X_1$, such that $f(x_1) = y_1, f(x_2) = y_2$, $y_1 * y_2 = f(x_1) * f(x_2) = f(x_1 * x_2)$, hence,

$$\begin{aligned} & H_2(y_1 * y_2) \cap H_2(y_2) \cap \alpha = H_2(f(x_1 * x_2)) \cap H_2(f(x_2)) \cap \alpha \\ & = f^{-1}(H_2)(x_1 * x_2) \cap f^{-1}(H_2)(x_2) \cap \alpha \subseteq f^{-1}(H_2)(x_1) \cup \beta \\ & = H_2(f(x_1)) \cup \beta = H_2(y_1) \cup \beta. \end{aligned} \tag{4.4}$$

Therefore, H_2 is an (α,β)-soft ideal of X_2. $\square$

Definition 4.3 *Let X_1, X_2 be two sets, $f : X_1 \to X_2$ be a mapping from X_1 to X_2, and H_1 be a soft set on X_1. If $H_1(x) = H_1(y)$ when $f(x) = f(y)$, for all $x, y \in X_1$, then H_1 is said to be invariant about f.*

Theorem 4.4 *Let X_1, X_2 be Weak-BCI-algebras, U be an initial universe set, $f : X_1 \to X_2$ be a homomorphism from X_1 to X_2, $H_1 : X_1 \to P(U)$ be a soft set and H_1 be invariant about f. If $f(H_1)$ is an (α,β)-soft ideal of X_2, then H_1 is an (α,β)-soft ideal of X_1.*

Proof For all $x_1 \in X_1$, let $f(x_1) = y_1 \in X_2$. Since $f(H_1)$ is an (α,β)-soft ideal of X_2 and H_1

is invariant about f, we have

$$\begin{aligned} H_1(x_1)\cap\alpha=&\bigcup_{f(x)=y_1}H(x)\cap\alpha=f(H_1)(y_1)\cap\alpha\\ &\subseteq f(H_1)(0)\cup\beta=\bigcup_{f(x)=0}H_1(x)\cup\beta=H_1(0)\cup\beta. \end{aligned}\tag{4.5}$$

For all $x_1,x_2\in X_1$, if $H_1(x_1*x_2)\cap H_1(x_2)\cap\alpha=\emptyset$, then $H_1(x_1*x_2)\cap H_1(x_2)\cap\alpha\subseteq H_1(x_1)\cup\beta$. If $H_1(x_1*x_2)\cap H_1(x_2)\cap\alpha\neq\emptyset$, let $f(x_1)=y_1,f(x_2)=y_2$. Since f is a homomorphism from X_1 to X_2, we have $f(x_1*x_2)=f(x_1)*f(x_2)=y_1*y_2$. Since $f(H_1)$ is an (α,β)-soft ideal of X_2 and H_1 is invariant about f, we have

$$\begin{aligned} H_1(x_1*x_2)\cap H_1(x_2)\cap\alpha=&(\bigcup_{f(x)=y_1*y_2}H_1(x))\cap(\bigcup_{f(x)=y_2}H_1(x))\cap\alpha\\ =&f(H_1)(y_1*y_2)\cap f(H_1)(y_2)\cap\alpha\\ \subseteq&f(H_1)(y_1)\cup\beta\\ =&\bigcup_{f(x)=y_1}H_1(x)\cup\beta=H_1(x_1)\cup\beta. \end{aligned}\tag{4.6}$$

Therefore, H_1 is an (α,β)-soft ideal of X_1. $\Box$

Theorem 4.5 *Let X_1, X_2 be Weak-BCI-algebras, U be an initial universe set, $f:X_1\to X_2$ be a surjective homomorphism from X_1 to X_2, $H_1:X_1\to P(U)$ be a soft set and H_1 be invariant about f. Then $f(H_1)$ is an (α,β)-soft ideal of X_2 if and only if H_1 is an (α,β)-soft ideal of X_1.*

Proof Necessity is given by Theorem 4.4.

Conversely, assume that H_1 is an (α,β)-soft ideal of X_1. For all $y_1\in X_2$, since f is a surjective homomorphism, there exists $x_1\in X_1$, such that $y_1=f(x_1)$. Since H_1 is invariant about f, we have

$$f(H_1)(y_1)\cap\alpha=\bigcup_{f(x)=y_1}H_1(x)\cap\alpha\subseteq H_1(0)\cup\beta\subseteq\bigcup_{f(x)=0}H_1(x)\cup\beta=f(H_1)(0)\cup\beta.\tag{4.7}$$

For all y_1, $y_2\in X_2$, if $f(H_1)(y_1*y_2)\cap f(H_1)(y_2)\cap\alpha=\emptyset$, then $f(H_1)(y_1*y_2)\cap f(H_1)(y_2)\cap\alpha\subseteq f(H_1)(y_1)\cup\beta$. If $f(H_1)(y_1*y_2)\cap f(H_1)(y_2)\cap\alpha\neq\emptyset$, since f is a surjective homomorphism, there exists x_1, $x_2\in X_1$, such that $f(x_1)=y_1$, $f(x_2)=y_2$ and $f(x_1*x_2)=f(x_1)*f(x_2)=y_1*y_2$. Since H_1 is invariant about f, we have

$$f(H_1)(y_1*y_2)=\bigcup_{f(x)=y_1*y_2}H_1(x)=H_1(x_1*x_2),\ f(H_1)(y_2)=\bigcup_{f(x)=y_2}H_1(x)=H_1(x_2),$$

hence,

$$\begin{aligned} f(H_1)(y_1*y_2)\cap f(H_1)(y_2)\cap\alpha=&H_1(x_1*x_2)\cap H_1(x_2)\cap\alpha\\ \subseteq&H_1(x_1)\cup\beta=f(H_1)(y_1)\cup\beta. \end{aligned}\tag{4.8}$$

Therefore, $f(H_1)$ is an (α,β)-soft ideal of X_2. $\Box$

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