Journal of Mathematical Research with Applications
Jan., 2019, Vol. 39, No. 1, pp. 10–22
DOI:10.3770/j.issn:2095-2651.2019.01.002
Http://jmre.dlut.edu.cn

A Fourth-Order Convergent Iterative Method by Means of Thiele's Continued Fraction for Root-Finding Problem

Shengfeng LI

Institute of Applied Mathematics, Bengbu University, Anhui 233030*, P. R. China*

Abstract In this paper, we propose a new single-step iterative method for solving non-linear equations in a variable. This iterative method is derived by using the approximation formula of truncated Thiele's continued fraction. Analysis of convergence shows that the order of convergence of the introduced iterative method for a simple root is four. To illustrate the efficiency and performance of the proposed method we give some numerical examples.

Keywords non-linear equation; Thiele's continued fraction; Viscovatov algorithm; iterative method; order of convergence

MR(2010) Subject Classification 30B70; 65H05; 26A18

1. Introduction

Solving a non-linear equation $f(x) = 0$ in a single variable efficiently is a main research direction in numerical analysis and has lots of applications in the field of natural science and engineering. As is known to all, it is impossible to solve these equations analytically in most of the cases. Hence, when an analytic solution, or root, of the equation is difficult to obtain, we can employ an numerical iterative scheme to find approximate solution of the non-linear equation. To find a single solution x^* of non-linear equation $f(x) = 0$, where $f : X \to R$, $X \subseteq R$, is a scalar function on an interval X, a most extensively used and best-known iterative method for solving a non-linear equation is Newton's method (NM for short) as below

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}. \tag{1.1}$$

There are many ways of obtaining Newton's method (1.1). Based on Taylor polynomial, let us recall how to derive Newton's method. Suppose that $f \in C^n[a, b]$, $[a, b] \subseteq X$. Expanding $f(x)$ into Taylor series about the point $x_k \in [a, b]$ yields

$$f(x) = f(x_k) + (x - x_k)f'(x_k) + \frac{1}{2!}(x - x_k)^2 f''(x_k) + \cdots .$$

Received April 4, 2018; Accepted August 12, 2018
Supported by the National Natural Science Foundation of China (Grant No. 11571071), the Natural Science Key Foundation of Education Department of Anhui Province (Grant No. KJ2013A183), the Project of Leading Talent Introduction and Cultivation in Colleges and Universities of Education Department of Anhui Province (Grant No. gxfxZD2016270) and the Incubation Project of the National Scientific Research Foundation of Bengbu University (Grant No. 2018GJPY04).
E-mail address: lsf7679@163.com; lsf@bbc.edu.cn

Since $f(x^*) = 0$, the above expansion with $x = x^*$ gives

$$0 = f(x_k) + (x^* - x_k)f'(x_k) + \frac{1}{2!}(x^* - x_k)^2 f''(x_k) + \cdots . \tag{1.2}$$

Newton's method is proposed by recognizing that if $|x^* - x_k|$ is small, then the term involving $(x^* - x_k)^2$ is much smaller. Therefore, if $f'(x_k) \neq 0$, by substituting the linear part of above expansion (1.2) for the function $f(x)$, we can obtain the following approximate expression of the equation $f(x) = 0$.

$$0 \approx f(x_k) + (x^* - x_k)f'(x_k). \tag{1.3}$$

Solving for x^* gives

$$x^* \approx x_k - \frac{f(x_k)}{f'(x_k)}.$$

This sets the stage for Newton's method. Given an initial approximation value $x_0 \in [a, b]$, we can obtain the iterative sequence $\{x_k\}_{k=0}^{\infty}$ by

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$

for all $k \geq 1$. The order of convergence of Newton's method (1.1) for root-finding of non-linear equation is quadratical [1, 2].

In the last few decades, the problems about finding an approximation to the root of an equation have been extensively studied. Some surveys and complete literatures for this direction could be found in Argyros [3], Abbasbandy [4], Chun [5], Kou et al. [6], Wang et al. [7], Noor and Shah [8], Shah and Noor [9], Kwun et al. [10] and the references therein. Halley's method (HM for short) is another well-known iterative method for solving a non-linear equation and its iterative scheme is written as

$$x_{k+1} = x_k - \frac{2f(x_k)f'(x_k)}{2f'^2(x_k) - f(x_k)f''(x_k)}. \tag{1.4}$$

Similarly, we can derive Halley's method by using Taylor polynomials for $f(x)$ about $x_k \in [a, b]$. Thinking back the approximate expression (1.3), it can be rewritten as

$$x^* - x_k \approx -\frac{f(x_k)}{f'(x_k)}. \tag{1.5}$$

By substituting the approximate expression (1.5) into the equation (1.2), we get the following second Taylor polynomials

$$0 \approx f(x_k) + (x^* - x_k)f'(x_k) - \frac{1}{2}(x^* - x_k)f''(x_k)\frac{f(x_k)}{f'(x_k)}. \tag{1.6}$$

Solving for x^* of the approximate expression (1.6) gives

$$x^* \approx x_k - \frac{2f(x_k)f'(x_k)}{2f'^2(x_k) - f(x_k)f''(x_k)}.$$

This sets the stage for Halley's method, which starts with an initial approximation value x_0 and generates the iterative sequence $\{x_k\}_{k=0}^{\infty}$ by Halley's iterative scheme (1.4), for $k \geq 1$. The order of convergence of Halley's method (1.4) is cubic [3].

Many iterative methods with high-order convergence are introduced in some literatures [4–7]. Abbasbandy presented a new iterative method in [4], for convenience, we call the method as Abbasbandy's method (AM for short) provisionally. The method has the following iterative scheme

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} - \frac{f^2(x_k)f''(x_k)}{2f'^3(x_k)} - \frac{f^3(x_k)f'''(x_k)}{6f'^4(x_k)}. \tag{1.7}$$

It is pointed out in the literature [4] that the order of convergence of AM is nearly supercubic. Moreover, Matinfar et al. put forward some higher-order methods in [11, 12], but most of which are multi-step iterative methods. Kou et al. [13] suggested a class of methods with fourth-order convergence. These fourth-order convergent iterative schemes can be written as follows:

$$x_{k+1} = x_k - (1 + \frac{1}{2}\bar{L}_f(x_k) + \frac{1}{2}\bar{L}_f(x_k)^2 + \gamma\bar{L}_f(x_k)^3 + O(\bar{L}_f(x_k)^4))\frac{f(x_k)}{f'(x_k)}, \tag{1.8}$$

where $\gamma \in R$ and $\bar{L}_f(x_n)$ is defined by the following equation

$$\bar{L}_f(x_k) = \frac{f''(x_k - f(x_k)/(3f'(x_k)))f(x_k)}{f'(x_k)^2}. \tag{1.9}$$

For the equation (1.8), it gives the general form of a class of iterative methods. For the sake of comparison later, we select a specific scheme from these iterative schemes (1.8) as below:

$$x_{k+1} = x_k - \frac{2}{1 + \sqrt{1 - 2\bar{L}_f(x_k)}}\frac{f(x_k)}{f'(x_k)}. \tag{1.10}$$

The iterative scheme (1.10) is called Kou's method (KM for short).

It is well known that continued fraction is a special type of rational approximations, which has many applications in various computation problems of science and technology. In this paper, based on Thiele's continued fraction, a new single-step iterative method can be constructed and be shown that the order of convergence of the proposed method is at least four. Some numerical examples are given to demonstrate the performance and to show that the iteration scheme is efficient and superior to other compared methods to some extent.

This paper is organized as follows. In Section 2 we present some preliminaries for Thiele's continued fraction and iterative method. In Section 3, we derive a new iterative scheme based on Thiele's continued fraction. In Section 4, we investigate the convergence analysis of the iterative method. In Section 5 we give numerical examples. In Section 6 we draw conclusions from the experiment results of numerical examples.

2. Preliminaries

In this section, we briefly recall some basic definitions and results for Thiele's continued fraction and the relative speed of convergence of an iterative scheme. Some surveys and complete literatures for continued fraction and the speed of convergence of the iterative method could be found in Tan [14, 15], Kincaid et al. [1] and Burden et al. [2]. For simplicity throughout this paper we let R stand for the set of real numbers.

Definition 2.1 *Suppose that* $\{x_i | x_i \in R,\ i = 0, 1, 2, \dots\}$ *and* $\{a_j | a_j \in R,\ j = 0, 1, 2, \dots\}$ *are* *two sets of real numbers. The following continued fraction*

$$a_0 + \frac{x - x_0|}{|\ a_1} + \frac{x - x_1|}{|\ a_2} + \cdots + \frac{x - x_{n-1}|}{|\ a_n} + \cdots \tag{2.1}$$

is called Thiele's continued fraction [14, 15].

Definition 2.2 *For (2.1) in Definition 2.1, the following continued fraction*

$$a_0 + \frac{x - x_0|}{|\ a_1} + \frac{x - x_1|}{|\ a_2} + \cdots + \frac{x - x_{n-1}|}{|\ a_n} \tag{2.2}$$

is called the n-th truncated Thiele's continued fraction [14, 15].

For the relation between the coefficients $a_i \in R$, $i = 0, 1, 2, \ldots$, of Thiele's continued fraction and the coefficients $C_i^{(0)} = \frac{f^{(i)}(x_k)}{i!}$, $i = 0, 1, 2, \ldots$, of Taylor's expansion, we provide straightforwardly a lemma as below without trying to prove it.

Lemma 2.3 (Viscovatov Algorithm) *Assume that the function $f(x)$ has n-th derivative in an interval $X \subseteq R$. If $f(x)$ can be expanded into the following Thiele's continued fraction about the point $x_k \in X$*

$$f(x) = a_0 + \frac{x - x_k|}{|\ a_1} + \frac{x - x_k|}{|\ a_2} + \cdots + \frac{x - x_k|}{|\ a_n} + \cdots,$$

then the coefficients a_n, $n = 0, 1, 2, \ldots$, can be calculated by using Viscovatov algorithm as follows

$$\begin{cases} a_0 = C_0^{(0)}, \\ a_1 = 1/C_1^{(0)}, \\ C_i^{(1)} = -C_{i+1}^{(0)}/C_1^{(0)}, \quad i \geq 1, \\ a_l = C_1^{(l-2)}/C_1^{(l-1)}, \quad l \geq 2, \\ C_i^{(l)} = C_{i+1}^{(l-2)} - a_l C_{i+1}^{(l-1)}, \quad i \geq 1, \ l \geq 2, \end{cases}$$

where $C_i^{(0)} = \frac{f^{(i)}(x_k)}{i!}$, $i = 0, 1, 2, \ldots$ (see [14, 15]).

On the other hand, we recall the relative speed of convergence of the iterative scheme. Firstly, we need a procedure for measuring how rapidly a sequence converges. Secondly, we require one judgement lemma for determining what speed is the sequence generated by the iterative scheme.

Definition 2.4 *Suppose that a sequence $\{r_i\}_{i=0}^{\infty}$ converges to r, with $r_i \neq r$ for all i. If there are positive constants λ and μ such that*

$$\lim_{i \to \infty} \frac{|r_i - r|}{|r_{i-1} - r|^{\lambda}} = \mu, \tag{2.3}$$

then $\{r_i\}_{i=0}^{\infty}$ is said to converge to r of order λ and μ is called asymptotic error constant [1, 2].

Definition 2.5 *Assume that $x^* \in [a, b]$ is a solution of the equation $f(x) = 0$. And for $x \neq x^*$, suppose that $f(x) = (x - x^*)^{\tau} h(x)$, where $\lim_{x \to x^*} h(x) \neq 0$. Then x^* is called a solution of multiplicity τ of the equation $f(x) = 0$ (see [1, 2]).*

Lemma 2.6 *Assume that $x^* \in [a, b]$ is a single solution of the equation $g(x) = x - \varphi(x) = 0$,* *where $\varphi(x) = x - \zeta(g(x))$ and ζ is a continuous function with $\zeta(0) = 0$. Suppose that $\varphi(x)$ is ν times differentiable on neighborhood of x^*, where $\nu \geq 2$. And let $\varphi(x)$ satisfy that*

$$\varphi^{(j)}(x^*) = 0, \quad j = 1, 2, \dots, \nu - 1, \varphi^{(\nu)}(x^*) \neq 0.$$

Then the convergence order of fixed-point iteration $x_k = \varphi(x_{k-1})$, $k \geq 1$, is at least ν.

Proof A detailed proof can be found in the references [1, 2]. □

3. The new iterative method

Now, we derive a new iterative scheme by means of Thiele's continued fraction as shown below. Considering the first truncated Thiele's continued fraction for $f(x)$ expanded about x_k, we have

$$f(x) \approx a_0 + \frac{x - x_k|}{|a_1}.$$

Since $f(x^*) = 0$, the above expression with $x = x^*$ gives

$$a_0 + \frac{x^* - x_k|}{|a_1} \approx 0. \tag{3.1}$$

Solving for $x^* - x_k$ yields

$$x^* - x_k \approx -a_0 a_1. \tag{3.2}$$

On the other hand, we consider the third truncated Thiele's continued fraction for $f(x)$ expanded about x_k. Then we have the following approximation formula

$$f(x) \approx a_0 + \frac{x - x_k|}{|a_1} + \frac{x - x_k|}{|a_2} + \frac{x - x_k|}{|a_3}, \tag{3.3}$$

so (3.3) with $x = x^*$ gives

$$0 \approx a_0 + \frac{x^* - x_k|}{|a_1} + \frac{x^* - x_k|}{|a_2} + \frac{x^* - x_k|}{|a_3}. \tag{3.4}$$

Substituting (3.2) into (3.4) gets

$$a_0 + \frac{x^* - x_k|}{|a_1} + \frac{x^* - x_k|}{|a_2} + \frac{-a_0 a_1|}{|a_3} \approx 0. \tag{3.5}$$

Solving (3.5) for x^* yields

$$x^* \approx x_k - \frac{a_0 a_1 - a_2 a_3}{1 - \frac{(a_0 + a_2) a_3}{a_0 a_1}}. \tag{3.6}$$

Let us set this inequality (3.6) as the stage for new method. Then we can start with an initial approximation value x_0 and generate the sequence $\{x_k\}_{k=0}^{\infty}$ by using the following iterative scheme

$$x_k = x_{k-1} - \frac{a_0 a_1 - a_2 a_3}{1 - \frac{(a_0 + a_2) a_3}{a_0 a_1}}, \tag{3.7}$$

for all $k \geq 1$.

It follows from Lemma 2.3 (Viscovatov algorithm) that

$$a_0 = f(x_k), \tag{3.8}$$

$$a_1 = \frac{1}{f'(x_k)}, \tag{3.9}$$

$$a_2 = -\frac{2(f'(x_k))^2}{f''(x_k)} \tag{3.10}$$

and

$$a_3 = \frac{3(f''(x_k))^2}{2(f'(x_k))^2 f'''(x_k) - 3f'(x_k)(f''(x_k))^2}. \tag{3.11}$$

Replacing (3.7) with (3.8)–(3.11), we have

$$x_{k+1} = x_k - \frac{f(x_k)(6f'^2(x_k)f''(x_k) - 3f(x_k)f''^2(x_k) + 2f(x_k)f'(x_k)f'''(x_k))}{2f'(x_k)(3f'^2(x_k)f''(x_k) - 3f(x_k)f''^2(x_k) + f(x_k)f'(x_k)f'''(x_k))}. \tag{3.12}$$

For convenience, let $\tilde{x}_k$ denote $6f'^2(x_k)f''(x_k) - 3f(x_k)f''^2(x_k) + 2f(x_k)f'(x_k)f'''(x_k)$. Then Eq. (3.12) can be written as follows

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \frac{\tilde{x}_k}{\tilde{x}_k - 3f(x_k)f''^2(x_k)}, \tag{3.13}$$

where $\tilde{x}_k = 6f'^2(x_k)f''(x_k) - 3f(x_k)f''^2(x_k) + 2f(x_k)f'(x_k)f'''(x_k)$.

Thus, the new iterative method (3.13) is obtained by using the approximation formula of Thiele's continued fraction. For further use in the next section, we define temporarily the iterative method as Thiele's method (TM for short).

4. Convergence analysis

In the section, we will prove that the iterative method (3.13) has biquadratic convergence.

Theorem 4.1 *Let $\delta(x) = f'(x)(3f'^2(x)f''(x) - 3f(x)f''^2(x) + f(x)f'(x)f'''(x))$. Suppose that x^* is a solution of the equation $f(x) = 0$. We then have that*

(i) x^* is a single solution of the equation $f(x) = 0$ if $\delta(x^*) \neq 0$.

(ii) x^* is a multiple solution of the equation $f(x) = 0$ if $\delta(x^*) = 0$.

Proof Assume that x^* is a solution of multiplicity n of $f(x) = 0$. For $x \neq x^*$, then we can write $f(x)$ as the following form

$$f(x) = (x - x^*)^n h(x), \tag{4.1}$$

where $\lim_{x \to x^*} h(x) \neq 0$. Successive differentiation of (4.1) with respect to x gives

$$f'(x) = (x - x^*)^n h'(x) + n(x - x^*)^{n-1} h(x), \tag{4.2}$$

$$f''(x) = (x - x^*)^n h''(x) + 2n(x - x^*)^{n-1} h'(x) + n(n-1)(x - x^*)^{(n-2)} h(x). \tag{4.3}$$

Let us substitute (4.1), (4.2) and (4.3) into $\delta(x)$ and notice that $f(x^*) = 0$. Then we obtain the following conclusions.

Case 1 When $n = 1$, we have $\delta(x^*) \neq 0$, which implies that x^* is a single solution of the equation $f(x) = 0$.

Case 2 When $n \geq 2$, we verify that $\delta(x^*) = 0$, which means that x^* is a multiple root of the equation $f(x) = 0$.

Thus, we have completed the proof of Theorem 4.1. $\square$

Theorem 4.2 *Let $\delta(x) = f'(x)(3f'^2(x)f''(x) - 3f(x)f''^2(x) + f(x)f'(x)f'''(x)) \neq 0$ for an arbitrary point $x \in [a,b] \subseteq X$. Then there is a single solution of the equation $f(x) = 0$ at most in the interval $[a,b]$.*

Proof By Theorem 4.1 and the condition $\delta(x) \neq 0$, it is obvious to see that $f(x) = 0$ has only one single solution if any. Suppose that

$$\varphi(x) = f(x)e^{\int_a^x \frac{f'(t)f'''(t)-3f''^2(t)}{3f'(t)f''(t)}\mathrm{d}t}. \tag{4.4}$$

Clearly, the root-finding problem $f(x) = 0$ can be transformed into the equivalent problem $\varphi(x) = 0$. Differentiating (4.4) with respect to x, we obtain

$$\varphi'(x) = K(x)e^{\int_a^x \frac{f'(t)f'''(t)-3f''^2(t)}{3f'(t)f''(t)}\mathrm{d}t}, \tag{4.5}$$

where $K(x) = \frac{3f'^2(x)f''(x)-3f(x)f''^2(x)+f(x)f'(x)f'''(x)}{3f'(x)f''(x)}$. It follows from $\delta(x) \neq 0$ that $K(x) \neq 0$, which means that

$$\varphi'(x) \neq 0. \tag{4.6}$$

Suppose that the equation $f(x) = 0$ has two different solutions x_1 and x_2 on the interval $[a,b]$ and let x_1 be less than x_2. By Rolle mean-value theorem, there exists a point $\xi \in (x_1, x_2) \subseteq [a,b]$ at least such that $f'(\xi) = 0$ and $\varphi'(\xi) = 0$, which contradicts the inequality (4.6) on the interval $[a,b]$. Thus there is a unique single solution on the closed interval $[a,b]$. This completes the proof of Theorem 4.2. $\square$

Theorem 4.3 *Let $x^* \in [a,b]$ be a solution of the equation $f(x) = 0$ and suppose that $f'(x^*) \neq 0$. If $f(x)$ is sufficiently smooth in a neighborhood of the point x^*, then the order of convergence of TM defined by the iterative scheme (3.13) is at least four.*

Proof By the hypothesis, x^* is a root of $f(x) = 0$ and $f'(x^*) \neq 0$, so we know that x^* is a unique single solution of $f(x) = 0$ according to Case 2 in Theorem 4.1. Hence, for any positive integer $i \geq 1$, we have that the derivatives $f^{(i)}(x^*) \neq 0$.

For the iterative scheme (3.13) of TM, we can write its corresponding iterative function easily as shown below:

$$\psi(x) = x - \frac{f(x)}{f'(x)}\frac{\tilde{x}}{\tilde{x} - 3f(x)f''^2(x)}, \tag{4.7}$$

where $\tilde{x} = 6f''(x) - 3f(x)f''^2(x) + 2f(x)f'(x)f'''(x)$.

For the iterative function (4.7), by calculating its first and high-order derivatives with respect to x at the point x^*, we have that

$$\psi'(x^*) = 0,\ \psi''(x^*) = 0,\ \psi'''(x^*) = 0$$

and

$$\psi^{(4)}(x^*) = \frac{9f''^4(x^*) - 6f'(x^*)f''^2(x^*)f'''(x^*) - 4f'^2(x^*)f'''^2(x^*) + 3f'^2(x^*)f''(x^*)f^{(4)}(x^*)}{3f'^3(x^*)f''(x^*)} \neq 0.$$

Therefore, it follows from Lemma 2.6 that the convergence order of TM defined by the iterative scheme (3.13) is at least four. We have showed Theorem 4.3. □

Method	k	x_k	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$
NM	1	2.467181467181467	1.032818532818532	14.365632650989149
	2	2.089122633298695	0.378058833882772	1.575570057176709
	3	2.036262296688809	0.052860336609886	0.028541532156055
	4	2.035268828673700	0.000993468015109	9.976×10^{-6}
	5	2.035268481182002	3.475×10^{-7}	1.222×10^{-12}
	6	2.035268481181959	4.263×10^{-14}	0.000000000000000
HM	1	2.156430593802118	1.343569406197883	3.628589872876624
	2	2.035415128933690	0.121015464868428	0.004210342482438
	3	2.035268481182240	0.000146647751449	8.065×10^{-12}
	4	2.035268481181959	2.811×10^{-13}	0.000000000000000
AM	1	2.211288387964784	1.288711612035216	5.371935193532352
	2	2.036083977721529	0.175204410243255	0.023418893952496
	3	2.035268481278497	0.000815496443032	2.771×10^{-9}
	4	2.035268481181960	9.654×10^{-11}	1.066×10^{-14}
	5	2.035268481181959	4.441×10^{-16}	0.000000000000000
KM	1	1.997896462406664	1.502103537593337	1.058854813217238
	2	2.035268456802230	0.037371994395566	6.999×10^{-7}
	3	2.035268481181959	2.438×10^{-8}	0.000000000000000
TM	1	2.074711925032943	1.425288074967057	1.148169650726839
	2	2.035268546349990	0.039443378682954	1.871×10^{-6}
	3	2.035268481181959	6.517×10^{-8}	0.000000000000000

Table 1 Comparison of NM, HM, AM, KM and TM for $f_1 = 0$ with an initial value $x_0 = 3.5$

5. Numerical examples

In order to check the performance of the new fourth-order method (TM) defined by (3.13), we have given numerical results on some test equation. Meanwhile, we have also compared its results with Newton's method (NM), Halley's method (HM), Abbasbandy's method (AM) and Kou's method (KM). All numerical computations have been carried out on Mathematica software. All problems have been solved by using a given initial guess value x_0. We have chosen

$|x_{k+1} - x_k| < \varepsilon$, where $\varepsilon = 10^{-14}$ and $|f(x_k)| < \epsilon$, where $\epsilon = 10^{-15}$ as stopping criteria so that the iterative process is terminated when the criteria are satisfied simultaneously. The test equations and their solutions x^* used as numerical examples are presented as below. Most of the equations could be also found in the literatures [16, 17] or some papers mentioned previously.

$$f_1(x) = x^3 + 4x^2 - 25 = 0, \quad x^* = 2.035268481181959.$$

$$f_2(x) = x^2 - e^x - 3x + 2 = 0, \quad x^* = 0.257530285439861.$$

$$f_3(x) = x^3 - 10 = 0, \quad x^* = 2.154434690031884.$$

$$f_4(x) = \cos x - x = 0, \quad x^* = 0.739085133215161.$$

$$f_5(x) = x^2 + \sin(x/5) - 1/4 = 0, \quad x^* = 0.409992017989137.$$

$$f_6(x) = x^2 - xe^x + \cos x = 0, \quad x^* = 0.639154096332008.$$

Method	k	x_k	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$
NM	1	2.598765364820434	1.001234635179567	12.489840160774131
	2	1.488517357724619	1.110248007095815	4.680389916998399
	3	0.437567874388413	1.050949483336206	0.670173406579613
	4	0.255148182223531	0.182419692164882	0.009003189707467
	5	0.257529753834991	0.002381571611460	2.009×10^{-6}
	6	0.257530285439834	5.316×10^{-7}	9.992×10^{-14}
	7	0.257530285439861	2.642×10^{-14}	0.000000000000000
HM	1	1.448603749486122	2.151396250513879	4.504524718339946
	2	0.038275355674870	1.410328393811252	0.847621643026989
	3	0.257884466670478	0.219609110995608	0.001338289850562
	4	0.257530285437712	0.000354181232766	8.118×10^{-12}
	5	0.257530285439861	2.148×10^{-12}	0.000000000000000
AM	1	1.874522622233996	1.725477377766004	6.627439774595892
	2	−0.076733918624507	1.951256540858503	1.309953601709611
	3	0.259186828933947	0.335920747558453	0.006258563833084
	4	0.257530284999094	0.001656543934853	1.666×10^{-9}
	5	0.257530285439861	4.408×10^{-10}	2.220×10^{-16}
KM	1	$2.26195 + 0.94288i$	1.636889576133769	8.862348042644750
	2	$0.85916 + 0.14948i$	1.611614980617898	2.262682468525830
	3	$0.25839 + 0.00085i$	0.618881024024911	0.004563277093256
	4	0.257530285439839	1.208×10^{-3}	8.107×10^{-14}
	5	0.257530285439861	2.143×10^{-14}	3.726×10^{-31}
TM	1	0.712770914049040	2.887229085950961	1.669905455507468
	2	0.250585902887892	0.462185011161147	0.026257634740378
	3	0.257530285561798	0.006944382673906	4.608×10^{-10}
	4	0.257530285439861	1.219×10^{-10}	2.220×10^{-16}

Table 2 Comparison of NM, HM, AM, KM and TM for $f_2 = 0$ with an initial value $x_0 = 3.6$

Method	k	x_k	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$
NM	1	13.000000000000000	13.500000000000000	2187.000000000000000
	2	8.686390532544380	4.313609467455621	645.417529883614700
	3	5.835104410594629	2.851286121949751	188.676222738856300
	4	3.987969334404295	1.847135076190334	53.424263153100846
	5	2.868238427146630	1.119730907257665	13.596400014251902
	6	2.317339180253315	0.550899246893315	2.444252469081436
	7	2.165618214366820	0.151720965886495	0.156537737060408
	8	2.154492343740757	0.011125870626063	0.000802835917861
	9	2.154434691574670	0.000057652166087	2.148×10^{-8}
	10	2.154434690031884	1.543×10^{-9}	1.776×10^{-15}
	11	2.154434690031884	0.000000000000000	1.776×10^{-15}
HM	1	-1.019230769230769	0.519230769230769	11.058808887118799
	2	-2.449189120140852	1.429958350910083	24.691527913220398
	3	0.670763794725106	3.119952914865958	9.698207224599528
	4	1.284254981157929	0.613491186432823	7.881864316896416
	5	1.995278482284260	0.711023501126331	2.056524561467152
	6	2.153788429062883	0.158509946778622	0.008996333941973
	7	2.154434689993099	0.000646260930216	5.401×10^{-10}
	8	2.154434690031884	3.878×10^{-11}	1.776×10^{-15}
	9	2.154434690031884	0.000000000000000	1.776×10^{-15}
AM	1	3658.000000000000500	658.500000000000500	4.895×10^{10}
	2	1987.061728837924400	1670.938271162076000	7.846×10^{9}
	3	1079.391557906623000	907.670170931301500	1.258×10^{9}
	4	586.336159998495200	493.055397908127700	2.016×10^{8}
	5	318.503610322648970	267.832549675846200	3.231×10^{7}
	6	173.014365257347700	145.489245065301280	5.179×10^{6}
	7	93.983309958134880	79.031055299212810	830131.658918804600000
	8	51.053333092755180	42.929976865379700	133057.593470544120000
	9	27.734948292080063	23.318384800675120	21324.480964712802000
	10	15.073600672854237	12.661347619225825	3414.924620532807700
	11	8.214190605591135	6.859410067263102	544.235485600313800
	12	4.549460619324603	3.664729986266532	84.162879385761000
	13	2.750087681279972	1.799372938044631	10.798864332466088
	14	2.187010770907556	0.563076910372416	0.460507754511978
	15	2.154444254024738	0.032566516882817	0.000133176958515
	16	2.154434690031884	9.564×10^{-6}	1.776×10^{-15}
	17	2.154434690031884	0.000000000000000	1.776×10^{-15}
KM	1	0.387840073115796	0.887840073115796	9.941661126331486
	2	1.032877532772214	0.645037459656417	8.898089066817242
	3	2.020701291331406	0.987823758559193	1.749004371843331
	4	2.154423058659121	0.133721767327715	0.000161963275399
	5	2.154434690031884	0.000011631372763	1.776×10^{-15}
	6	2.154434690031884	0.000000000000000	1.776×10^{-15}
TM	1	4.715909090909092	5.215909090909092	94.880868179470380
	2	2.431698025032435	2.284211065876657	4.379008028114857
	3	2.154684928821315	0.277013096211120	0.003484921456504
	4	2.154434690031884	0.000250238789431	1.776×10^{-15}
	5	2.154434690031884	0.000000000000000	1.776×10^{-15}

Table 3 Comparison of NM, HM, AM, KM and TM for $f_3 = 0$ with an initial value $x_0 = -0.5$.

Method	k	x_k	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$
NM	1	0.750363867840244	0.249636132159756	0.018923073822117
	2	0.739112890911362	0.011250976928882	0.000046455898991
	3	0.739085133385284	0.000027757526078	2.847×10^{-10}
	4	0.739085133215161	1.701×10^{-10}	1.070×10^{-20}
HM	1	0.740873995080344	0.259126004919656	0.002995042639237
	2	0.739085133877582	0.001788861202762	1.109×10^{-9}
	3	0.739085133215161	6.624×10^{-10}	5.635×10^{-29}
AM	1	0.742406339898200	0.257593660101800	0.005562483546949
	2	0.739085141675681	0.003321198222518	1.415962917393062
	3	0.739085133215161	8.461×10^{-9}	2.348×10^{-25}
KM	1	0.739121683853787	0.260878316146213	0.000061172082163
	2	0.739085133215161	0.000036550638626	2.592×10^{-20}
TM	1	0.738895238700841	0.261104761299159	0.000317796416994
	2	0.739085133215161	0.000189894514319	2.870×10^{-17}

Table 4 Comparison of NM, HM, AM, KM and TM for $f_4 = 0$ with an initial value $x_0 = 1$

Method	k	x_k	$\|x_k - x_{k-1}\|$	$\|f(x_k)\|$
NM	1	0.568003834917740	0.431996165082260	0.185984942219348
	2	0.428659893753630	0.139343941164110	0.019376300575553
	3	0.410321287614389	0.018338606139241	0.000335736733870
	4	0.409992124110730	0.000329163503659	2.082×10^{-7}
	5	0.409992017989148	1.061×10^{-7}	1.124×10^{-14}
	6	0.409992017989137	1.103×10^{-14}	1.215×10^{-28}
HM	1	0.462733613815932	0.537266386184068	0.056537067840756
	2	0.410113176115303	0.052620437700628	0.000123512591844
	3	0.409992017990845	0.000121158124458	1.741×10^{-12}
	4	0.409992017989137	1.708×10^{-12}	4.478×10^{-36}
AM	1	0.483407890383375	0.516592109616625	0.080214217877317
	2	0.410554645973505	0.072853244409870	0.000573809509301
	3	0.409992018330462	0.000562627643044	3.479×10^{-10}
	4	0.409992017989137	3.143×10^{-10}	7.787×10^{-29}
KM	1	0.410063334932167	0.589936665067833	0.00007269929638
	2	0.409992017989137	0.000071316943030	3.371×10^{-20}
TM	1	0.428685989222670	0.571314010777330	1.669905455507468
	2	0.409992124840071	0.018693864382599	1.089×10^{-7}
	3	0.409992017989137	1.069×10^{-7}	1.250×10^{-28}

Table 5 Comparison of NM, HM, AM, KM and TM for $f_5 = 0$ with an initial value $x_0 = 1$

Method	k	x_k	$\lvert x_k - x_{k-1}\rvert$	$\lvert f(x_k)\rvert$
NM	1	1.021633181400203	0.478366818599797	1.272110715210767
	2	0.734166225881035	0.287466955519168	0.248424731113762
	3	0.645921966357842	0.088244259523193	0.016494134114442
	4	0.639189915536399	0.006732050821443	0.000086835514916
	5	0.639154097338489	0.000035818197910	2.440×10^{-9}
	6	0.639154096332008	1.006×10^{-9}	1.110×10^{-16}
HM	1	0.755365976764605	0.744634023235395	0.309115181154623
	2	0.639465946337427	0.115900030427178	0.000756173149435
	3	0.639154096337539	0.000311849999888	1.341×10^{-11}
	4	0.639154096332008	5.531×10^{-12}	1.110×10^{-16}
AM	1	0.812569058471290	0.687430941528710	0.483375202928945
	2	0.641248728470113	0.171320330001177	0.005086170058611
	3	0.639154099692217	0.002094628777896	8.146×10^{-9}
	4	0.639154096332008	3.360×10^{-9}	1.110×10^{-16}
KM	1	$0.66294 + 0.31650i$	0.894900473582751	0.774480089053256
	2	$0.63637 + 0.00065i$	0.316967593962079	0.006902029252616
	3	0.639154096320110	0.002853343277613	4.720×10^{-11}
	4	0.639154096332008	1.947×10^{-11}	1.110×10^{-16}
TM	1	0.684274000198468	0.815725999801532	0.113349272540104
	2	0.639154312606719	0.045119687591749	5.243×10^{-7}
	3	0.639154096332008	2.163×10^{-7}	1.110×10^{-16}

Table 6 Comparison of NM, HM, AM, KM and TM for $f_6 = 0$ with an initial value $x_0 = 1.5$

6. Conclusion

From Section 3, it is evident that when the iteration is performed by the approximation formula of Thiele's continued fraction, we obtained a biquadratically convergent iterative method. This result is also supported by some numerical examples. It is shown in Tables 1–6 that Thiele's method (TM) really converges more rapidly than Newton's method (NM), Halley's method (HM) and Abbasbandy's method (AM). As far as function evaluations are considered, the numerical results also show that for most of the cases, TM requires less or equal number of function evaluations as required in other compared iterative methods, such as NM, HM and AM. The main merits of TM is fast convergence. If we choose a fourth-order convergent iterative method, such as Kou's method (KM), to compare with the new method (TM), then we see that the results in Tables 1–6 illustrate the rapid convergence of both KM and TM in the case of a simple zero. In the majority of cases, we note that the proposed TM gives about the same accuracy as the KM. In the iterative process by using KM, we also notice that the KM may require to calculate the square root of a negative number with a certain initial approximation x_0, then the approximation might encounter some complex numbers at times (see Tables 2 and 6). But it is obvious to see that the iterative process by applying TM is free from calculation of square root. So there does not appear any complex number in the iterative process, which is another major advantage of TM. However, one disadvantage is that TM needs to calculate third derivative. To circumvent the problem of the high-order derivative evaluation in TM, we will discuss its variation in future studies. As a result, TM which we have derived in this paper is not only faster but also requires no more effort than the methods mentioned here.

Acknowledgements The author would like to thank the anonymous referees and the editor for their careful reading of the manuscript and many constructive comments and suggestions.

References

[1] D. KINCAID, W. CHENEY. *Numerical Analysis: Mathematics of Scientific Computing (Third Ed.)*. American Mathematical Society, Providence (RI), 2009.

[2] R. L. BURDEN, J. D. FAIRES. *Numerical Analysis (Tenth Ed.)*, Brooks/Cole, Pacific Grove(CA), 2015.

[3] I. K. ARGYROS. *A note on the Halley method in Banach spaces*. Appl. Math. Comput., 1993, **58**(2-3): 215–224.

[4] S. ABBASBANDY. *Improving Newton-Raphson method for nonlinear equations by modified Adomian decomposition method*. Appl. Math. Comput., 2003, **145**(2): 887–893.

[5] C. CHUN. *Iterative methods improving Newtons method by the decomposition method*. Comput. Math. Appl., 2005, **50**(10): 1559–1568.

[6] Jisheng KOU, Yitian LI, Xiuhua WANG. *A composite fourth-order iterative method for solving non-linear equations*. Appl. Math. Comput., 2007, **184**(2): 471–475.

[7] Xiuhua WANG, Jisheng KOU, Chuanqing GU. *A new modified secant-like method for solving nonlinear equations*. Comput. Math. Appl., 2010, **60**(6): 1633–1638.

[8] M. A. NOOR, F. A. SHAH. *A family of iterative schemes for finding zeros of nonlinear equations having unknown multiplicity*. Appl. Math. Inf. Sci., 2014, **8**(5): 2367–2373.

[9] F. A. SHAH, M. A. NOOR. *Some numerical methods for solving nonlinear equations by using decomposition technique*. Appl. Math. Comput., 2015, **251**: 378–386.

[10] Y. C. KWUN, M. SAQIB, M. FAHAD, et al. *A new cubically convergent iterative method for solving nonlinear equations*. Int. J. Pure Appl. Math., 2016, **111**(1): 67–76.

[11] Jisheng KOU, Xiuhua WANG, Yitian LI. *Some eighth-order root-finding three-step methods*. Commun. Nonlinear Sci. Numer. Simul., 2010, **15**(3): 536–544.

[12] M. MATINFAR, M. AMINZADEH. *An iterative method with six-order convergence for solving nonlinear equations*. Int. J. Math. Modell. Comput., 2012, **2**(1): 45–51.

[13] Jisheng KOU. *Some variants of Cauchys method with accelerated fourth-order convergence*. J. Comput. Appl. Math., 2008, **213**(1): 71–78.

[14] Jieqing TAN. *The limiting case of Thiele's interpolating continued fraction expansion*. J. Comput. Math., 2001, **19**(4): 433–444.

[15] Jieqing TAN. *The Theory of Continued Fraction and Its Applications*. Science Press, Beijing, 2007. (in Chinese)

[16] S. WEERAKOON, T. G. I. FERNANDO. *A variant of Newton's method with accelerated third-order convergence*. Appl. Math. Lett., 2000, **13**(8): 87–93.

[17] V. DAFTARDAR-GEJJI, H. JAFARI. *An iterative method for solving nonlinear functional equations*. J. Math. Anal. Appl., 2006, **316**(2): 753–763.