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Certain Subclasses of Harmonic Univalent Functions Defined by Convolution and Subordination

Shuhai LI*, **Huo TANG**, **En AO**
School of Mathematics and Statistics, Chifeng University, Inner Mongolia 024000*, P. R. China*

Abstract Let S_H be the class of functions $f = h + \bar{g}$ that are harmonic univalent and sense-preserving in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ for which $f(0) = f'(0) - 1 = 0$. In the present paper, we introduce some new subclasses of S_H consisting of univalent and sense-preserving functions defined by convolution and subordination. Sufficient coefficient conditions, distortion bounds, extreme points and convolution properties for functions of these classes are obtained. Also, we discuss the radii of starlikeness and convexity.

Keywords Harmonic univalent functions; subordination; convolution; radius

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1. Introduction and preliminaries

A complex valued harmonic function f in a simply connected domain $\mathbb{D} \subset \mathbb{C}$ has the canonica representation $f = h+\overline{g}$, where h and g are analytic in $\mathbb{D}$ and $g(z_0) = 0$ for some prescribed point $z_0 \in \mathbb{D}$. A necessary and sufficient condition for f to be locally univalent and sense preserving in $\mathbb{D}$ is that $|h'(z)| > |g'(z)|$ in $\mathbb{D}$ (see [1]; also see [2–5]).

Denote by S_H the class of univalent and harmonic functions f that are sense preserving in $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and have the form

$$f = h + \overline{g}, \tag{1.1}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad |b_1| < 1. \tag{1.2}$$

In [2–6], many authors further investigated various subclasses of S_H and obtained some important results.

For $0 \le \beta < 1$, we let $S^*_{\mathcal{H}}(\beta)$ and $S^c_{\mathcal{H}}(\beta)$, respectively, denote the subclasses of S_H consisting of harmonic starlike and harmonic convex functions of order β, that is [2]

$$f \in S^*_{\mathcal{H}}(\beta) \Longleftrightarrow \frac{\partial}{\partial\theta}\big(\arg f(re^{i\theta})\big) > \beta, \quad 0 \le \theta < 2\pi, \ |z| = r < 1$$

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* Corresponding author
E-mail address: lishms66@sina.com (Shuhai LI); thth2009@163.com (Huo TANG); cfxyae09@163.com (En AO)

and

$$f\in S^c_{\mathcal{H}}(\beta)\Longleftrightarrow \frac{\partial}{\partial\theta}\big(\arg\frac{\partial}{\partial\theta}f(re^{i\theta})\big)>\beta,\quad 0\le\theta<2\pi,\ |z|=r<1.$$

We say that an analytic function $f:\mathbb{U}\to\mathbb{C}$ is subordinate to an analytic function $g:\mathbb{U}\to\mathbb{C}$, and write $f(z)\prec g(z)$, if there exists a complex value function ω which maps $\mathbb{U}$ into itself with $\omega(0)=0$ and $|\omega(z)|<1$ $(z\in\mathbb{U})$, such that $f(z)=g(\omega(z))$ $(z\in\mathbb{U})$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence [7]:

$$f(z)\prec g(z)\Longleftrightarrow f(0)=g(0)\ \text{ and }\ f(\mathbb{U})\subset g(\mathbb{U}).$$

More recently, various differential and integral operators for harmonic functions have been studied by Jahangiri et al. [8], Cotirla [9], El-Ashwah and Aouf [10], Yalçin and Altinkaya [11] and by using convolution, some subclasses of harmonic functions have been studied by Ahuja [12], Ali et al. [13], Nagpal and Ravichandran [14], Li et al. [15, 16] and Çakmak et al. [17].

Let F be fixed harmonic function given by

$$F=H(z)+\overline{G(z)}=z+\sum_{k=2}^{\infty}A_kz^k+\overline{\sum_{k=1}^{\infty}B_kz^k},\quad |B_1|<1. \tag{1.3}$$

We define the convolution (or Hadamard product) of F and f by

$$(F*f)(z):=z+\sum_{k=2}^{\infty}a_kA_kz^k+\overline{\sum_{k=1}^{\infty}b_kB_kz^k}=(f*F)(z). \tag{1.4}$$

Also, we denote by T_H the class of harmonic functions $f(z)$ and

$$f(z)=h(z)+\overline{g(z)}=z-\sum_{k=2}^{\infty}|a_k|z^k+\sum_{k=1}^{\infty}|b_k|\overline{z}^k. \tag{1.5}$$

Now we introduce the following two classes.

Definition 1.1 *Let the function $f\in S_H$ of the form (1.1), and $i,j\in\{0,1\}$, $A,B\in R$; $-1\le B<A\le 1$. The function $f(z)\in S_H(\phi_i,\psi_j;A,B)$ if and only if*

$$\frac{(f*\phi_i)(z)}{(f*\psi_j)(z)}\prec\frac{1+Az}{1+Bz}, \tag{1.6}$$

also, the function $f(z)\in K_H(\phi_i,\psi_j;A,B)$ if and only if

$$\frac{(f*\phi_i)'(z)}{(f*\psi_j)'(z)}\prec\frac{1+Az}{1+Bz}, \tag{1.7}$$

where $z=re^{i\theta}, f'(z)=\frac{\partial}{\partial\theta}f(re^{i\theta}),0\le\theta<2\pi$ and

$$\phi_i(z)=z+\sum_{k=2}^{\infty}p_kz^k+(-1)^i\sum_{k=1}^{\infty}q_kz^k,\ \ \psi_j(z)=z+\sum_{k=2}^{\infty}u_kz^k+(-1)^j\sum_{k=1}^{\infty}v_kz^k \tag{1.8}$$

for $p_k\ge u_k\ge 0$, $q_k\ge v_k\ge 0$, $k\ge 2$.

We let

$$\overline{S}_H(\phi_i,\psi_j;A,B)=T_H\bigcap S_H(\phi_i,\psi_j;A,B)$$

and

$$\overline{K}_H(\phi_i,\psi_j;A,B)=T_H\bigcap K_H(\phi_i,\psi_j;A,B).$$

The set classes $S_H(\phi_i,\psi_j;A,B)$ and $\overline{S}_H(\phi_i,\psi_j;A,B)$ are comprehensive family that contains several previously studied subclasses of T_H :

$$\begin{aligned}
&\overline{S}_H\Big(\frac{z+z^2}{(1-z)^3}+\frac{\overline{z}+\overline{z}^2}{(1-\overline{z})^3},\frac{z}{(1-z)^2}-\frac{\overline{z}}{(1-\overline{z})^2};1-2\beta,-1\Big)\\
&\quad=\overline{K}_H(\beta)=\big\{f\in H:\Re\big(1+\frac{zf''(z)}{f'(z)}\big)>\beta\big\}\ \text{(see [3])};\\
&\overline{S}_H(\phi_i,\psi_j;1-2\beta,-1)\\
&\quad=\mathcal{TH}(\phi_i,\psi_j;\beta)=\big\{f\in H:\Re\frac{(f*\phi_i)(z)}{(f*\psi_j)(z)}>\beta,0\le\beta<1\big\}\ \text{(see [14])};\\
&\overline{S}_H\Big(\frac{z}{(1-z)^2}-\frac{\overline{z}}{(1-\overline{z})^2},\frac{z}{1-z}-\frac{\overline{z}}{1-\overline{z}};1-2\beta,-1\Big)\\
&\quad=\overline{S}_H(\beta)=\big\{f\in H:\Re\frac{zf'(z)}{f(z)}>\beta\big\}\ \text{(see [4, 18])}.
\end{aligned}$$

Making use of the techniques and methods used by the paper [19], in this paper, we find sufficient coefficient conditions, distortion bounds, extreme points, convolution and radii of starlikeness and convexity for the above-defined class $\overline{S}_H(\phi_i,\psi_j;A,B)$.

2. Basic properties

Firstly, we give the sufficient coefficient conditions for functions of these classes.

Theorem 2.1 *Let $f=h+\overline{g}$ be such that h and g are given by (1.2). Also, suppose that $A,B\in R$ and $-1\le B<A\le 1$. If*

$$\sum_{k=2}^{\infty}\lambda_k|a_k|+\sum_{k=1}^{\infty}\mu_k|b_k|\le 1, \tag{2.1}$$

where

$$\begin{cases} k\le\lambda_k=\frac{(1-B)p_k-(1-A)u_k}{A-B}, & k\ge 2;\\ k\le\mu_k=\frac{(1-B)q_k-(-1)^{j-i}(1-A)v_k}{A-B}, & k\ge 1,\end{cases} \tag{2.2}$$

then $f(z)$ is sense-preserving harmonic univalent in $\mathbb{U}$ and $f\in S_H(\phi_i,\psi_j;A,B)$.

Proof If $z_1\ne z_2$, then

$$\begin{aligned}
\Big|\frac{f(z_1)-f(z_2)}{h(z_1)-h(z_2)}\Big| &\ge 1-\Big|\frac{g(z_1)-g(z_2)}{h(z_1)-h(z_2)}\Big|\\
&=1-\Big|\frac{\sum_{k=1}^{\infty}|b_k|(z_1^k-z_2^k)}{(z_1-z_2)+\sum_{k=2}^{\infty}|a_k|(z_1^k-z_2^k)}\Big|\\
&>1-\frac{\sum_{k=1}^{\infty}k|b_k|}{1-\sum_{k=2}^{\infty}k|a_k|}\\
&\ge 1-\frac{\sum_{k=1}^{\infty}\mu_k|b_k|}{1-\sum_{k=2}^{\infty}\lambda_k|a_k|}\ge 0,
\end{aligned}$$

which proves univalent. Note that f is sense-preserving harmonic in $\mathbb{U}$. This is because

$$|h'(z)| \geq 1-\sum_{k=1}^{\infty} k|b_k||z|^{k-1} > 1-\sum_{k=1}^{\infty}\mu_k|b_k|$$
$$\geq \sum_{k=1}^{\infty}\mu_k|b_k| > \sum_{k=2}^{\infty} k|a_k||z|^{k-1} \geq |g'(z)|.$$

We first show that if the inequality (2.1) holds for the coefficients of $f = h+\overline{g}$, then the required condition (1.6) is satisfied. The function $f \in S_H(\phi_i, \psi_j; A, B)$ if and only if there exists an analytic function $\omega(z), \omega(0) = 0, |\omega(z)| < 1 (z \in \mathbb{U})$ such that

$$\frac{F(z)}{G(z)} = \frac{1+A\omega(z)}{1+B\omega(z)}, \quad \phi \in R, \ z \in \mathbb{U},$$

where

$$F(z) = (f * \phi_i)(z) = z - \sum_{k=2}^{\infty}|a_k|p_k z^k + (-1)^i \sum_{k=1}^{\infty}|b_k|q_k\overline{z}^k$$

and

$$G(z) = (f * \psi_j)(z) = z - \sum_{k=2}^{\infty}|a_k|u_k z^k + (-1)^j \sum_{k=1}^{\infty}|b_k|v_k\overline{z}^k,$$

or equivalently

$$\Big|\frac{F(z)-G(z)}{AG(z)-BF(z)}\Big| < 1 \ (z \in \mathbb{U}), \tag{2.3}$$

it suffices to show that

$$|AG(z)-BF(z)| - |F(z)-G(z)| > 0. \tag{2.4}$$

Therefore, we get

$$\begin{aligned}
&|AG(z)-BF(z)| - |F(z)-G(z)|\\
&= \Big|(A-B)z - \sum_{k=2}^{\infty}(Au_k - Bp_k)a_k z^k + (-1)^i\sum_{k=1}^{\infty}((-1)^{j-i}Av_k - Bq_k)\overline{b_k z^k}\Big| -\\
&\quad \Big| -\sum_{k=2}^{\infty}(p_k - u_k)a_k z^k + (-1)^i \sum_{k=1}^{\infty}(q_k - (-1)^{j-i}v_k)\overline{b_k z^k}\Big|\\
&\geq (A-B)|z| - \sum_{k=2}^{\infty}(Au_k - Bp_k)|a_k||z|^k - \sum_{k=1}^{\infty}((-1)^{j-i}Av_k - Bq_k)|b_k||z|^k -\\
&\quad \sum_{k=2}^{\infty}(p_k-u_k)|a_k||z|^k - \sum_{k=1}^{\infty}(q_k - (-1)^{j-i}v_k)|b_k||z|^k\\
&= (A-B)|z|\Big[1 - \sum_{k=2}^{\infty}\lambda_k|a_k||z|^{k-1} - \sum_{k=1}^{\infty}\mu_k|b_k||z|^{k-1}\Big]\\
&> (A-B)|z|\Big[1 - \sum_{k=2}^{\infty}\lambda_k|a_k| - \sum_{k=1}^{\infty}\mu_k|b_k|\Big] \geq 0.
\end{aligned}$$

By hypothesis the last expression is nonnegative. Thus the proof is completed. $\Box$

Using the same method as Theorem 2.1, we can get

Theorem 2.2 *Let $f = h + \overline{g}$ be such that h and g are given by (1.2). Also, suppose that $A, B \in R$ and $-1 \leq B < A \leq 1$. If*

$$\sum_{k=2}^{\infty} k\lambda_k|a_k| + \sum_{k=1}^{\infty} k\mu_k|b_k| \leq 1, \tag{2.5}$$

where λ_k and μ_k are defined by (2.2). Then $f(z)$ is sense-preserving harmonic univalent in $\mathbb{U}$ and $f \in K_H(\phi_i, \psi_j; A, B)$.

Theorem 2.3 *Let $f = h + \overline{g}$ be given by (1.5). Then $f \in \overline{S}_H(\phi_i, \psi_j; A, B)$ if and only if the condition (2.1) holds true.*

Proof Since $\overline{S}_H(\phi_i, \psi_j; A, B) \subset S_H(\phi_i, \psi_j; A, B)$. According to Theorem 2.1, we only need to prove the "only if" part of the theorem. Let $f \in \overline{S}_H(\phi_i, \psi_j; A, B)$, $-1 \leq B < A \leq 1$. Then it satisfies (1.6) or equivalently

$$\left|\frac{\sum_{k=2}^{\infty}(p_k - u_k)a_k z^k - (-1)^i \sum_{k=1}^{\infty}(q_k - (-1)^{j-i}v_k)\overline{b_k z^k}}{(A-B)z + \sum_{k=2}^{\infty}(Au_k - Bp_k)a_k z^k + (-1)^i \sum_{k=1}^{\infty}((-1)^{j-i}Av_k - Bq_k)\overline{b_k z^k}}\right| < 1. \tag{2.6}$$

From (2.6), we have

$$\Re\left\{\frac{\sum_{k=2}^{\infty}(p_k - u_k)a_k z^k - (-1)^i \sum_{k=1}^{\infty}(q_k - (-1)^{j-i}v_k)\overline{b_k z^k}}{(A-B)z - [\sum_{k=2}^{\infty}(Au_k - Bp_k)a_k z^k - (-1)^i \sum_{k=1}^{\infty}((-1)^{j-i}Av_k - Bq_k)\overline{b_k z^k}]}\right\} < 1, \tag{2.7}$$

which is equivalent to

$$\Re\left\{1 - \frac{\sum_{k=2}^{\infty}(p_k - u_k)a_k z^k - (-1)^i \sum_{k=1}^{\infty}(q_k - (-1)^{j-i}v_k)\overline{b_k z^k}}{(A-B)z - [\sum_{k=2}^{\infty}(Au_k - Bp_k)a_k z^k - (-1)^i \sum_{k=1}^{\infty}((-1)^{j-i}Av_k - Bq_k)\overline{b_k z^k}]}\right\} > 0$$

or

$$\begin{aligned} &\Re\{\rho(A,B)\} \\ &= \Re\left\{\frac{(A-B)z - [\sum_{k=2}^{\infty}(Au_k - Bp_k)a_k z^k - (-1)^i \sum_{k=1}^{\infty}((-1)^{j-i}Av_k - Bq_k)\overline{b_k z^k}]}{(A-B)z - [\sum_{k=2}^{\infty}(Au_k - Bp_k)a_k z^k - (-1)^i \sum_{k=1}^{\infty}((-1)^{j-i}Av_k - Bq_k)\overline{b_k z^k}]} - \right. \\ &\left. \frac{\sum_{k=2}^{\infty}(p_k - u_k)a_k z^k - (-1)^i \sum_{k=1}^{\infty}(q_k - (-1)^{j-i}v_k)\overline{b_k z^k}}{(A-B)z - [\sum_{k=2}^{\infty}(Au_k - Bp_k)a_k z^k - (-1)^i \sum_{k=1}^{\infty}((-1)^{j-i}Av_k - Bq_k)\overline{b_k z^k}]}\right\} > 0, \end{aligned} \tag{2.8}$$

which yields

$$\begin{aligned} &\Re\{\rho(A,B)\} \\ &\geq \left\{\frac{(A-B) - [\sum_{k=2}^{\infty}(Au_k - Bp_k)|a_k||z|^{k-1} - (-1)^i \sum_{k=1}^{\infty}((-1)^{j-i}Av_k - Bq_k)|b_k||z|^{k-1}]}{(A-B) + \sum_{k=2}^{\infty}|Au_k - Bp_k||a_k||z|^{k-1} + \sum_{k=1}^{\infty}|(-1)^{j-i}Av_k - Bq_k||b_k|z|^{k-1}} - \right. \\ &\left. \frac{\sum_{k=2}^{\infty}(p_k - u_k)|a_k||z|^{k-1} - (-1)^i \sum_{k=1}^{\infty}(q_k - (-1)^{j-i}v_k)|b_k||z|^{k-1}}{(A-B) + \sum_{k=2}^{\infty}|Au_k - Bp_k||a_k||z^{k-1}| + \sum_{k=1}^{\infty}|(-1)^{j-i}Av_k - Bq_k||b_k||z|^{k-1}}\right\} > 0. \end{aligned} \tag{2.9}$$

The above inequality must hold for all $z \in \mathbb{U}$. Taking $|z| = r$ $(0 < r < 1)$, then (2.9) gives

$$\sum_{k=2}^{\infty} \lambda_k|a_k|r^{k-1} + \sum_{k=1}^{\infty} \mu_k|b_k|r^{k-1} < 1. \tag{2.10}$$

Letting $r \to 1^-$ in (2.10), we will get (2.1). □

Applying the same method as Theorem 2.3, we can obtain

Theorem 2.4 *Let $f = h + \overline{g}$ be given by (1.5). Then $f \in \overline{K}_H(\phi_i, \psi_j; A, B)$ if and only if the condition (2.5) holds true.*

Obviously, from Theorems 2.3 and 2.4, we have

$$\bar{K}_H(\phi_i, \psi_j; A, B) \subset \bar{S}_H(\phi_i, \psi_j; A, B). \tag{2.11}$$

Next, using Theorems 2.3 and 2.4, we give the distortion theorems for functions of these classes.

Theorem 2.5 *Let $f \in \overline{S}_H(\phi_i, \psi_j; A, B), \lambda_k$ and μ_k be given by (2.2). If $\{\lambda_k\}$ and $\{\mu_k\}$ are non-decreasing seqences, then*

$$(1-|b_1|)r - \frac{(A-B)(1-\mu_1|b_1|)}{\tau}r^2 \le |f(z)| \le (1+|b_1|)r + \frac{(A-B)(1-\mu_1|b_1|)}{\tau}r^2, \quad |z| = r,$$

for all $z \in \mathbb{U}$, where $\tau = \min\{\lambda_2, \mu_2\}$ and $b_1 = f_{\overline{z}}(0)$.

Proof Since $f \in \overline{S}_H(\phi_i, \psi_j; A, B)$, using (1.5) and Theorem 2.3, we have

$$\begin{aligned}
|f(z)| &= |z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z^k}| \\
&\le (1+|b_1|)r + \sum_{k=2}^{\infty} |a_k| r^2 + \sum_{k=2}^{\infty} |b_k| r \\
&\le (1+|b_1|)r + \frac{A-B}{\tau} \sum_{k=2}^{\infty} \frac{\tau}{A-B} (|a_k| + |b_k|) r^2 \\
&\le (1+|b_1|)r + \frac{A-B}{\tau} \sum_{k=2}^{\infty} \big(\frac{\tau}{A-B}|a_k| + \frac{\tau}{A-B}|b_k|\big) r^2 \\
&\le (1+|b_1|)r + \frac{A-B}{\tau} \sum_{k=2}^{\infty} \big(\lambda_k |a_k| + \mu_k |b_k|\big) r^2 \\
&\le (1+|b_1|)r + \frac{A-B}{\tau}(1-\mu_1|b_1|) r^2.
\end{aligned}$$

The bounds given in Theorem 2.5 are respectively attained for the following functions

$$f(z) = (1-|b_1|)z - \frac{(A-B)(1-\mu_1|b_1|)}{\tau} z^2$$

and

$$f(z) = (1+|b_1|)\bar{z} + \frac{(A-B)(1-\mu_1|b_1|)}{\tau} \bar{z}^2. \quad \Box$$

Using Theorem 2.5, we obtain the following covering result.

Corollary 2.6 *Let $f \in \overline{S}_H(\phi_i, \psi_j; A, B)$. Then*

$$\big\{w : |w| < (1-|b_1|) - \frac{(A-B)(1-\mu_1|b_1|)}{\tau}\big\} \subset f(\mathbb{U}).$$

Similarly, we can obtain

Theorem 2.7 *Let $f \in \overline{K}_H(\phi_i, \psi_j; A, B), \lambda_k$ and μ_k be given by (2.2). If $\{\lambda_k\}$ and $\{\mu_k\}$ are non-decreasing sequences, then*

$$(1-|b_1|)r - \frac{(A-B)(1-\mu_1|b_1|)}{2\tau}r^2 \le |f(z)| \le (1+|b_1|)r + \frac{(A-B)(1-\mu_1|b_1|)}{2\tau}r^2 \ (|z|=r),$$

for all $z \in \mathbb{U}$, where $\tau = \min\{\lambda_2, \mu_2\}$ and $b_1 = f_{\overline{z}}(0)$.

Next, we give the extreme points of these classes.

Theorem 2.8 *Let $f \in \overline{S}_H(\phi_i, \psi_j; A, B), \lambda_k$ and μ_k be given by (2.2). Then $f \in \mathrm{clco}\overline{S}_H(\phi_i, \psi_j; A, B)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty}[X_k h_k + Y_k g_k] \ \ z \in U, \tag{2.12}$$

where

$$h_1 = z, \quad h_k = z - \frac{1}{\lambda_k}z^k, \ k \ge 2, \quad g_k = z + \frac{1}{\mu_k}\overline{z}^k, \ k \ge 1$$

and

$$X_1 \equiv 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k, \quad X_k \ge 0, \ Y_k \ge 0; \ k = 1, 2, \ldots.$$

Proof Let $-1 \le B < A \le 1$. We get

$$f(z) = \Big(\sum_{k=1}^{\infty}[X_k + Y_k]\Big)z - \sum_{k=2}^{\infty}\frac{1}{\lambda_k}X_k z^k + \sum_{k=1}^{\infty}\frac{1}{\mu_k}Y_k\overline{z^k}.$$

Since, $0 \le X_k \le 1 \ \ (k = 1, 2, \ldots)$, we obtain

$$\sum_{k=2}^{\infty}\lambda_k\frac{1}{\lambda_k}X_k z^k + \sum_{k=1}^{\infty}\mu_k\frac{1}{\mu_k}Y_k\overline{z^k} = \sum_{k=2}^{\infty}X_k + \sum_{k=1}^{\infty}Y_k = 1 - X_1 \le 1.$$

Consequently, using Theorem 2.3, we have $f \in \overline{S}_H(\phi_i, \psi_j; A, B)$.

Conversely, if $f \in \overline{S}_H(\phi_i, \psi_j; A, B)$, then

$$|a_k| \le \frac{1}{\lambda_k}, \quad |b_k| \le \frac{1}{\mu_k}.$$

Putting

$$X_k = \lambda_k|a_k|, \quad Y_k = \mu_k|b_k|$$

and

$$X_1 = 1 - \sum_{k=2}^{\infty}X_k - \sum_{k=1}^{\infty}Y_k \ge 0,$$

we obtain

$$\begin{aligned} f(z) &= z - \sum_{k=2}^{\infty}|a_k|z^k + \sum_{k=1}^{\infty}|b_k|\overline{z}^k \\ &= \Big(\sum_{k=1}^{\infty}X_k + \sum_{k=1}^{\infty}Y_k\Big)z - \sum_{k=2}^{\infty}\frac{1}{\lambda_k}X_k z^k + \sum_{k=1}^{\infty}\frac{1}{\mu_k}Y_k\overline{z}^k \\ &= \sum_{k=1}^{\infty}[h_k(z)X_k + g_k(z)Y_k]. \end{aligned}$$

Thus f can be expressed in the form (2.12). $\square$

Theorem 2.9 *Let $f \in \overline{K}_H(\phi_i, \psi_j; A, B)$, λ_k and μ_k be given by (2.2). Then $f \in \mathrm{clco}\overline{K}_H(\phi_i, \psi_j; A, B)$ if and only if*

$$f(z) = \sum_{k=1}^{\infty}[X_k h_k + Y_k g_k], \quad z \in U,$$

where

$$h_1 = z, \quad h_k = z - \frac{1}{k\lambda_k}z^k, \ k \geq 2, \quad g_k = z + \frac{1}{k\mu_k}\bar{z}^k, \ k \geq 1$$

and

$$X_1 \equiv 1 - \sum_{k=2}^{\infty} X_k - \sum_{k=1}^{\infty} Y_k, \quad X_k \geq 0, \ Y_k \geq 0; \ k = 1, 2, \ldots.$$

Theorem 2.10 *The classes $\overline{S}_H(\phi_i, \psi_j; A, B)$ and $\overline{K}_H(\phi_i, \psi_j; A, B)$ are closed under convex combinations.*

Remark 2.11 If $A = 1-2\beta$, $B = -1$, then Theorems 2.1, 2.3, 2.5 and 2.8, respectively, coincide with [14, Theorems 2.1, 2.5, 2.9 and 2.11].

3. Convolution properties

Firstly, we give the convolution properties for functions of these classes.

Theorem 3.1 *Let the functions $f(z), F(z) \in T_H$ with $|A_k| \leq 1$ and $|B_k| \leq 1$.*

(i) If $f(z) \in \overline{S}_H(\phi_i, \psi_j; A, B)$, then $(f * F)(z) \in \overline{S}_H(\phi_i, \psi_j; A, B)$;

(ii) If $f(z) \in \overline{K}_H(\phi_i, \psi_j; A, B)$, then $(f * F)(z) \in \overline{K}_H(\phi_i, \psi_j; A, B)$.

Proof In view of Theorems 2.3 and 2.4, it suffices to show that the coefficients of $f * F$ satisfy the conditions (2.1) and (2.5). Since

$$\sum_{k=2}^{\infty}\lambda_k|a_k||A_k| + \sum_{k=1}^{\infty}\mu_k|b_k||B_k| \leq \sum_{k=2}^{\infty}\lambda_k|a_k| + \sum_{k=1}^{\infty}\mu_k|b_k| \leq 1$$

and

$$\sum_{k=2}^{\infty}k\lambda_k|a_k||A_k| + \sum_{k=1}^{\infty}k\mu_k|b_k||B_k| \leq 1,$$

the results follow immediately. $\square$

Recently, El-Ashwah and Frasin [20] have studied the Hadamard product (or convolution) of harmonic univalent meromorphic functions. In this section, we establish certain results concerning the convolution properties of functions belonging to the classes $\overline{S}_H(\phi_i, \psi_j; A, B)$ and $\overline{K}_H(\phi_i, \psi_j; A, B)$. In order to obtain that, we now introduce a new class of analytic functions.

Definition 3.2 *Let $\delta \geq 0$, $1 \leq B < A \leq 1$. The function $f = h + \overline{g}$, where h and g are given by (1.5), belongs to the class $f \in \overline{S}^{\delta}_H(\phi_i, \psi_j; A, B)$ if and only if*

$$\sum_{k=2}^{\infty}k^{\delta}\lambda_k|a_k| + \sum_{k=1}^{\infty}k^{\delta}\mu_k|b_k| \leq A - B, \tag{3.1}$$

where λ_k and μ_k are defined by (2.2).

Obviously, for any positive integer δ, we have the following inclusion relation:

$$\overline{S}_H^{\delta}(\phi_i,\psi_j;A,B)\subset\overline{K}_H(\phi_i,\psi_j;A,B)\subset\overline{S}_H(\phi_i,\psi_j;A,B).$$

Let the harmonic functions f_i $(i=1,2,\ldots,p)$ and F_j $(j=1,2,\ldots,q)$ have the form

$$f_i=h_i(z)+\overline{g_i(z)}=z+\sum_{k=2}^{\infty}|a_{k,i}|z^k+(-1)^i\overline{\sum_{k=1}^{\infty}|b_{k,i}|z^k},\quad |b_{k,1}|<1 \tag{3.2}$$

and

$$F_j=H_j(z)+\overline{G_j(z)}=z+\sum_{k=2}^{\infty}|A_{k,j}|z^k+(-1)^j\overline{\sum_{k=1}^{\infty}|B_{k,j}|z^k},\quad |B_{k,1}|<1. \tag{3.3}$$

We define the Hadamard product (or convolution) of f_i and F_j by

$$(f_i*F_j)(z):=z+\sum_{k=2}^{\infty}|a_{k,i}||A_{k,j}|z^k(-1)^{i+j}\overline{\sum_{k=1}^{\infty}|b_{k,i}||B_{k,j}|z^k}=:(F_j*f_i)(z), \tag{3.4}$$

where $i=1,2,\ldots,p$ and $j=1,2,\ldots,q$.

Using Theorems 2.3 and 2.4, we obtain the following theorem.

Theorem 3.3 *Let the functions f_i defined by (3.2) be in the class $\overline{K}_H(\phi_i,\psi_j;A,B)$ for every $i=1,2,\ldots,p$; and let the functions F_j defined by (3.3) be in the class $\overline{S}_H(\phi_i,\psi_j;A,B)$ for every $j=1,2,\ldots,q$. Then the Hadamard product $(f_1*f_2*\cdots*f_p*F_1*F_2*\cdots*F_q)(z)$ belongs to the class $\overline{S}_H^{2p+q-1}(\phi_i,\psi_j;A,B)$.*

Proof Putting

$$\xi(z)=(f_1*f_2*\cdots*f_p*F_1*F_2*\cdots*F_q)(z), \tag{3.5}$$

from (3.5) we have

$$\xi(z)=z-\sum_{k=2}^{\infty}\Big(\prod_{i=1}^{p}|a_{k,i}|\prod_{j=1}^{q}|A_{k,i}|\Big)z^k-\overline{\sum_{k=1}^{\infty}\Big(\prod_{i=1}^{p}|b_{k,i}|\prod_{j=1}^{q}|B_{k,j}|\Big)z^k}. \tag{3.6}$$

To prove the theorem, we need to show that

$$\sum_{k=2}^{\infty}k^{2p+q-1}\lambda_k\Big(\prod_{i=1}^{p}|a_{k,i}|\prod_{j=1}^{q}|A_{k,i}|\Big)+\sum_{k=1}^{\infty}k^{2p+q-1}\mu_k\Big(\prod_{i=1}^{p}|b_{k,i}|\prod_{j=1}^{q}|B_{k,j}|\Big)\le 1, \tag{3.7}$$

where λ_k and μ_k are defined by (2.2).

Since $f_i\in\overline{K}_H(\phi_i,\psi_j;A,B)$, we obtain

$$\sum_{k=2}^{\infty}k\lambda_k|a_{k,i}|+\sum_{k=1}^{\infty}k\mu_k|b_{k,i}|\le 1, \tag{3.8}$$

for every $i=1,2,\ldots,p$. Therefore

$$k\lambda_k|a_{k,i}|\le 1\quad\text{or}\quad |a_{k,i}|\le\frac{1}{k\lambda_k} \tag{3.9}$$

and

$$k\mu_k|b_{k,i}|\le 1\quad\text{or}\quad |b_{k,i}|\le\frac{1}{k\mu_k}. \tag{3.10}$$

Further, since $\lambda_k \geq k$ and $\mu_k \geq k$, we get

$$|a_{k,i}| \leq k^{-2} \quad \text{and} \quad |b_{k,i}| \leq k^{-2}, \tag{3.11}$$

for every $i = 1, 2, \ldots, p$. Also, since $F_j \in \overline{S}_H(\phi_i, \psi_j; A, B)$, we have

$$\sum_{k=2}^{\infty} \lambda_k |A_{k,j}| + \sum_{k=1}^{\infty} \mu_k |B_{k,j}| \leq 1, \tag{3.12}$$

for every $j = 1, 2, \ldots, q$. Hence we obtain

$$|A_{k,j}| \leq k^{-1} \quad \text{and} \quad |B_{k,j}| \leq k^{-1} \tag{3.13}$$

for every $j = 1, 2, \ldots, q$.

Using (3.11) for $i = 1, 2, \ldots, p$; (3.13) for $j = 1, 2, \ldots, q-1$ and (3.12) for $j = q$, we obtain

$$\begin{aligned}
&\sum_{k=2}^{\infty} k^{2p+q-1} \lambda_k \Big(\prod_{i=1}^{p} |a_{k,i}| \prod_{j=1}^{q-1} |A_{k,i}| \Big) |A_{k,q}| + \sum_{k=1}^{\infty} k^{2p+q-1} \mu_k \Big(\prod_{i=1}^{p} |b_{k,i}| \prod_{j=1}^{q-1} |B_{k,j}| \Big) |B_{k,q}| \\
&\quad \leq \sum_{k=2}^{\infty} k^{2p+q-1} (\lambda_k k^{-2p} k^{-(q-1)}) |A_{k,q}| + \sum_{k=1}^{\infty} k^{2p+q-1} (\mu_k k^{-2p} k^{-(q-1)}) |B_{k,q}| \\
&\quad = \sum_{k=2}^{\infty} \lambda_k |A_{k,j}| + \sum_{k=1}^{\infty} \mu_k |B_{k,j}| \leq 1,
\end{aligned}$$

and therefore $\xi(z) \in \overline{S}_H^{2p+q-1}(\phi_i, \psi_j; A, B)$. We note that the required estimate can also be obtained by using (3.11) for $i = 1, 2, \ldots, p-1$; (3.13) for $j = 1, 2, \ldots, q$ and (3.8) for $i = p$. □

Taking into account the Hadamard product of functions $f_1 * f_2 * \cdots * f_p$ only, in the proof of Theorem 3.3, and using (3.11) for $i = 1, 2, \ldots, p-1$; and relation (3.8) for $i = p$, we are led to

Corollary 3.4 *Let the functions f_i defined by (3.2) be in the class $\overline{K}_H(\phi_i, \psi_j; A, B)$ for every $i = 1, 2, \ldots, p$. Then the Hadamard product $(f_1 * f_2 * \cdots * f_p)(z)$ belongs to the class $\overline{S}_H^{2p-1}(\phi_i, \psi_j; A, B)$.*

Also, taking into account the Hadamard product of functions $F_1 * F_2 * \cdots * F_q$ only, in the proof of Theorem 3.3, and using (3.13) for $j = 1, 2, \ldots, q-1$; and relation (3.12) for $j = q$, we are led to

Corollary 3.5 *Let the functions F_j defined by (3.3) be in the class $\overline{S}_H(\phi_i, \psi_j; A, B)$ for every $j = 1, 2, \ldots, q$. Then the Hadamard product $(F_1 * F_2 * \cdots * F_q)(z)$ belongs to the class $\overline{S}_H^{q-1}(\phi_i, \psi_j; A, B)$.*

4. Radii of starlikeness and convexity

Let $Q \subseteq \mathcal{H}$. We define the radius of starlikeness and the radius of convexity of the class Q, respectively

$$R^*_\beta(Q) = \inf_{f \in Q} (\sup\{r \in (0,1] : f \text{ is starlike of order } \beta \text{ in } D(r)\})$$

and

$$R_\beta^c(Q) = \inf_{f\in Q}(\sup\{r \in (0,1] : f \text{ is convex of order } \beta \text{ in } D(r)\}),$$

where $D(r) = \{z \in \mathbb{C} : |z| < r \leq 1\}$ (see [21]).

Using [4, Theorem 2], we have

The function $f = h + \bar{g}$ is starlike of order β in $D(r)$ if and only if

$$\Big|\frac{(zh'(z) - \overline{zg'(z)}) - (1+\beta)(h(z) + \overline{g(z)})}{(zh'(z) - \overline{zg'(z)}) + (1-\beta)(h(z) + \overline{g(z)})}\Big| < 1, \quad |z| = r < 1. \tag{4.1}$$

Theorem 4.1 *Let $0 \leq \beta < 1$, $|b_1| < \min\{\frac{1}{\mu_1}, \frac{1-\beta}{1+\beta}\}$, λ_k $(k \geq 2)$ and μ_k $(k \geq 1)$ be given by (2.2). Then*

(i) $R_\beta^*(\overline{S}_H(\phi_i, \psi_j; A, B)) = \inf_{k\geq 2}[\frac{(1-\beta)-(1+\beta)|b_1|}{1-\mu_1|b_1|}\min\{\frac{\lambda_k}{k-\beta}, \frac{\mu_k}{k+\beta}\}]^{\frac{1}{k-1}}$*;*

(ii) $R_\beta^c(\overline{K}_H(\phi_i, \psi_j; A, B)) = \inf_{k\geq 2}[\frac{(1-\beta)-(1+\beta)|b_1|}{1-\mu_1|b_1|}\min\{\frac{\lambda_k}{k(k-\beta)}, \frac{\mu_k}{k(k+\beta)}\}]^{\frac{1}{k-1}}$.

Proof (i) Let $f \in \overline{S}_H(\phi_i, \psi_j; A, B)$, $|z| = r < 1$. Then using (1.1) we have

$$\begin{aligned}
&\Big|\frac{(zh'(z) - \overline{zg'(z)}) - (1+\beta)(h(z) + \overline{g(z)})}{(zh'(z) - \overline{zg'(z)}) + (1-\beta)(h(z) + \overline{g(z)})}\Big| \\
&= \Big|\frac{-\beta z + \Sigma_{k=2}^{\infty}((k-1-\beta)a_k z^k - (k+1+\beta)b_k\bar{z}^k)}{(2-\beta)z + \Sigma_{k=2}^{\infty}((k+1-\beta)a_k z^k - (k-1+\beta)b_k\bar{z}^k)}\Big| \\
&\leq \frac{\beta + \Sigma_{k=2}^{\infty}((k-1-\beta)|a_k| - (k+1+\beta)|b_k|)r^{k-1}}{(2-\beta) - \Sigma_{k=2}^{\infty}((k+1-\beta)|a_k| + (k-1+\beta)|b_k|)r^{k-1}}.
\end{aligned}$$

From (4.1), we get $f \in S_{\mathcal{H}}^*(\beta)$ if and only if

$$\sum_{k=2}^{\infty}\Big[\frac{k-\beta}{(1-\beta)-(1+\beta)|b_1|}|a_k| + \frac{k+\beta}{(1-\beta)-(1+\beta)|b_1|}|b_k|\Big] \leq 1. \tag{4.2}$$

Also, by Theorem 2.3, we have

$$\sum_{k=2}^{\infty}\lambda_k|a_k| + \sum_{k=1}^{\infty}\mu_k|b_k| \leq 1.$$

The condition (4.2) is true if

$$\frac{k-\beta}{(1-\beta)-(1+\beta)|b_1|}r^{k-1} \leq \frac{\lambda_k}{1-\mu_1|b_1|}$$

and

$$\frac{k+\beta}{(1-\beta)-(1+\beta)|b_1|}r^{k-1} \leq \frac{\mu_k}{1-\mu_1|b_1|}, \quad k = 2, 3, \ldots,$$

or if

$$r \leq \Big[\frac{(1-\beta)-(1+\beta)|b_1|}{1-\mu_1|b_1|}\min\{\frac{\lambda_k}{k-\beta}, \frac{\mu_k}{k+\beta}\}\Big]^{\frac{1}{k-1}}, \quad k = 2, 3, \ldots.$$

It follows that the function f is starlike of order β in the disk $U(r^*)$ where

$$r^* = \inf_{k\geq 2}\Big[\frac{(1-\beta)-(1+\beta)|b_1|}{1-\mu_1|b_1|}\min\Big\{\frac{\lambda_k}{k-\beta}, \frac{\mu_k}{k+\beta}\Big\}\Big]^{\frac{1}{k-1}}.$$

Using a similar argument as above we can obtain (ii). □

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