Journal of Mathematical Research with Applications
Jan., 2019, Vol. 39, No. 1, pp. 43–50
DOI:10.3770/j.issn:2095-2651.2019.01.005
Http://jmre.dlut.edu.cn

Infimum of the Spectrum of Laplace-Beltrami Operator on Classical Bounded Symmetric Domains with Bergman Metric

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Abstract In this paper, we estimate the infimum of the spectrum of the Laplace-Beltrami operator with Kähler metric on the classical bounded symmetric domains. We will give an explicit range for the infimum of the spectrum of the Laplace-Beltrami operator on the second type classical bounded symmetric domains. In particular, for those domains with rank 1, we obtain an explicit formula, which agrees with a previously known result.

Keywords Laplace-Beltrami operator; classical bounded symmetric domains; Kähler metric

MR(2010) Subject Classification 32Q15; 47F05

1. Introduction

Let (M, g) be a Kähler manifold of complex dimension n with Kähler metric

$$g = \sum_{i,j=1}^{n} g_{i\bar{j}} \mathrm{d}z_i \otimes \mathrm{d}\bar{z}_j.$$

The Laplace-Beltrami operator with respect to the Kähler metric g is defined by

$$\Delta_g = -4 \sum_{i,j=1}^{n} g^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}, \tag{1.1}$$

where $[g^{i\bar{j}}]^t = [g_{i\bar{j}}]^{-1}$. Let

$$\lambda_1(\Delta_g) = \inf \Big\{ 4 \int_M g^{i\bar{j}} \frac{\partial h}{\partial z_i} \frac{\partial h}{\partial \bar{z}_j} \mathrm{d}V_g : h \in C_0^\infty(M), \int_M |h|^2 \mathrm{d}V_g = 1 \Big\}. \tag{1.2}$$

Here $\mathrm{d}V_g$ is the volume measure of M with respect to the Kähler metric g.

When (M, g) is compact and Δ_g is uniformly elliptic, $\lambda_1(\Delta_g)$ is the first eigenvalue of Δ_g. When (M, g) is a complete noncompact manifold, $\lambda_1(\Delta_g)$ is not an eigenvalue of Δ_g. However,

Received July 19, 2017; Accepted September 10, 2018
Supported by the Natural Science Foundation of Guangxi Province (Grant Nos. 2015GXNSFBA139019; 2016GXNSFDA380031), Guangxi Key Laboratory of Cryptography and Information Security (Grant No. GCIS201612), National Natural Science Foundation of China (Grant No. 11662001), Research Foundation of Fujian Province for Young and Middle-aged Teachers (Grant No. JAT170467) and the Research Foundation of Minjiang University for the Introduction of Talents (Grant No. MJY1706).
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$\lambda_1(\Delta_g)$ is the infimum of the positive spectrum of Δ_g. Estimation of the spectrum of the Laplace-Beltrami operator on Riemannian and Kähler manifolds have been studied by many authors, including Cheng, Li, Wang and Li, etc. There have been many breakthrough results [1–12]. Since the spectral property of Δ_g on M will reflect the geometric properties of the manifold M, many people have studied the geometric properties of manifolds by estimating $\lambda_1(\Delta_g)$ (see [1–6,10–12]).

In the complete noncompact case, Li and Wang [5] and Munteanu [8] gave upper bounds for $\lambda_1(\Delta_g)$. Li and Wang [5] proved that if the holomorphic bisectional curvature of a manifold has negative lower bound -1, then $\lambda_1(\Delta_g) \le n^2$. Munteanu in [8] proved that $\lambda_1(\Delta_g) \le n^2$ with the assumption that the Ricci curvature satisfies $R_{i\bar{j}} \ge -(n+1)g_{i\bar{j}}$. Both Li and Wang's and Munteanu's results are sharp.

By estimating the upper bound and lower bound of $\lambda_1(\Delta_g)$, Li and Tran [9] gave the infimum of the spectrum of the Laplace-Beltrami operator with Bergman metric on a bounded pseudoconvex domain.

Let D be a bounded pseudoconvex domain in $\mathbb{C}^n$ and $u(z) \in C^\infty(D)$ be a strictly plurisubharmonic exhaustion function for D. Let $g = \sum_{i,j=1}^n \frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \mathrm{d}z_i \times \mathrm{d}\bar{z}_j$ be the Kähler metric induced by u. Let

$$|\partial u|_g^2 = \sum_{i,j=1}^n g^{i\bar{j}} \frac{\partial u}{\partial z_i} \frac{\partial u}{\partial \bar{z}_j} \tag{1.3}$$

and

$$\alpha_g = \sup\{\alpha : \int_D (\det[g_{i\bar{j}}])^\alpha \mathrm{d}v < \infty\}. \tag{1.4}$$

Theorem 1.1 ([9]) *Let u be a strictly plurisubharmonic exhaustion function for D and g be the Kähler metric induced by u. Assuming that $|\partial u|_g^2 \le \beta$, one has*

(i) $\lambda_1(\Delta_g) \ge n^2/\beta$;

(ii) $\lambda_1(\Delta_g) \le \beta c_D^2 (1-\alpha_g)^2$, where c_D is constant which is dependent on D.

(ii) If $\rho = -e^{-u}$ is a strictly plurisubharmonic function for D, then $\lambda_1 = n^2$.

Li and Tran [9] gave many examples of bounded strongly pseudoconvex domains with smooth boundary, where $\lambda_1(\Delta_g)$ can be explicitly formulated. However, for most non-smooth domains, like the classical bounded symmetric domains, the value of $\lambda_1(\Delta_g)$ is not clear. In this paper, we will give a sharp range of $\lambda_1(\Delta_g)$ on the second type classical bounded symmetric domains.

2. Classical bounded symmetric domains

Let $M^{(m,n)}$ be the set of all $m \times n$ matrices with entries in $\mathbb{C}$. For any $A = [a_{ij}] \in M^{(m,n)}$, let

$$A^* = \overline{A}' = [\overline{a_{ji}}].$$

Let I_m be the $m \times m$ identity matrix. The second type classical bounded symmetric domains can be represented by [13],

$$R_{II} := R_{II}(m) = \{Z \in M^{(m,m)} : Z = Z', I_m - ZZ^* > 0\}. \tag{2.1}$$

Let $A = [a_{ij}]$ be an $m \times m$ matrix. Then we define the $\frac{m(m+1)}{2} \times \frac{m(m+1)}{2}$ matrix $(A \dot{\times} A)_s$ as follows:

$$(A\dot{\times}A)_s = [a_{(jr)(ks)}]_{\frac{m(m+1)}{2} \times \frac{m(m+1)}{2}}, \tag{2.2}$$

$$a_{(jr)(ks)} = p_{jr}p_{ks}(a_{jk}a_{rs} + a_{js}a_{rk}) \quad j \le r, k \le s \tag{2.3}$$

where

$$p_{ij} = \begin{cases} \frac{1}{\sqrt{2}}, & i = j; \\ 1, & i \neq j. \end{cases} \tag{2.4}$$

According to Lu [14], one has the following proposition:

Proposition 2.1 *Let A and B be $m \times m$ and $n \times n$ matrices. Then*

$$(A\dot{\times}A)_s(B\dot{\times}B)_s = (AB\dot{\times}AB)_s, \quad [(A\dot{\times}A)_s]' = (A'\dot{\times}A')_s \tag{2.5}$$

and

$$(A\dot{\times}A)_s^{-1} = (A^{-1}\dot{\times}A^{-1})_s, \quad \det(A\dot{\times}A)_s = (\det A)^{m+1}. \tag{2.6}$$

Let $C = [c_{pq}]$ be an $s \times s$ matrix where c_{pq} is a function of z_k. Then

$$\frac{\partial \log \det C}{\partial z_k} = \sum_{p,q=1}^{s} c^{pq} \frac{\partial c_{pq}}{\partial z_k} \tag{2.7}$$

and

$$\frac{\partial^2 \log \det C}{\partial z_k \partial \overline{z}_\ell} = \sum_{p,q=1}^{s} c^{pq} \frac{\partial^2 c_{pq}}{\partial z_k \partial \overline{z}_\ell} - \sum_{ij,pq=1}^{s} c^{iq} c^{pj} \frac{\partial c_{pq}}{\partial z_k} \frac{\partial c_{ij}}{\partial \overline{z}_\ell} \tag{2.8}$$

where

$$\sum_{j=1}^{s} c^{ij} c_{kj} = \delta_{ik}. \tag{2.9}$$

For $Z \in R_{II}$, we denote

$$Z = [\frac{z_{jk}}{\sqrt{2}p_{jk}}], \quad p_{ij} = \begin{cases} \frac{1}{\sqrt{2}}, & i = j; \\ 1, & i \neq j. \end{cases} \tag{2.10}$$

Let

$$z = (z_{11}, \dots, z_{1m}, z_{22}, \dots, z_{2m}, \dots, z_{qq}) \in \mathbb{C}^{\frac{m(m+1)}{2}}. \tag{2.11}$$

Then $\|z\|^2 = \mathrm{tr}(ZZ^*)$. Let $K_{II}(Z, Z)$ be the Bergman kernel function of R_{II}, and $K_{II}(Z) := K_{II}(Z, Z)$, and $V(R_{II})$ be the volume of R_{II}. The Bergman kernel function was discovered by Hua [13], the details can be found in [14, Section 3.3], and

$$K_{II}(Z) = \frac{1}{V(R_{II})} \cdot \frac{1}{\det(I - ZZ^*)^{m+1}}. \tag{2.12}$$

We consider the Bergman metric g_{II} which is induced by

$$u_{II}(Z) = \frac{1}{m+1} \log K_{II}(Z), \quad Z \in R_{II}. \tag{2.13}$$

Lu proved in [14] the following:

Proposition 2.2 *The complex Hessian matrix for R_{II} can be stated by*

$$H(u_{II})(Z)=[(I-\overline{ZZ}^*)^{-1}\dot{\times}(I-\overline{ZZ}^*)^{-1}]_s \tag{2.14}$$

and

$$H(u_{II})^{-1}(Z)=[(I-\overline{ZZ}^*)\dot{\times}(I-\overline{ZZ}^*)]_s. \tag{2.15}$$

Proof For $Z=[\frac{z_{ij}}{\sqrt{2}p_{ij}}]_{m\times m}\in R_{II}$, let $w_{ij}=\frac{z_{ij}}{p_{ij}}$. We have

$$\begin{aligned}\frac{\partial(\frac{z_{hs}}{p_{hs}}\frac{\overline{z}_{s\ell}}{p_{s\ell}})}{\partial z_{j\alpha}}=&\frac{1}{\sqrt{2}p_{j\alpha}}\frac{\partial(w_{hs}\overline{w}_{s\ell})}{\partial w_{j\alpha}}\\ =&\frac{1}{\sqrt{2}p_{j\alpha}}\Big[p_{j\alpha}p_{j\alpha}\sum_s(\delta_{hj}\delta_{s\alpha}\overline{w}_{s\ell}+\delta_{h\alpha}\delta_{js}\overline{w}_{s\ell})\Big]\\ =&\frac{1}{\sqrt{2}p_{j\alpha}}\Big[p_{j\alpha}p_{j\alpha}(\delta_{hj}\overline{w}_{\alpha\ell}+\delta_{h\alpha}\overline{w}_{j\ell})\Big],\\ \frac{\partial^2(\sum_{s=1}^n\frac{z_{hs}}{p_{hs}}\frac{\overline{z}_{s\ell}}{p_{s\ell}})}{\partial z_{j\alpha}\partial\overline{z}_{k\beta}}=&\frac{p_{j\alpha}}{\sqrt{2}}\frac{\partial}{\partial\overline{z}_{k\beta}}(\delta_{hj}\overline{w}_{\alpha\ell}+\delta_{h\alpha}\overline{w}_{j\ell})\\ =&\frac{p_{j\alpha}}{\sqrt{2}}\frac{1}{\sqrt{2}p_{k\beta}}\frac{\partial}{\partial\overline{w}_{k\beta}}(\delta_{hj}\overline{w}_{\alpha\ell}+\delta_{h\alpha}\overline{w}_{j\ell})\\ =&\frac{p_{j\alpha}}{\sqrt{2}}\frac{1}{\sqrt{2}p_{k\beta}}\Big[\delta_{hj}p_{k\beta}p_{k\beta}(\delta_{k\alpha}\delta_{\beta\ell}+\delta_{k\ell}\delta_{\alpha\beta})+\\ &\delta_{h\alpha}p_{k\beta}p_{k\beta}(\delta_{jk}\delta_{\beta\ell}+\delta_{j\beta}\delta_{k\ell})\Big],\end{aligned}$$

by (2.7) and (2.8), one has

$$\begin{aligned}\frac{\partial^2\log K_{II}}{\partial z_{j\alpha}\partial\overline{z}_{k\beta}}\Big|_{Z=0}=&-(m+1)\frac{\partial^2\log\det(I_m-ZZ^*)}{\partial z_{j\alpha}\partial\overline{z}_{k\beta}}\Big|_{Z=0}\\ =&(m+1)\sum_{h,\ell=1}^m\Big((I_m-Z\overline{Z})^{-1}\Big)_{h\ell}\frac{\partial^2(\sum_{s=1}^n\frac{z_{hs}}{p_{hs}}\frac{\overline{z}_{s\ell}}{p_{s\ell}})}{\partial z_{j\alpha}\partial\overline{z}_{k\beta}}\Big|_{Z=0}\\ =&(m+1)\frac{p_{j\alpha}p_{k\beta}}{2}\sum_{h,\ell=1}^m\delta_{h\ell}\Big[\delta_{hj}(\delta_{k\alpha}\delta_{\beta\ell}+\delta_{k\ell}\delta_{\alpha\beta})+\\ &\delta_{h\alpha}(\delta_{jk}\delta_{\beta\ell}+\delta_{j\beta}\delta_{k\ell})\Big]\\ =&(m+1)p_{j\alpha}p_{k\beta}(\delta_{jk}\delta_{\alpha\beta}+\delta_{k\alpha}\delta_{j\beta}).\end{aligned}$$

Thus,

$$H(u_{II})(0)=[\frac{\partial^2u_{II}}{\partial z_{j\alpha}\partial\overline{z}_{k\beta}}]|_{Z=0}=[I_m\dot{\times}I_m]_s.$$

(2.14) can be proved by reference to the proof of (3.3.33) in [14, P116-125]. We omit the details here. By (2.14) and Proposition 2.1, (2.15) follows. □

Proposition 2.3 *With the notations above, one has*

$$\det H(u_{II})(Z)=\frac{1}{\det(I-ZZ^*)^{m+1}}. \tag{2.16}$$

Proof By (2.6) and (2.14), we have

$$\begin{aligned}\det H(u_{II})(z) &= \det[(I-\overline{ZZ}^*)^{-1}\dot{\times}(I-\overline{ZZ}^*)^{-1}]_s\\ &= \det(I-\overline{ZZ}^*)^{-(m+1)} = \det(I-ZZ^*)^{-(m+1)}. \ \square\end{aligned}$$

3. Main results

Let

$$\frac{\partial}{\partial z} = (\frac{\partial}{\partial z_1}, \ldots, \frac{\partial}{\partial z_n}). \tag{3.1}$$

Since

$$(\frac{\partial u}{\partial z})'\frac{\partial u}{\partial \overline{z}} = \begin{pmatrix} \frac{\partial u}{\partial z_1}\frac{\partial u}{\partial \overline{z}_1} & \cdots & \frac{\partial u}{\partial z_1}\frac{\partial u}{\partial \overline{z}_n} \\ \vdots & & \vdots \\ \frac{\partial u}{\partial z_n}\frac{\partial u}{\partial \overline{z}_1} & \cdots & \frac{\partial u}{\partial z_n}\frac{\partial u}{\partial \overline{z}_n} \end{pmatrix}, \tag{3.2}$$

we have

$$|\partial u|_g^2 = \sum_{i,j=1}^n g^{i\overline{j}}\frac{\partial u}{\partial z_i}\frac{\partial u}{\partial \overline{z}_j} = \mathrm{tr}\Big[[g^{i\overline{j}}](\frac{\partial u}{\partial z})'\frac{\partial u}{\partial \overline{z}}\Big]. \tag{3.3}$$

Proposition 3.1 *On R_{II}, one has the following estimate:*

$$|\partial u_{II}|_{g_{II}}^2 \leq m. \tag{3.4}$$

Proof For $Z = [\frac{z_{jk}}{\sqrt{2}p_{jk}}] \in R_{II}$, since $z_{jk} = z_{kj}$ and

$$\begin{aligned}\mathrm{tr}\{[(A\dot{\times}A)_s]\frac{\partial u}{\partial z'}\frac{\partial u}{\partial \overline{z}}\} &= \sum_{j,k=1}^m \sum_{j\leq r,k\leq s} a_{(jr)(ks)}\frac{\partial u}{\partial z_{jr}}\frac{\partial u}{\partial \overline{z}_{ks}}\\ &= \sum_{j,k=1}^m \sum_{j\leq r,k\leq s} (a_{jk}a_{rs}+a_{js}a_{rk})p_{jr}p_{ks}\frac{\partial u}{\partial z_{jr}}\frac{\partial u}{\partial \overline{z}_{ks}}\\ &= \frac{1}{2}\sum_{j,k=1}^m \sum_{j\leq r,k\leq s} 2(a_{jk}a_{rs}+a_{js}a_{rk})p_{jr}p_{ks}\frac{\partial u}{\partial z_{jr}}\frac{\partial u}{\partial \overline{z}_{ks}}\\ &= \frac{1}{2}\sum_{j,k=1}^m\sum_{k\leq s}\Big[\Big(\sum_{j<r}a_{jk}a_{rs}\frac{\partial u}{\partial z_{jr}}+\sum_{j>r}a_{rs}a_{jk}\frac{\partial u}{\partial z_{jr}}+\sqrt{2}\sum_{j=r}a_{jk}a_{rs}\frac{\partial u}{\partial z_{jr}}\Big)+\\ &\quad\Big(\sum_{j>r}a_{rk}a_{js}\frac{\partial u}{\partial z_{jr}}+\sum_{j<r}a_{js}a_{rk}\frac{\partial u}{\partial z_{jr}}+\sqrt{2}\sum_{j=r}a_{js}a_{rk}\frac{\partial u}{\partial z_{jr}}\Big)\Big]p_{ks}\frac{\partial u}{\partial \overline{z}_{ks}}\\ &= \frac{1}{2}\sum_{j,k,r=1}^m\Big(\sum_{k<s}a_{jk}a_{rs}\frac{\partial u}{\partial \overline{z}_{ks}}+\sum_{k>s}a_{rk}a_{js}\frac{\partial u}{\partial \overline{z}_{ks}}+\sqrt{2}\sum_{k=s}a_{jk}a_{rs}\frac{\partial u}{\partial \overline{z}_{ks}}\Big)\frac{1}{p_{jr}}\frac{\partial u}{\partial z_{jr}}\\ &= \sum_{j,r=1}^m\sum_{k,s=1}^m a_{jk}a_{rs}\frac{1}{2p_{jr}p_{ks}}\frac{\partial u}{\partial z_{jr}}\frac{\partial u}{\partial \overline{z}_{ks}},\end{aligned} \tag{3.5}$$

by (3.2) and (3.5), one has

$$\begin{aligned}|\partial u_{II}|^2_{g_{II}} =&\sum_{i,j=1}^{m}\sum_{k\geq i,\ell\geq j} u_{II}^{ik,j\ell}\frac{\partial u_{II}}{\partial z_{ik}}\frac{\partial u_{II}}{\partial \overline{z}_{j\ell}}\\ =&\mathrm{tr}\{[(I-\overline{ZZ}^*)\dot{\times}(I-\overline{ZZ}^*)]_s\frac{\partial u_{II}}{\partial z'}\frac{\partial u_{II}}{\partial \overline{z}}\}\\ =&\sum_{i,j=1}^{m}\sum_{k,\ell=1}^{m}(I-ZZ^*)_{(ij)}(I-ZZ^*)_{(k\ell)}\frac{1}{2p_{ik}p_{j\ell}}\frac{\partial u_{II}}{\partial z_{ik}}\frac{\partial u_{II}}{\partial \overline{z}_{j\ell}}.\end{aligned}\tag{3.6}$$

Let $D[\lambda_1,\dots,\lambda_m]$ be $m\times m$ diagonal matrix with all diagonal entries being $\lambda_1,\dots,\lambda_m$. For $Z\in R_{II}$, since ZZ^* is the Hermite matrix, there exists $m\times m$ unitary matrix U such that

$$UZZ^*U^*=D[\lambda_1,\dots,\lambda_m]\tag{3.7}$$

and $\lambda_j\in[0,1),\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_m\geq 0$. Without loss of generality, we assume that $ZZ^*=D[\lambda_1,\dots,\lambda_m]$, otherwise we let $Z=UZU'$. Then

$$(I_m-ZZ^*)^{-1}=D[\sum_{k=0}^{\infty}\lambda_1^k,\dots,\sum_{k=0}^{\infty}\lambda_m^k].\tag{3.8}$$

When $i=k$,

$$\begin{aligned}\frac{\partial u_{II}}{\partial z_{ik}}=&-\frac{\partial\log\det(I-ZZ^*)}{\partial z_{ik}}\\ =&-\sum_{s,t=1}^{m}((I_m-ZZ^*)^{-1})_{st}\frac{\partial}{\partial z_{ik}}\Big(\delta_{st}-\sum_{\ell=1}^{m}\frac{z_{s\ell}}{\sqrt{2}p_{s\ell}}\frac{\overline{z}_{t\ell}}{\sqrt{2}p_{t\ell}}\Big)\\ =&-\frac{1}{\sqrt{2}p_{ik}}\sum_{s,t=1}^{m}((I_m-ZZ^*)^{-1})_{st}\Big(-\sum_{\ell=1}^{m}\delta_{is}\delta_{k\ell}\frac{\overline{z}_{t\ell}}{\sqrt{2}p_{t\ell}}\Big)\\ =&\frac{1}{\sqrt{2}p_{ik}}\frac{\frac{\overline{z}_{ik}}{\sqrt{2}p_{ik}}}{1-\lambda_i}.\end{aligned}\tag{3.9}$$

When $i<k$,

$$\begin{aligned}\frac{\partial u_{II}}{\partial z_{ik}}=&-\frac{\partial\log\det(I-ZZ^*)}{\partial z_{ik}}\\ =&\frac{1}{\sqrt{2}p_{ik}}\sum_{s,t=1}^{m}\Big((I_m-ZZ^*)^{-1}\Big)_{st}\sum_{\ell=1}^{m}\Big(\delta_{is}\delta_{k\ell}\frac{\overline{z}_{t\ell}}{\sqrt{2}p_{t\ell}}+\delta_{i\ell}\delta_{ks}\frac{\overline{z}_{t\ell}}{\sqrt{2}p_{t\ell}}\Big)\\ =&\frac{1}{\sqrt{2}p_{ik}}\Big(\frac{\frac{\overline{z}_{ik}}{\sqrt{2}p_{ik}}}{1-\lambda_i}+\frac{\frac{\overline{z}_{ki}}{\sqrt{2}p_{ki}}}{1-\lambda_k}\Big).\end{aligned}\tag{3.10}$$

Thus,

$$\frac{\partial u_{II}}{\partial z_{ik}}=\frac{p_{ik}}{\sqrt{2}}\frac{\overline{z}_{ik}}{\sqrt{2}p_{ik}}\Big(\frac{1}{1-\lambda_i}+\frac{1}{1-\lambda_k}\Big).\tag{3.11}$$

Similarly,

$$\frac{\partial u_{II}}{\partial \overline{z}_{j\ell}}=\frac{p_{j\ell}}{\sqrt{2}}\frac{z_{j\ell}}{\sqrt{2}p_{j\ell}}\Big(\frac{1}{1-\lambda_j}+\frac{1}{1-\lambda_\ell}\Big).\tag{3.12}$$

Therefore,

$$
\begin{aligned}
|\partial u_{II}|^2_{g_{II}} &= \sum_{i,j=1}^{m} \sum_{k\geq i,\ell\geq j} u_{II}^{ik,j\ell} \frac{\partial u_{II}}{\partial z_{ik}} \frac{\partial u_{II}}{\partial \overline{z}_{j\ell}} \\
&= \frac{1}{4} \sum_{i=1}^{m}\sum_{k=1}^{m} (1-\lambda_i)(1-\lambda_k)\Big(\frac{1}{1-\lambda_i} + \frac{1}{1-\lambda_k}\Big)^2 \Big|\frac{z_{ik}}{\sqrt{2}p_{ik}}\Big|^2 \\
&= \frac{1}{2} \sum_{i,k=1}^{m} \Big(\frac{1-\lambda_k}{1-\lambda_i} + 1\Big) \Big|\frac{z_{ik}}{\sqrt{2}p_{ik}}\Big|^2 \leq m. \qquad (3.13)
\end{aligned}
$$

For the convenience of reader, we state the following proposition [13, Theorem 2.2.1]:

Proposition 3.2 ([13]) *With the notations above, one has,*

$$
J_m(\lambda) := \int_{R_{II}} \det(I - ZZ^*)^{\lambda} dZ < +\infty \iff \lambda > -1. \qquad (3.14)
$$

Finally, we will be able to prove the following main theorem of this article.

Theorem 3.3 *Let $\Delta_{g_{II}}$ be the Laplace-Beltrami operator with respect to g_{II}, and $\lambda_1(\Delta_{g_{II}})$ be the infimum of the spectrum of the Laplace-Beltrami operator on R_{II} with Bergman metric g_{II}. Then*

$$
\lambda_1(\Delta_{II}) \in \Big[\frac{m(m+1)^2}{4}, m^3\Big]. \qquad (3.15)
$$

Proof By statement (i) of Theorem 1.1 ([9, Proposition 2.1]) and Proposition 3.1, one has

$$
\lambda_1(\Delta_{II}) \geq \frac{(\frac{m(m+1)}{2})^2}{m} = \frac{m(m+1)^2}{4}.
$$

For $f(Z) = e^{-\tau u_{II}(z)}$, by Proposition 2.3, one has

$$
\begin{aligned}
\int_{R_{II}} |f(Z)|^2 \mathrm{d}V_{u_{II}}(Z) &= \int_{R_{II}} K_{II}(Z,Z)^{\frac{-2\tau}{c_{II}}} \mathrm{d}V_{u_{II}}(Z) \\
&= C_{II} \int_{R_{II}} K_{II}(Z,Z)^{(1-2\frac{\tau}{c_{II}})} \mathrm{d}v(z), \qquad (3.16)
\end{aligned}
$$

where $\mathrm{d}V_{u_{II}} = \det[H(u_{II})]\mathrm{d}v$, $c_{II} = m+1$ and C_{II} is a constant which depends on R_{II}.

By Faraut and Koranyi [15] and Proposition 3.2,

$$
\int_{R_{II}} K_{II}(Z,Z)^{\alpha} \mathrm{d}v \begin{cases} = +\infty & \alpha \geq \alpha_{II} =: \frac{1}{m+n}, \\ < +\infty & \alpha < \alpha_{II} = \frac{1}{m+n}. \end{cases} \qquad (3.17)
$$

Now choose τ such that

$$
1 - 2\frac{\tau}{c_{II}} < \alpha_{II} \iff \tau > \frac{1}{2} c_{II}(1 - \alpha_{II}). \qquad (3.18)
$$

Applying the argument of the proof of [9, Theorem 2.2] and Proposition 3.1, one has

$$\begin{aligned}\lambda_1(\Delta_{g_{II}}) \leq & 4\frac{\int_{R_{II}} \sum u_{II}^{ik,j\ell} \frac{\partial f}{\partial z_{ik}} \frac{\partial f}{\partial \bar{z}_{j\ell}} \mathrm{d}V_{u_{II}}}{\int_{R_{II}} |f|^2 \mathrm{d}V_{u_{II}}} \\ =& 4\tau^2 \frac{\int_{R_{II}} |f|^2 \sum u_{II}^{ik,j\ell} \frac{\partial u_{II}}{\partial z_{ik}} \frac{\partial u_{II}}{\partial \bar{z}_{j\ell}} \mathrm{d}V_{u_{II}}}{\int_{R_{II}} |f|^2 \mathrm{d}V_{u_{II}}} \\ =& 4\tau^2 \frac{\int_{R_{II}} |f|^2 |\partial u_{II}|^2_{g_{II}} \mathrm{d}V_{u_{II}}}{\int_{R_{II}} |f|^2 \mathrm{d}V_{u_{II}}} \leq 4\tau^2 m. \end{aligned} \tag{3.19}$$

Let $\tau \to \frac{1}{2}c_{II}(1-\alpha_{II})$. Then

$$\lambda_1(\Delta_{g_I}) \leq 4m[\frac{1}{2}c_{II}(1-\alpha_{II})]^2 = mc_{II}^2(1-\alpha_{II})^2. \tag{3.20}$$

Thus,

$$\lambda_1(\Delta_{g_I}) \leq mc_{II}^2(1-\alpha_{II})^2 = m(m+1)^2(1-\frac{1}{m+1})^2 = m^3. \quad \square$$

Remark 3.4 In particular, when R_{II} has rank 1, the upper bound and lower bound of $\lambda_1(\Delta_{g_{II}})$ are equal, and $\lambda_1(\Delta_{g_{II}}) = 1$, which agrees with the known result in [9].

Acknowledgements The authors gratefully acknowledge their appreciations to Professor Songying LI for his supervision.

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