Journal of Mathematical Research with Applications
Jan., 2019, Vol. 39, No. 1, pp. 51–60
DOI:10.3770/j.issn:2095-2651.2019.01.006
Http://jmre.dlut.edu.cn

Relative Property A for Discrete Metric Space

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Abstract Yu introduced Property A on discrete metric spaces. In this paper, a relative Property A for a discrete metric space X with respect to a set Y and a map $\rho_{X,Y}$ is defined. Some characterizations for relative Property A are given. In particular, a discrete metric space with relative Property A can be coarse embedding into a Hilbert space under certain condition.

Keywords relative Property A; coarse embedding; mazur map

MR(2010) Subject Classification 46B85; 46C05

1. Introduction

Coarse embeddings were introduced by Gromov in [1]. A function $f : X \to Y$ between metric spaces is a coarse embedding if there exist two non-decreasing maps $\theta_1, \theta_2 : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\theta_1(t) \to +\infty$ as $t \to +\infty$ and $\theta_1(d_X(x, y)) \le d_Y(f(x), f(y)) \le \theta_2(d_X(x, y))$ for all $x, y \in X$. The readers can refer to the book [2] for a self-contained introduction to coarse geometry.

Property A is a weak version of amenability for discrete metric space which was introduced by Yu [3], who claimed that a metric space satisfies this property can be coarse embedding into a Hilbert space. For metric spaces with bounded geometry it implies the coarse Baum Connes conjecture, and for a finitely generated group with word-length metric it implies the strong Novikov conjecture. Kasparov and Yu treated the case when the Hilbert space is replaced with a uniformly convex Banach space [4]. In [5], Nowak constructed some metric spaces which do not satisfy Property A but embed coarsely into a Hilbert space. So coarse embedding into a Hilbert space and Property A are not equivalent. In [6], Higson and Roe gave a useful equivalent definition of Property A. They claimed that a discrete metric space with bounded geometry with Property A if and only if for every $R > 0$ and $\varepsilon > 0$, there exists a map $\xi : X \to \ell^1(X)^+$ and an $S \in \mathbb{R}^+$ such that $\|\xi_x\|_{\ell^1} = 1$ and $\operatorname{supp} \xi_x \subseteq B(x, S)$ for every $x \in X$ and $\|\xi_x - \xi_y\|_{\ell^1} < \varepsilon$ whenever $d(x, y) < R$. In [7], Ji, Ogle and Ramsey defined relative Property A for a discrete group G relative to a finite family of subgroups $\mathscr{H}$, and they showed that if G has Property A

Received May 1, 2018; Accepted August 12, 2018
Supported by the National Natural Science Foundation of China (Grant No. 11701301).
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relative to a family of subgroups $\mathscr{H}$ and if each $H \in \mathscr{H}$ has Property A then G has Property A. Readers can refer to [8] for more details.

In this paper, we define a relative Property A for metric space with respect to a set Y and a map $\rho_{X,Y}$, which is a generalization of Yu's Property A. Several examples and equivalent characterizations of relative Property A are given. In particular, we show that relative Property A implies coarse embedding into a Hilbert space if $d(x_1,x_2) \le \rho_{X,Y}(x_1,y) + \rho_{X,Y}(x_2,y)$ for all $x_1,x_2 \in X$ and $y \in Y$.

2. Relative Property A

Recall that a discrete metric space (X,d) has Property A if for all $R,\varepsilon > 0$, there exists a family $\{A_x\}_{x\in X}$ of finite non-empty subsets of $X \times \mathbb{N}$ such that (1) for all $x,y \in X$ with $d(x,y) \le R$, we have $\frac{|A_x \triangle A_y|}{|A_x \cap A_y|} < \varepsilon$; (2) there exists an S such that for each $x \in X$ if $(y,n) \in A_x$, then $d(x,y) \le S$.

Definition 2.1 *Let X be a discrete metric space (X,d) and Y be a set with $\rho_{X,Y} : X\times Y \to \mathbb{R}^+$. We say X has relative Property A with respect to Y and $\rho_{X,Y}$ if the following are satisfied: for any $R > 0$ and $\varepsilon > 0$, there exists an $S > 0$ and a collection $\{A_x\}_{x\in X}$ of finite nonempty subsets of $Y \times \mathbb{N}$ such that:*

(1) For each $x \in X$ if $(y,n) \in A_x$, then $y \in B_{\rho_{X,Y}}(x,S)$, where $B_{\rho_{X,Y}}(x,S) = \{y \in Y | \rho_{X,Y}(x,y) \le S\}$;

(2) If $d(x_1,x_2) < R$, then $\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} < \varepsilon$.

The following proposition gives the relationships between Property A and relative Property A.

Proposition 2.2 *Suppose (Z,d) is a discrete metric space, X is a subspace of Z, Y is a subset of Z and the map $\rho_{X,Y}(x,y) = d(x,y)$ for all $x \in X, y \in Y$, then*

(1) If $X \subseteq Y$ and X has Property A, then X has relative Property A with respect to Y and $\rho_{X,Y}$;

(2) If $Y \subseteq X$ and X has relative Property A with respect to Y and $\rho_{X,Y}$, then X has Property A;

(3) If $X = Y$, X has Property A if and only if X has relative property A with respect to Y and $\rho_{X,Y}$.

Proof (1) Suppose X has Property A. Then for any $R > 0$ and $\varepsilon > 0$, there exists a collection $\{A_x\}_{x\in X}$ and an S satisfying the definition of Property A. Since $X \subseteq Y$, we can see A_x as subset of $Y \times \mathbb{N}$. Then if $(y,n) \in A_x$, we have $d(x,y) \le S$. So $y \in B_{\rho_{X,Y}}(x,S)$. For any $x_1,x_2 \in X$ with $d(x_1,x_2) < R$, we have

$$\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} < \varepsilon.$$

So X has relative Property A with respect to Y and $\rho_{X,Y}$.

(2) The proof is similar to that in (1).

(3) It is clear from (1) and (2). □

Now we give some examples about relative Property A.

Example 2.3 Let X be a finite metric space with a set Y and a map $\rho_{X,Y} : X \times Y \to \mathbb{R}^+$. For any $R > 0$ and $\varepsilon > 0$, fix $y_0 \in Y$, let $S = \max_{x \in X} \rho_{X,Y}(x, y_0)$ and $A_x = (y_0, 1) \subseteq Y \times \mathbb{N}$ for any $x \in X$. If $(y, n) \in A_x$, then $y = y_0$, $n = 1$. So $y \in B_{\rho_{X,Y}}(x, S)$. For any $x_1, x_2 \in X$ with $d(x_1, x_2) < R$, we have $\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = 0 < \varepsilon$. So X has relative Property A with respect to Y and $\rho_{X,Y}$.

Example 2.4 Let X be a discrete metric space with a set Y and a map $\rho_{X,Y} : X \times Y \to \mathbb{R}^+$. If $\rho_{X,Y}$ is uniformly bounded, then there exists an S such that $\rho_{X,Y}(x, y) \leq S$ for all $x \in X$ and $y \in Y$. For any $R > 0$ and $\varepsilon > 0$, we fix a $y_0 \in Y$ and let $A_x = (y_0, 1) \subseteq Y \times \mathbb{N}$ for all $x \in X$. It is clear that if $(y, n) \in A_x$, we have $\rho_{X,Y}(x, y) \leq S$. For any $x_1, x_2 \in X$ with $d(x_1, x_2) < R$, we have $\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = 0 < \varepsilon$. So X has relative Property A with respect to Y and $\rho_{X,Y}$.

Example 2.5 If $X = \mathbb{Z}$ and Y is a countable set. Let f be a bijection from Y to $\mathbb{Z}$. We define $\rho_{X,Y}(x, y) = |x - f(y)|$. Fix $R > 0$ and $\varepsilon > 0$ where $\varepsilon < 1$ and choose an $S \in \mathbb{N}$ such that $S > 2R\varepsilon^{-1}$. Define $A_x = \{(y, 1) \in Y \times \mathbb{N} | \rho_{X,Y}(x, y) \leq S\}$. It is clear that each A_x is a finite subset in $Y \times \mathbb{N}$. By the definition of A_x, $\rho_{X,Y}(x, y) \leq S$ if $(y, n) \in A_x$. For any $x_1, x_2 \in X$ with $|x_1 - x_2| < R$, we have $|A_{x_1} \cup A_{x_2}| = 2S + |x_1 - x_2| + 1$, $|A_{x_1} \triangle A_{x_2}| = 2|x_1 - x_2|$ and $|A_{x_1} \cap A_{x_2}| = 2S - |x_1 - x_2| + 1$. So

$$\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = \frac{2|x_1 - x_2|}{2S - |x_1 - x_2| + 1} < \frac{2R}{2S - R + 1} < \frac{2R}{2R\varepsilon^{-1} - R} = \frac{2\varepsilon}{2 - \varepsilon}.$$

So $\mathbb{Z}$ has relative Property A with respect to a countable set Y and $\rho_{X,Y}$. Let $Y = \mathbb{Z}$ and $\rho_{X,Y}(x, y) = |x - y|$. From Proposition 2.2, we conclude that $\mathbb{Z}$ has Property A.

Next we give an example of discrete space with Property A but without relative Property A with a map $\rho_{X,Y}$ and some set Y.

Example 2.6 If $X = \mathbb{Z}$ and $Y = \{0\} \in \mathbb{Z}$ is a single point set. We define $\rho_{X,Y}(x, y) = |x - y|$ for all $x \in X$ and $y \in Y$. For any $S > 0$, there exists an $x_0 \in \mathbb{Z}$ such that $|x_0 - 0| = |x_0| > S$. Since A_{x_0} is nonempty, there exists $n \in \mathbb{N}$ such that $(0, n) \in A_{x_0}$ and $\rho_{X,Y}(x_0, 0) > S$. So $\mathbb{Z}$ dose not have relative Property A with respect to $Y = \{0\}$ and $\rho_{X,Y}(x, y) = |x - y|$.

Proposition 2.7 *If a discrete metric space X has relative Property A with respect to Y and $\rho_{X,Y}$, then any subspace of X also has relative Property A with respect to Y and $\rho_{X,Y}$.*

Proof Suppose X has relative Property A with respect to Y and $\rho_{X,Y}$ and X' is a subspace of X. Let $\{A_x\}_{x \in X}$ be the sets satisfying the definition of relative Property A with respect to Y and $\rho_{X,Y}$. We can choose the subfamilies $\{A_x\}_{x \in X'}$. It is easy to check these sets satisfying the definition of X' having relative Property A with respect to Y and $\rho_{X,Y}$. □

Proposition 2.8 *Let X be a discrete metric space (X, d) and Y_1, Y_2 be subsets of Y. If X has relative Property A with respect to both Y_1 and Y_2 and $\rho_{X,Y}$, then X has relative Property A* *with respect to* $Y_1 \cup Y_2$ *and* $\rho_{X,Y}$.

Proof Suppose that X has relative Property A with respect to both Y_1 and Y_2 and $\rho_{X,Y}$. For any $R > 0$ and $\varepsilon > 0$, let $\{A'_x\}_{x\in X}$ and $\{A''_x\}_{x\in X}$ be the sets satisfying the definition of relative Property A with respect to Y_1 and Y_2 and $\rho_{X,Y}$, and S', S'' be the relevant constants. Let $A_x = A'_x \cup A''_x$. If $(y,n) \in A_x$, then $(y,n) \in A'_x$ or $(y,n) \in A''_x$. In the first case, $\rho_{X,Y}(x,y) \le S'$; in the second case, $\rho_{X,Y}(x,y) \le S''$. So $\rho_{X,Y}(x,y) \le S$ where $S = \max\{S', S''\}$.

For each $x_1, x_2 \in X$ with $d(x_1,x_2) < R$, from the condition (2) of relative Property A, we get $\frac{|A'_{x_1}\triangle A'_{x_2}|}{|A'_{x_1}\cap A'_{x_2}|} < \varepsilon$ and $\frac{|A''_{x_1}\triangle A''_{x_2}|}{|A''_{x_1}\cap A''_{x_2}|} < \varepsilon$. Then

$$\begin{aligned}\frac{|A_{x_1}\triangle A_{x_2}|}{|A_{x_1}\cap A_{x_2}|} &= \frac{|(A'_{x_1}\cup A''_{x_1})\triangle(A'_{x_2}\cup A''_{x_2})|}{|(A'_{x_1}\cup A''_{x_1})\cap(A'_{x_2}\cup A''_{x_2})|}\\ &= \frac{|(A'_{x_1}\triangle A'_{x_2})\cup(A''_{x_1}\triangle A''_{x_2})|}{|(A'_{x_1}\cap A'_{x_2})\cup(A'_{x_1}\cap A''_{x_2})\cup(A''_{x_1}\cap A'_{x_2})\cup(A''_{x_1}\cap A''_{x_2})|}\\ &\le \frac{|A'_{x_1}\triangle A'_{x_2}|}{|(A'_{x_1}\cap A'_{x_2})\cup(A'_{x_1}\cap A''_{x_2})\cup(A''_{x_1}\cap A'_{x_2})\cup(A''_{x_1}\cap A''_{x_2})|}+\\ &\quad \frac{|A''_{x_1}\triangle A''_{x_2}|}{|(A'_{x_1}\cap A'_{x_2})\cup(A'_{x_1}\cap A''_{x_2})\cup(A''_{x_1}\cap A'_{x_2})\cup(A''_{x_1}\cap A''_{x_2})|}\\ &\le \frac{|A'_{x_1}\triangle A'_{x_2}|}{|A'_{x_1}\cap A'_{x_2}|}+\frac{|A''_{x_1}\triangle A''_{x_2}|}{|A''_{x_1}\cap A''_{x_2}|}\\ &< 2\varepsilon.\end{aligned}$$

So the proposition is proved. □

Proposition 2.9 *If* X_1 *is a discrete metric space with relative Property A with respect to* Y *and* $\rho_{X_1,Y}$ *and* X_2 *is a discrete metric space with relative Property A with respect to* Y *and* $\rho_{X_2,Y}$. *Then* $X_1 \times X_2$ *is a discrete metric space with relative Property A with respect to* Y *and* $\rho_{X_1\times X_2,Y}$, *where* $\rho_{X_1\times X_2,Y}((x_1,x_2),y) = \min\{\rho_{X_1,Y}(x_1,y), \rho_{X_2,Y}(x_2,y)\}$ *for all* $x_1 \in X_1, x_2 \in X_2$ *and* $y \in Y$, *and* $d_{X_1\times X_2}((x'_1,x'_2),(x''_1,x''_2)) = d_{X_1}(x'_1,x''_1) + d_{X_2}(x'_2,x''_2)$ *for all* $x'_1, x''_1 \in X_1$, $x'_2, x''_2 \in X_2$.

Proof Suppose X_1 and X_2 have relative Property A with respect to Y and $\rho_{X_1,Y}$ and $\rho_{X_2,Y}$. For any $R > 0$ and $\varepsilon > 0$, let $\{A_{x_1}\}_{x_1\in X_1}$, $\{A_{x_2}\}_{x_2\in X_2}$ be the sets satisfying the definition of relative Property A and S_1, S_2 be the relevant constants. Set $A_{(x_1,x_2)} = A_{x_1} \cup A_{x_2}$. Then if $(y,n) \in A_{(x_1,x_2)}$, we have $(y,n) \in A_{x_1}$ or $(y,n) \in A_{x_2}$. In the first case, $\rho_{X_1,Y}(x_1,y) \le S_1$; in the second case, $\rho_{X_2,Y}(x_2,y) \le S_2$. Since $\rho_{X_1\times X_2,Y}((x_1,x_2),y) = \min\{\rho_{X_1,Y}(x_1,y), \rho_{X_1,Y}(x_1,y)\}$, so $\rho_{X_1\times X_2,Y}((x_1,x_2),y) \le S$ where $S = \max\{S_1, S_2\}$.

Suppose $x'_1, x''_1 \in X_1$ and $x'_2, x''_2 \in X_2$ with $d_{X_1\times X_2}((x'_1,x'_2),(x''_1,x''_2)) < R$. We have $d_{X_1}(x'_1,x''_1) < R$ and $d_{X_2}(x'_2,x''_2) < R$, so $\frac{|A_{x'_1}\triangle A_{x''_1}|}{|A_{x'_1}\cap A_{x''_1}|} < \varepsilon$ and $\frac{|A_{x'_2}\triangle A_{x''_2}|}{|A_{x'_2}\cap A_{x''_2}|} < \varepsilon$. As the proof of above proposition, we can see that

$$\frac{|A_{(x'_1,x'_2)}\triangle A_{(x''_1,x''_2)}|}{|A_{(x'_1,x'_2)}\cap A_{(x''_1,x''_2)}|} = \frac{|(A_{x'_1}\cup A_{x'_2})\triangle(A_{x''_1}\cup A_{x''_2})|}{|(A_{x'_1}\cup A_{x'_2})\cap(A_{x''_1}\cup A_{x''_2})|} < 2\varepsilon.$$

So $X_1 \times X_2$ is a metric space with relative Property A with respect to Y and $\rho_{X_1\times X_2,Y}$. □

One of Yu's main motivations for introducing Property A was that it implies coarse embedding into Hilbert space. Now we prove that a discrete metric space with relative Property A with respect to Y and $\rho_{X,Y}$ can be coarse embedding into a Hilbert space under the condition of $d(x_1, x_2) \leq \rho_{X,Y}(x_1, y) + \rho_{X,Y}(x_2, y)$ for all $x_1, x_2 \in X$ and $y \in Y$. The main idea of the following theorem comes from [8].

Theorem 2.10 *Suppose (X, d) is a discrete metric space with relative Property A with respect to Y and $\rho_{X,Y}$. If $d(x_1, x_2) \leq \rho_{X,Y}(x_1, y) + \rho_{X,Y}(x_2, y)$ for all $x_1, x_2 \in X$ and $y \in Y$, then X can be coarse embedding into a Hilbert space.*

Proof First we define a Hilbert space

$$\mathcal{H} = \bigoplus_{k=1}^{\infty} \ell^2(Y \times \mathbb{N}).$$

Since X is a discrete metric space with relative Property A with respect to Y and $\rho_{X,Y}$, then for $R = k \in \mathbb{N}$ and $\varepsilon = \frac{1}{2^{2k+1}} > 0$, we can define a sequence of sets $\{A_x^k\}_{x \in X} \subseteq Y \times \mathbb{N}$ satisfying:

(1) There exists an $S_k > k$, such that if $(y, n) \in A_x^k$, then $y \in B_{\rho_{X,Y}}(x, \frac{1}{2}S_k)$;

(2) For any $x_1, x_2 \in X$ with $d(x_1, x_2) < k$, we have $\frac{|A_{x_1}^k \triangle A_{x_2}^k|}{|A_{x_1}^k \cap A_{x_2}^k|} < \frac{1}{2^{2k+1}}$.

Let χ_{A_x} be the characteristic function of A_x. For any $x_1, x_2 \in X$ with $d(x_1, x_2) < k$, we have

$$\begin{aligned}
&\|\frac{\chi_{A_{x_1}^k}}{|A_{x_1}^k|^{\frac{1}{2}}} - \frac{\chi_{A_{x_2}^k}}{|A_{x_2}^k|^{\frac{1}{2}}}\|_{\ell^2(Y\times\mathbb{N})}^2 \\
&= \frac{|A_{x_1}^k \backslash A_{x_2}^k|}{|A_{x_1}^k|} + \frac{|A_{x_2}^k \backslash A_{x_1}^k|}{|A_{x_2}^k|} + |A_{x_1}^k \cap A_{x_2}^k|(\frac{1}{|A_{x_1}^k|^{\frac{1}{2}}} - \frac{1}{|A_{x_2}^k|^{\frac{1}{2}}})^2 \\
&= \frac{|A_{x_1}^k \backslash A_{x_2}^k|}{|A_{x_1}^k|} + \frac{|A_{x_2}^k \backslash A_{x_1}^k|}{|A_{x_2}^k|} + |A_{x_1}^k \cap A_{x_2}^k|(\frac{|A_{x_1}^k| + |A_{x_2}^k| - 2|A_{x_1}^k|^{\frac{1}{2}}|A_{x_2}^k|^{\frac{1}{2}}}{|A_{x_1}^k||A_{x_2}^k|}) \\
&\leq \frac{|A_{x_1}^k \triangle A_{x_2}^k|}{|A_{x_1}^k \cap A_{x_2}^k|} + |A_{x_1}^k \cap A_{x_2}^k|(\frac{|A_{x_1}^k| + |A_{x_2}^k| - 2|A_{x_1}^k|^{\frac{1}{2}}|A_{x_1}^k|^{\frac{1}{2}}}{|A_{x_2}^k||A_{x_2}^k|}) \\
&\leq \frac{|A_{x_1}^k \triangle A_{x_2}^k|}{|A_{x_1}^k \cap A_{x_2}^k|} + \frac{|A_{x_1}^k| + |A_{x_2}^k| - 2|A_{x_1}^k \cap A_{x_2}^k|}{|A_{x_1}^k \cap A_{x_2}^k|} \\
&= \frac{2|A_{x_1}^k \triangle A_{x_2}^k|}{|A_{x_1} \cap A_{x_2}|} < \frac{1}{2^{2k}}.
\end{aligned}$$

Hence

$$\|\frac{\chi_{A_{x_1}^k}}{|A_{x_1}^k|^{\frac{1}{2}}} - \frac{\chi_{A_{x_2}^k}}{|A_{x_2}^k|^{\frac{1}{2}}}\|_{\ell^2(Y\times\mathbb{N})} < \frac{1}{2^k}$$

for any $x_1, x_2 \in X$ with $d(x_1, x_2) < k$.

Now, choose $x_0 \in X$ and define $f : X \to \mathcal{H}$ by

$$f(x) = \bigoplus_{k=1}^{\infty}(\frac{\chi_{A_x^k}}{|A_x^k|^{\frac{1}{2}}} - \frac{\chi_{A_{x_0}^k}}{|A_{x_0}^k|^{\frac{1}{2}}}).$$

For any $x \in X$, there exists a $k' \in \mathbb{N} > 0$ such that $d(x, x_0) < k'$. We have

$$\|f(x) - f(x_0)\| = \sum_{k=1}^{\infty} \|\frac{\chi_{A_x^k}}{|A_x^k|^{\frac{1}{2}}} - \frac{\chi_{A_{x_0}^k}}{|A_{x_0}^k|^{\frac{1}{2}}}\|_{\ell^2(Y\times\mathbb{N})}$$

$$
\begin{aligned}
&= \sum_{k=1}^{k'-1} \| \frac{\chi_{A_x^k}}{|A_x^k|^{\frac{1}{2}}} - \frac{\chi_{A_{x_0}^k}}{|A_{x_0}^k|^{\frac{1}{2}}} \|_{\ell^2(Y\times\mathbb{N})} + \sum_{k=k'}^{\infty} \| \frac{\chi_{A_x^k}}{|A_x^k|^{\frac{1}{2}}} - \frac{\chi_{A_{x_0}^k}}{|A_{x_0}^k|^{\frac{1}{2}}} \|_{\ell^2(Y\times\mathbb{N})} \\
&\le \sum_{k=1}^{k'-1} \| \frac{\chi_{A_x^k}}{|A_x^k|^{\frac{1}{2}}} - \frac{\chi_{A_{x_0}^k}}{|A_{x_0}^k|^{\frac{1}{2}}} \|_{\ell^2(Y\times\mathbb{N})} + 1 < \infty.
\end{aligned}
$$

So f is well-defined.

For any $m \in \mathbb{N}$, if $x_1, x_2 \in X$ with $m \le d(x_1, x_2) < m+1$, we can get the estimate

$$
\begin{aligned}
\|f(x_1) - f(x_2)\|^2 &= \sum_{k=1}^{\infty} \| \frac{\chi_{A_{x_1}^k}}{|A_{x_1}^k|^{\frac{1}{2}}} - \frac{\chi_{A_{x_2}^k}}{|A_{x_2}^k|^{\frac{1}{2}}} \|^2_{\ell^2(Y\times\mathbb{N})} \\
&\le \sum_{k=1}^{m} \| \frac{\chi_{A_{x_1}^k}}{|A_{x_1}^k|^{\frac{1}{2}}} - \frac{\chi_{A_{x_2}^k}}{|A_{x_2}^k|^{\frac{1}{2}}} \|^2_{\ell^2(Y\times\mathbb{N})} + 1 \\
&\le \sum_{k=1}^{m} (\| \frac{\chi_{A_{x_1}^k}}{|A_{x_1}^k|^{\frac{1}{2}}} \|_{\ell^2(Y\times\mathbb{N})} + \| \frac{\chi_{A_{x_2}^k}}{|A_{x_2}^k|^{\frac{1}{2}}} \|_{\ell^2(Y\times\mathbb{N})})^2 + 1 \\
&= 4m + 1.
\end{aligned}
$$

Hence $\|f(x_1) - f(x_2)\| \le \sqrt{4m+1}$.

On the other hand, we define a map $Q: \mathbb{R}^+ \to \mathbb{N}$ by $Q(t) = |\{S_k | S_k < t\}|$ for $t \in \mathbb{R}^+$. From the choice of the families $\{A_x^k\}_{x\in X}$, $Q(t)$ exists for all $t \in \mathbb{R}^+$ since $S_k \ge k$, and $Q(t) \to \infty$ as $t \to \infty$. Now we claim that for any $x_1, x_2 \in X$ if $d(x_1, x_2) > S_k$, then $A_{x_1}^k \cap A_{x_2}^k = \emptyset$. Indeed, if there exists $(y, n) \in A_{x_1}^k \cap A_{x_2}^k$, then $\rho_{X,Y}(x_1, y) \le \frac{1}{2} S_k$ and $\rho_{X,Y}(x_2, y) \le \frac{1}{2} S_k$. Since for any $x_1, x_2 \in X$, we have $d(x_1, x_2) \le \rho_{X,Y}(x_1, y) + \rho_{X,Y}(x_2, y) \le S_k$, this is a contradiction. For any $m \in \mathbb{N}$, if $m \le d(x_1, x_2) < m+1$, we write $Q(d(x_1, x_2)) = r$. So the number of S_k satisfies $S_k < d(x_1, x_2)$ is r. Then we can write the set $\{S_k | S_k < d(x_1, x_2)\}$ as $\{S_{k_1}, S_{k_2}, ..., S_{k_r}\}$. Since $d(x_1, x_2) > S_{k_j}$, we have $A_{x_1}^{k_j} \cap A_{x_2}^{k_j} = \emptyset$ for $j = 1, \dots, r$. We have

$$
\begin{aligned}
\|f(x_1) - f(x_2)\|^2 &= \sum_{k=1}^{\infty} \| \frac{\chi_{A_{x_1}^k}}{|A_{x_1}^k|^{\frac{1}{2}}} - \frac{\chi_{A_{x_2}^k}}{|A_{x_2}^k|^{\frac{1}{2}}} \|^2_{\ell^2(Y\times\mathbb{N})} \\
&\ge \sum_{j=1}^{r} \| \frac{\chi_{A_{x_1}^{k_j}}}{|A_{x_1}^{k_j}|^{\frac{1}{2}}} - \frac{\chi_{A_{x_2}^{k_j}}}{|A_{x_2}^{k_j}|^{\frac{1}{2}}} \|^2_{\ell^2(Y\times\mathbb{N})} = 2r.
\end{aligned}
$$

Hence $\|f(x_1) - f(x_2)\| \ge \sqrt{2r}$.

Define the maps $\theta_1, \theta_2 : \mathbb{R}^+ \to \mathbb{R}^+$ by $\theta_1(t) = \sqrt{2Q(t)}$ and $\theta_2(t) = \sqrt{4t+1}$ for $t \in \mathbb{R}^+$. It is clear that both $\theta_1(t)$ and $\theta_2(t)$ are non-decreasing and $\theta_1(t) \to +\infty$ as $t \to +\infty$. We have

$$\theta_1(d(x_1, x_2)) \le d(f(x_1), f(x_2)) \le \theta_2(d(x_1, x_2))$$

for any $x_1, x_2 \in X$.

So X can be coarse embedding into a Hilbert space. $\square$

In [8], the author collected many equivalent formulations of Property A under the condition of bounded geometry. Recall that a metric space X is bounded geometry if for every $C > 0$, there is an absolute bound on the number of elements in any ball within X of radius C. In this paper, we need to restrict the discrete metric space with relative bounded geometry in order to prove the similar results.

Definition 2.11 *A discrete metric space (X, d) with respect Y and $\rho_{X,Y}$ is relative bounded geometry if for all $L > 0$, there exists an $N_L \in \mathbb{N}$ such that $|\{y \in Y | y \in B_{\rho_{X,Y}}(x, L)\}| < N_L$ for all $x \in X$.*

If we let $X = Y$ and $\rho_{X,Y} = d$, then it is exactly the definition of bounded geometry. In the following, we will give some equivalent formulations of relative Property A under this condition.

Theorem 2.12 *Let X be a discrete metric space with relative bounded geometry with respect to Y and $\rho_{X,Y}$. Then X has relative Property A with respect to Y and $\rho_{X,Y}$ if and only if for every $R > 0$ and $\varepsilon > 0$ there exists an $S > 0$ and $\xi : X \to \ell^1(Y)^+$ satisfying:*

(1) $\|\xi_x\|_{\ell^1} = 1$, for any $x \in X$;

(2) If $\xi_x(y) \neq 0$, then $y \in B_{\rho_{X,Y}}(x, S)$;

(3) If $d(x_1, x_2) < R$, then $\|\xi_{x_1} - \xi_{x_2}\|_{\ell^1} < \varepsilon$.

Proof Suppose that X has relative Property A with respect to Y and $\rho_{X,Y}$. For any $R > 0$ and $\varepsilon > 0$, let $\{A_x\}_{x \in X}$ and S satisfy the definition of relative Property A with respect to Y and $\rho_{X,Y}$.

For any $x \in X$, define

$$\xi_x : Y \to \mathbb{R}^+, \xi_x(y) = \frac{|A_x \cap (y \times \mathbb{N})|}{|A_x|}.$$

Then

$$\|\xi_x\|_{\ell^1} = \sum_{y \in Y} |\xi_x(y)| = \sum_{y \in Y} \frac{|A_x \cap (y \times \mathbb{N})|}{|A_x|} = 1.$$

For any $x_1, x_2 \in X$,

$$\begin{aligned} \|\xi_{x_1}|A_{x_1}| - \xi_{x_2}|A_{x_2}|\|_{\ell^1} &= \sum_{y \in Y} \big| |(y \times \mathbb{N}) \cap A_{x_1}| - |(y \times \mathbb{N}) \cap A_{x_2}| \big| \\ &= |A_{x_1} \triangle A_{x_2}|. \end{aligned}$$

Since for any $x_1, x_2 \in X$ with $d(x_1, x_2) < R$, we have

$$\varepsilon > \frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = \frac{|A_{x_1}| + |A_{x_2}| - 2|A_{x_1} \cap A_{x_2}|}{|A_{x_1} \cap A_{x_2}|}.$$

Whence

$$2 + \varepsilon > \frac{|A_{x_1}| + |A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} \geq \frac{|A_{x_1}| + |A_{x_2}|}{|A_{x_1}|} = 1 + \frac{|A_{x_2}|}{|A_{x_1}|}.$$

So by symmetry

$$1 + \varepsilon > \frac{|A_{x_2}|}{|A_{x_1}|} > \frac{1}{1 + \varepsilon}.$$

Combining these two comments, we conclude that

$$\begin{aligned} \|\xi_{x_1} - \xi_{x_2}\|_{\ell^1} &\leq \|\xi_{x_1} - \xi_{x_2}\frac{|A_{x_2}|}{|A_{x_1}|}\|_{\ell^1} + \|\xi_{x_2} - \xi_{x_2}\frac{|A_{x_2}|}{|A_{x_1}|}\|_{\ell^1} \\ &\leq \frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1}|} + |1 - \frac{|A_{x_2}|}{|A_{x_1}|}| \end{aligned}$$

$$\leq \frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} + |1 - \frac{|A_{x_2}|}{|A_{x_1}|}|$$
$$\leq 2\varepsilon.$$

Finally, note that if $\xi_x(y) \neq 0$, then $A_x \cap \{y \times \mathbb{N}\} \neq \emptyset$. So there exists an $n \in \mathbb{N}$ such that $(y, n) \in A_x$. By the definition of A_x, we have $y \in B_{\rho_{X,Y}}(x, S)$.

Conversely, suppose that for any $\varepsilon > 0$ and $R > 0$, there exists an S and ξ satisfying the conditions (1), (2) and (3). By relative bounded geometry, for all $x \in X$ the number of elements of y within the support of ξ_x is uniformly bounded. So we can use the same approximation method as proving [6, Lemma 3.5]. We may assume that there is a natural number M such that for all $x \in X$, the function $\xi_x \in \ell^1(Y)$ assumes only values in the range

$$\frac{0}{M}, \frac{1}{M}, \frac{2}{M}, \dots, \frac{M}{M}.$$

Define

$$A_x = \{(y, j) \in Y \times \mathbb{N} | \xi_x(y) \geq \frac{j}{M}, \quad j > 0, \ j \in \mathbb{N}\}.$$

Set

$$A_x^j = \{y \in Y | \xi_x(y) \geq \frac{j}{M}\}, \quad j = 0, \dots, M.$$

It is clear that $A_x^j \subseteq A_x^{j-1}$ for $j = 1, \dots, M$. Let $Z = \bigcup_{j=1}^M (A_x^{j-1} \backslash A_x^j) \cup A_x^M$. We can see that the sets $\{A_x^{j-1} \backslash A_x^j\}$ $(j = 1, \dots, M)$ and A_x^M are all disjoint. Since $\|\xi_x\| = 1$ for all $x \in X$, we have

$$1 = \sum_{y \in Y} \xi_x(y) = \sum_{j=1}^M (\big||A_x^{j-1}| \backslash |A_x^j|\big| \frac{j-1}{M} + |A_x^M| \frac{M}{M})$$
$$= \frac{1}{M} [\sum_{j=1}^M (|A_x^{j-1}| - |A_x^j|)(j-1) + M|A_x^M|]$$
$$= \frac{1}{M} (|A_x^1| + |A_x^2| + \dots + |A_x^M|).$$

So $\sum_{j=1}^M |A_x^j| = M$. Set

$$\tilde{A}_x^j = \{(y, j) \in Y \times \mathbb{N} | y \in A_x^j\}.$$

We have $|\tilde{A}_x^j| = |A_x^j|$ and $A_x = \bigcup_{j=1}^M \tilde{A}_x^j$. Since $\tilde{A}_x^j$ are all disjoint for $j = 1, \dots, M$. So $|A_x| = \sum_{j=1}^M |\tilde{A}_x^j| = \sum_{j=1}^M |A_x^j| = M$. If $(y, n) \in A_x$, then $\xi_x(y) \geq \frac{n}{M} > 0$. So $y \in B_{\rho_{X,Y}}(x, S)$. In addition, for any $x_1, x_2 \in X$, we have

$$|A_{x_1} \triangle A_{x_2}| = M \|\xi_{x_1} - \xi_{x_2}\|_{\ell^1} = |A_{x_1}| \|\xi_{x_1} - \xi_{x_2}\|_{\ell^1}.$$

So we obtain

$$\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1}|} < \varepsilon$$

when $d(x_1, x_2) < R$. Since

$$\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1}|} = \frac{|A_{x_1}| + |A_{x_2}| - 2|A_{x_1} \cap A_{x_2}|}{|A_{x_1}|},$$

we have

$$|A_{x_1} \cap A_{x_2}| > \frac{(2-\varepsilon)M}{2}.$$

Then

$$\frac{|A_{x_1} \triangle A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} = \frac{2M - 2|A_{x_1} \cap A_{x_2}|}{|A_{x_1} \cap A_{x_2}|} < \frac{2M-(2-\varepsilon)M}{\frac{(2-\varepsilon)M}{2}} = \frac{2\varepsilon}{2-\varepsilon}.$$

So X has relative Property A with respect to Y and $\rho_{X,Y}$. □

The result above can be generalized to the $\ell^p(Y)$, where $1 \le p < \infty$ and Y is a countable set. In order to obtain the next corollary, we introduce the Mazur map [9] first. The Mazur map $M_{p,q} : S(\ell^p) \to S(\ell^q)$ is defined by the formula

$$M_{p,q}(x) = \mathrm{sign}(x)|x|^{\frac{p}{q}}$$

where $S(\ell^p)$ is the unit sphere of ℓ^p and $x \in S(\ell^p)$. It is a uniform homeomorphism between unit spheres of ℓ^p and ℓ^q. More precisely, it satisfies the following inequalities:

$$\frac{p}{q}\|x-y\|_p \le \|M_{p,q}(x) - M_{p,q}(y)\|_q \le C\|x-y\|_p^{\frac{p}{q}}$$

for all $x, y \in S(\ell^p)$ and $p < q$, where the constant C depends only on $\frac{p}{q}$. We have the opposite inequalities if $p > q$.

Corollary 2.13 *Let Y be a countable set and X be a discrete metric space with relative bounded geometry with respect to Y and $\rho_{X,Y}$. Then X has relative Property A with respect to Y and $\rho_{X,Y}$ if and only if the following hold for any $1 \le p < \infty$: for every $R > 0$ and $\varepsilon > 0$ there exists an $S > 0$ and $\eta : X \to \ell^p(Y)^+$ satisfying:*

(1) $\|\eta_x\|_{\ell^p} = 1$ for any $x \in X$;

(2) If $\eta_x(y) \neq 0$, then $y \in B_{\rho_{X,Y}}(x, S)$;

(3) If $d(x_1, x_2) < R$, then $\|\eta_{x_1} - \eta_{x_2}\|_{\ell^p} \le \varepsilon$.

Proof We prove the sufficiency first. Suppose X has relative Property A with respect to Y and $\rho_{X,Y}$. For any $R > 0$ and $\varepsilon > 0$, there exists $\xi : X \to \ell^1(Y)^+$ and an S satisfying (1), (2) and (3) of the Theorem 2.12. Take any $1 \le p < \infty$ and define a function $\eta : X \to \ell^p(Y)^+$ satisfying

$$\eta_x(y) = \xi_x(y)^{\frac{1}{p}}.$$

Then

$$\|\eta_x\|_{\ell^p}^p = \sum_{y\in Y} |\eta_x(y)|^p = \sum_{y\in Y} \eta_x(y)^p = \sum_{y\in Y} \xi_x(y) = \sum_{y\in Y} |\xi_x(y)| = \|\xi_x\|_{\ell^1} = 1.$$

So $\|\eta_x\|_{\ell^p} = 1$. If $\eta_x(y) \neq 0$, then $\xi_x(y) \neq 0$. So $y \in B_{\rho_{X,Y}}(x, S)$.

For any $x_1, x_2 \in X$ with $d(x_1, x_2) < R$, we have

$$\begin{aligned}\|\eta_{x_1} - \eta_{x_2}\|_{\ell^p}^p &= \sum_{y\in Y} |\eta_{x_1}(y) - \eta_{x_2}(y)|^p \le \sum_{y\in Y} |\eta_{x_1}(y)^p - \eta_{x_2}(y)^p| \\ &= \sum_{y\in Y} |\xi_{x_1}(y) - \xi_{x_2}(y)| = \|\xi_{x_1} - \xi_{x_2}\|_{\ell^1} < \varepsilon.\end{aligned}$$

So $\|\eta_{x_1} - \eta_{x_2}\|_{\ell^p} < \varepsilon^{\frac{1}{p}}$.

Conversely, for any $R > 0$ and $\varepsilon > 0$ give a map $\eta : X \to \ell^p(Y)^+$ and an S which satisfy (1), (2), (3) of the corollary. Define $\xi : X \to \ell^1(Y)^+$ by the formula

$$\xi_x = M_{p,1}(\eta_x),$$

where $M_{p,1}$ is the Mazur map. Then for any $x \in X$, $\|\xi_x\|_{\ell^1} = 1$ and $\eta_x(y) = 0$ if and only if $\xi_x(y) = 0$ for all $y \in Y$. So if $\xi_x(y) \neq 0$, we have $y \in B_{\rho_{X,Y}}(x, S)$.

For any $x_1, x_2 \in X$ with $d(x_1, x_2) < R$, we have

$$\|\xi_{x_1} - \xi_{x_2}\|_{\ell^1} = \|M_{p,1}(\eta_{x_1}) - M_{p,1}(\eta_{x_2})\|_{\ell^p} \leq p\|\eta_{x_1} - \eta_{x_2}\|_{\ell^p} < p\varepsilon.$$

So X has relative Property A with respect to Y and $\rho_{X,Y}$. $\square$

Acknowledgements We thank the referees for their time and comments.

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