Journal of Mathematical Research with Applications
Jan., 2019, Vol. 39, No. 1, pp. 89–100
DOI:10.3770/j.issn:2095-2651.2019.01.009
Http://jmre.dlut.edu.cn

Ring Isomorphisms and Complete Preservers of Fixed Points for Multipliers

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Abstract Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two rings with unit I. We give some characterizations of ring homomorphisms and ring isomorphisms between $\mathcal{R}_1$ and $\mathcal{R}_2$ in term of complete preservers of fixed points of multipliers, under some mild assumption on $\mathcal{R}_1$. Applications to several kinds of operator algebras such as Banach algebras, nest algebras, matrix algebras and standard operator algebras are presented.

Keywords complete preserver problem; Banach algebra; ring isomorphism; operator on Banach space

MR(2010) Subject Classification 47B49; 16N60

1. Introduction

Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two rings. For any $A \in \mathcal{R}_1$, denote by L_A and R_A the left multiplier and the right multiplier defined, respectively, by $L_AT = AT$ and $R_AT = TA$ for all $T \in \mathcal{R}_1$. If $L_AT = T$ ($R_AT = T$), we say that T is a fixed point of left multiplier L_A (right multiplier R_A). Let $\pi : \mathcal{R}_1 \to \mathcal{R}_2$ be a ring homomorphism. Then it is clear that π preserves the fixed points of all multipliers, that is, $L_AT = T$ ($R_AT = T$) implies that $L_{\pi(A)}\pi(T) = \pi(T)$ ($R_{\pi(A)}\pi(T) = \pi(T)$). Much more can be said. For each positive integer n, denote by $\mathcal{M}_n(\mathcal{R})$ the set of all $n \times n$ matrices over $\mathcal{R}$, which is still a ring. Let $\Phi : \mathcal{R}_1 \to \mathcal{R}_2$ be a map. For each $n \in \mathbb{N}$, Φ can be extended naturally to a map Φ_n from $\mathcal{M}_n(\mathcal{R}_1)$ into $\mathcal{M}_n(\mathcal{R}_2)$ by defining

$$\Phi_n((S_{ij})_{n\times n}) = (\Phi(S_{ij}))_{n\times n}.$$

Note that, if $\pi : \mathcal{R}_1 \to \mathcal{R}_2$ is a ring homomorphism, then π_n is a ring homomorphism from $\mathcal{M}_n(\mathcal{R}_1)$ into $\mathcal{M}_n(\mathcal{R}_2)$ for each $n = 1, 2, \ldots$. Therefore, π_n preserves fixed points for all multipliers on $\mathcal{M}_n(\mathcal{R}_1)$. We are interested in the question of whether or not the converse is true.

Generally speaking, let (P) be a property that a ring may have. Then Φ is said to be n-(P) preserving if Φ_n preserves (P); Φ is said to be completely (P) preserving if Φ is n-(P) preserving for every positive integer n. The complete preserver problems ask how to characterize the maps on $\mathcal{R}$ that completely preserve some property (P), and then to get some characterizations of ring homomorphisms or ring isomorphisms.

Received February 20, 2018; Accepted November 8, 2018
Supported by the National Natural Science Foundation of China (Grant No. 11671294).
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The matrix structure is an intrinsic property for operator algebras and operator spaces. It is natural to consider the complete preserver problems [1], and find rigid properties of isomorphisms in the sense that such properties are preserved completely by a map will imply that such a map has "nice" structure. In this respect, some kinds of complete preserving linear maps are already intensively studied in operator algebras and operator spaces. For example, the study of completely positive linear maps and completely bounded linear maps are very important topics in operator algebra and operator space theory [2]. Hadwin and Larson introduced the notion of completely rank-nonincreasing linear maps on $\mathcal{B}(H)$ and characterized such maps in [3], later generalized to $\mathcal{B}(X)$ in [4], where H is a Hilbert space, X is a Banach space and $\mathcal{B}(X)$ is the algebra of all bounded linear operators acting on X. Completely invertibility preserving linear maps and completely trace-rank preserving linear maps were discussed and characterized in [5, 6]. General surjective maps between standard operator algebras that completely preserve idempotents, square-zero, commutativity, Jordan zero-products were characterized in [7, 8].

In the present paper we want to characterize the ring isomorphisms and ring homomorphisms in term of complete preservers of fixed points for multipliers. Here we define the complete preservers of fixed points in a simpler form.

We say that a map $\Phi : \mathcal{R}_1 \to \mathcal{R}_2$ is n-fixed points preserving (in both directions) for left multipliers if

$$(A_{ij})_{n\times n}(T_j)_{n\times 1} = (T_j)_{n\times 1} \tag{1.1}$$

implies that (if and only if)

$$(\Phi(A_{ij}))_{n\times n}(\Phi(T_j))_{n\times 1} = (\Phi(T_j))_{n\times 1}. \tag{1.2}$$

In this case, we also say that Φ_n is fixed points preserving (in both directions) for left multipliers. Φ is said to be completely fixed point preserving (in both directions) for left multipliers if it is n-fixed points preserving (in both directions) for left multipliers for every positive integer n. Similarly, one can define the complete preservers of fixed points for right multipliers.

In this paper, we give some characterizations of ring homomorphisms and ring isomorphisms for rings in term of complete preservers of fixed points for left multipliers. After that, we apply the general results to some operator algebras such as Banach algebras, nest algebras, matrix algebras and standard operator algebras. The case for preservers of fixed points for right multipliers are dealt with similarly.

2. General results for rings

We first discuss the problem of characterizing the complete preservers of fixed points for left multipliers. From now on, for convenience's sake, we simply say preservers of fixed points and omit "for left multipliers".

We begin with a general lemma.

Lemma 2.1 *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two unital rings with unit denoted by I for both, and let*

$\Phi : \mathcal{R}_1 \to \mathcal{R}_2$ *be a map. If the range of* Φ *contains* $0, I$ *and if* Φ_2 *is fixed points preserving, then the following statements are true:*

(i) $\Phi(0) = 0, \Phi(I) = I$.

(ii) Φ *is multiplicative.*

(iii) $\Phi(-2I) = -\Phi(2I)$.

(iv) $\Phi(S - T) = \Phi(S) - \Phi(T)$ *for any* $T \in \mathcal{R}_1$ *and invertible* $S \in \mathcal{R}_1$.

In addition, if Φ_2 *is fixed points preserving in both directions, then* Φ *is injective.*

Proof For any $T \in \mathcal{R}_1$, we have

$$\begin{pmatrix} I & I \\ 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ 0 \end{pmatrix} = \begin{pmatrix} T \\ 0 \end{pmatrix}.$$

Since Φ_2 is fixed points preserving, by the definition (see Eqs. (1.1) and (1.2)), we get

$$\begin{pmatrix} \Phi(I) & \Phi(I) \\ \Phi(0) & \Phi(0) \end{pmatrix} \begin{pmatrix} \Phi(T) \\ \Phi(0) \end{pmatrix} = \begin{pmatrix} \Phi(T) \\ \Phi(0) \end{pmatrix}.$$

It follows that

$$\Phi(I)\Phi(T) + \Phi(I)\Phi(0) = \Phi(T), \tag{2.1}$$

and

$$\Phi(0)\Phi(T) + \Phi(0)^2 = \Phi(0). \tag{2.2}$$

Since 0, I are in the range of Φ, there exists some T_0 and T_1 in $\mathcal{R}_1$ such that $\Phi(T_0) = 0$ and $\Phi(T_1) = I$. Taking $T = T_1$ in Eqs. (2.1) and (2.2) yields, respectively

$$\Phi(I) + \Phi(I)\Phi(0) = I \quad \text{and} \quad \Phi(0)^2 = 0. \tag{2.3}$$

Also, taking $T = T_0$ in Eqs. (2.1) and (2.2) yields respectively that $\Phi(I)\Phi(0) = 0$ and $\Phi(0)^2 = \Phi(0)$. These, together with Eq. (2.3), ensure that $\Phi(0) = 0$ and $\Phi(I) = I$. So (i) is true.

For any $T \in \mathcal{R}_1$, since

$$\begin{pmatrix} T & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ I-T \end{pmatrix} = \begin{pmatrix} I \\ I-T \end{pmatrix},$$

we see that

$$\begin{pmatrix} \Phi(T) & I \\ 0 & I \end{pmatrix} \begin{pmatrix} I \\ \Phi(I-T) \end{pmatrix} = \begin{pmatrix} I \\ \Phi(I-T) \end{pmatrix},$$

which gives

$$\Phi(I-T) = I - \Phi(T). \tag{2.4}$$

For any $T, S \in \mathcal{R}_1$, as

$$\begin{pmatrix} I-T & TS \\ 0 & I \end{pmatrix} \begin{pmatrix} S \\ I \end{pmatrix} = \begin{pmatrix} S \\ I \end{pmatrix},$$

by Eq. (2.4) one gets

$$\begin{pmatrix} I-\Phi(T) & \Phi(TS) \\ 0 & I \end{pmatrix} \begin{pmatrix} \Phi(S) \\ I \end{pmatrix} = \begin{pmatrix} \Phi(S) \\ I \end{pmatrix}.$$

This implies that $\Phi(TS)=\Phi(T)\Phi(S)$ holds for any $T,S\in\mathcal{R}_1$, and hence, (ii) is true.

For any $T\in\mathcal{R}_1$, since

$$\begin{pmatrix} 2I & I \\ 0 & I \end{pmatrix}\begin{pmatrix} -T \\ T \end{pmatrix}=\begin{pmatrix} -T \\ T \end{pmatrix},$$

we see that

$$\begin{pmatrix} \Phi(2I) & I \\ 0 & I \end{pmatrix}\begin{pmatrix} \Phi(-T) \\ \Phi(T) \end{pmatrix}=\begin{pmatrix} \Phi(-T) \\ \Phi(T) \end{pmatrix}.$$

So

$$\Phi(2I)\Phi(-T)+\Phi(T)=\Phi(-T). \tag{2.5}$$

Taking $T=I$ in Eq. (2.5) and using (ii), we have $\Phi(-2I)+I=\Phi(-I)$. Taking $T=-I$ in Eq. (2.4) yields that $\Phi(2I)=I-\Phi(-I)$. These ensure that $\Phi(-2I)=-\Phi(2I)$. So the statement (iii) is true.

For any $T\in\mathcal{R}_1$ and invertible $S\in\mathcal{R}_1$, by (ii) and (2.4), we have

$$\begin{aligned}\Phi(S-T)=&\Phi((I-TS^{-1})S)=\Phi(I-TS^{-1})\Phi(S)\\ =&(I-\Phi(T)\Phi(S^{-1}))\Phi(S)=\Phi(S)-\Phi(T),\end{aligned}$$

hence (iv) is true.

Now assume further that Φ_2 is fixed point preserving in both directions. For $T,S\in\mathcal{R}_1$, assume $\Phi(T)=\Phi(S)$. Since

$$\begin{pmatrix} T & I-T \\ T & I-T \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}=\begin{pmatrix} I \\ I \end{pmatrix},$$

by (i) we have

$$\begin{pmatrix} \Phi(T) & \Phi(I-T) \\ \Phi(T) & \Phi(I-T) \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}=\begin{pmatrix} I \\ I \end{pmatrix}.$$

Then, $\Phi(T)=\Phi(S)$ gives

$$\begin{pmatrix} \Phi(S) & \Phi(I-T) \\ \Phi(T) & \Phi(I-T) \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}.$$

As Φ_2 preserves the fixed points in both directions, we get

$$\begin{pmatrix} S & I-T \\ T & I-T \end{pmatrix}\begin{pmatrix} I \\ I \end{pmatrix}=\begin{pmatrix} I \\ I \end{pmatrix},$$

which entails that $T=S$. So Φ is an injection. $\Box$

Recall that a ring $\mathcal{R}$ is prime if for any $A,B\in\mathcal{R}$, $A\mathcal{R}B=\{0\}$ implies either $A=0$ or $B=0$. The following lemma comes from [9].

Lemma 2.2 *Let $\mathcal{R}$ be a prime ring containing an idempotent $e\neq 0,I$ ($\mathcal{R}$ need not have an identity). Then every multiplicative bijection from $\mathcal{R}$ onto an arbitrary ring $\mathcal{S}$ is additive.*

Theorem 2.3 *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two unital rings with units denoted by I for both, and let $\Phi:\mathcal{R}_1\to\mathcal{R}_2$ be a surjective map. If $\mathcal{R}_1$ is prime, then the following statements are equivalent:*

(1) Φ is completely fixed points preserving in both directions.

(2) Φ_4 is fixed points preserving in both directions.

(3) Φ is a ring isomorphism.

Proof Clearly, every ring isomorphism is completely fixed points preserving in both directions. So (3)⇒(1)⇒(2). We only need to prove (2)⇒(3).

Assume that Φ is surjective and 4-fixed points preserving in both directions. By the assumption, Φ_2 is also surjective and 2-fixed points preserving in both directions on $M_2(\mathcal{R}_1)$. So by Lemma 2.1 we conclude that $\Phi_2(0) = 0$, $\Phi_2(I) = I$, Φ_2 is bijective and multiplicative. In addition, $\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ is a nontrivial idempotent in $M_2(\mathcal{R}_1)$, which implies by Lemma 2.2 that Φ_2 is additive. Hence Φ_2 is a ring isomorphism, and consequently, Φ is a ring isomorphism. □

Corollary 2.4 *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two unital rings with units denoted by I for both, and let $\Phi : \mathcal{R}_1 \to \mathcal{R}_2$ be a surjective map. If $\mathcal{R}_1$ is prime and contains a nontrivial idempotent, then the following statements are equivalent:*

(1) Φ is completely fixed points preserving in both directions.

(2) Φ_2 is fixed points preserving in both directions.

(3) Φ is a ring isomorphism.

Proof This is obvious by Lemmas 2.1 and 2.2. □

The following Theorem gives a characterization of (injective) ring homomorphisms for some rings that may not be prime.

Theorem 2.5 *Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be two unital rings with units denoted by I for both, and let $\Phi : \mathcal{R}_1 \to \mathcal{R}_2$ be a map so that $0, I$ are contained in the range of Φ. If $\frac{1}{2}I \in \mathcal{R}_1$ and for any $S \in \mathcal{R}_1$, there exists some positive integer k such that $2^k I + S \in \mathcal{R}_1$ is invertible in $\mathcal{R}_1$. Then the following statements are equivalent:*

(1) Φ is completely fixed points preserving (in both directions).

(2) Φ is 2-fixed points preserving (in both directions).

(3) Φ is a (an injective) ring homomorphism with $\Phi(I) = I$.

Proof Clearly, every ring homomorphism is completely fixed points preserving. So (3)⇒(1)⇒(2). We only need to prove (2)⇒(3).

Assume that Φ_2 preserves fixed points. Then by Lemma 2.1, Φ is a multiplicative map so that $\Phi(-2I) = -\Phi(2I)$ and $\Phi(S-T) = \Phi(S) - \Phi(T)$ for any $T \in \mathcal{R}_1$ and invertible $S \in \mathcal{R}_1$. On account of $\frac{1}{2}I \in \mathcal{R}_1$, we have $\Phi(-I) = \Phi(\frac{1}{2}I \cdot (-2I)) = \Phi(\frac{1}{2}I)\Phi(-2I) = -\Phi(I)$. Consequently, $\Phi(-T) = -\Phi(T)$ holds for all $T \in \mathcal{R}_1$. So, $\Phi(S+T) = \Phi(S) - \Phi(-T) = \Phi(S) + \Phi(T)$ holds for any $T \in \mathcal{R}_1$ and invertible $S \in \mathcal{R}_1$. Now, if $S \in \mathcal{R}_1$ is not invertible, by the assumption, there exists some positive integer k such that $2^k I + S$ is invertible in $\mathcal{R}_1$. Also. note that $2^k I$ is invertible in $\mathcal{R}_1$. Hence we have $\Phi(S+T) = \Phi(2^k I + S) + \Phi(-2^k I + T) = \Phi(2^k I) + \Phi(S) + \Phi(-2^k I) + \Phi(T) = \Phi(S) + \Phi(T)$. It follows that Φ is additive. Therefore, Φ is a ring homomorphism.

Moreover, if Φ_2 preserves fixed points in both directions, by Lemma 2.1, Φ is an injective ring homomorphism. $\Box$

3. Application to operator algebras

In this section, we apply the general results in Section 2 to some operator algebras.

Theorem 3.1 *Let $\mathcal{A}$ and $\mathcal{B}$ be real or complex Banach algebras with identity I and let $\Phi : \mathcal{A} \to \mathcal{B}$ be a map so that $0, I$ are contained in the range of Φ. Then the following statements are equivalent:*

(1) Φ is completely fixed points preserving (in both directions).

(2) Φ_2 is fixed points preserving (in both directions).

(3) Φ is a (an injective) ring homomorphism.

Proof Since $\mathcal{A}$ is a Banach algebra, for any nonzero scalar λ, we have $\lambda I, \lambda^{-1} I \in \mathcal{A}$ and $\lambda I + S$ is invertible in $\mathcal{A}$ whenever $|\lambda| > \|S\|$. Now, the theorem follows from Theorem 2.5 immediately. $\Box$

Let X be a real or complex Banach space and $\mathcal{A}$ be a subalgebra (not assumed to be closed under any operator topology) in $\mathcal{B}(X)$. Recall that, $\mathcal{A}$ is called a standard operator algebra if $\mathcal{A}$ contains the identity I and $\mathcal{F}(X)$, the set of all finite rank operators.

Theorem 3.2 *Let X, Y be Banach spaces over the real or complex field $\mathbb{F}$ with $\dim X \geq 2$. Let $\mathcal{A}, \mathcal{B}$ be standard operator algebras on X, Y, respectively. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a map. If the range of Φ contains $0, I$ and all rank-1 idempotents, then the following statements are equivalent:*

(1) Φ is completely fixed points preserving in both directions.

(2) Φ_2 is fixed points preserving in both directions.

(3) There exists a ring automorphism $\tau : \mathbb{F} \to \mathbb{F}$ and a τ-linear bijective transformation $T : X \to Y$ with $T\mathcal{A}T^{-1} = \mathcal{B}$ such that $\Phi(A) = TAT^{-1}$ for all $A \in \mathcal{A}$. Moreover, if X, Y are real, then T is a linear bounded invertible operator; if X, Y are complex and $\dim X = \infty$, then T is a linear or conjugate linear bounded invertible operator.

(4) Φ is a ring isomorphism.

Proof Note that the maps of the form (3) are ring isomorphisms from $\mathcal{A}$ onto $\mathcal{B}$. If the spaces are real or if the spaces are complex and $\dim X = \infty$, then Φ is continuous under the weak operator topology (WOT, briefly). On the other hand, since the range of Φ contains all finite rank operators as an injective algebraic homomorphism or conjugate algebraic homomorphism, this implies that Φ is surjective because $\mathcal{F}(Y)$ is dense in $\mathcal{B}$ under WOT. If $\dim X < \infty$, Φ is an injective τ-algebraic homomorphism. Thus the range of Φ contains all finite rank operators on Y and consequently, $\dim Y = \dim X$ and Φ is surjective.

So the implications (3)$\Rightarrow$(4)$\Rightarrow$(1)$\Rightarrow$(2) are obvious. We only need to prove (2)$\Rightarrow$(3). Assume (2). Note that, $\mathcal{A}$ may not be closed in any operator topology and thus Theorem 3.1 is unapplicable. Also, Corollary 2.4 is not applicable as Φ is not assumed surjective. However, one can use Lemma 2.1 to see that Φ is a multiplicative injection, $\Phi(0) = 0$, $\Phi(I) = I$, $\Phi(-2I) = -\Phi(2I)$

and $\Phi(A-B)=\Phi(A)-\Phi(B)$ whenever A is invertible in $\mathcal{A}$. Moreover, as $\frac{1}{2}I\in\mathcal{A}$, we get $\Phi(-I)=-\Phi(I)$ and consequently $\Phi(A+B)=\Phi(A)+\Phi(B)$ for any $A,B\in\mathcal{A}$ with A invertible in $\mathcal{A}$.

For any $F\in\mathcal{F}(X)$, take a positive integer k so that $2^k>\|F\|$. Then 2^kI-F is invertible in $\mathcal{B}(X)$. But the fact that F is of finite rank ensures that $(2^kI-F)^{-1}=\lambda I+E$ for some scalar λ and some operator $E\in\mathcal{F}(X)$. Therefore, the subalgebra $\mathbb{F}I+\mathcal{F}(X)\subseteq\mathcal{A}$ satisfies the assumptions on $\mathcal{R}_1$ in Theorem 2.5 and consequently, the restriction $\Phi|_{\mathbb{F}I+\mathcal{F}(X)}:\mathbb{F}I+\mathcal{F}(X)\to\mathcal{B}$ is an injective ring homomorphism.

Note that a nonzero operator $A\in\mathcal{A}$ is of rank one if and only if, for any rank-1 idempotents B,C, $BAC=0$ implies that $BA=0$ or $AC=0$. By this fact, if A is of rank-one, so is $\Phi(A)$. For any $\lambda\in\mathbb{F}$, the multiplicativity of Φ implies that $\Phi(\lambda I)\Phi(A)=\Phi(A)\Phi(\lambda I)$. Hence $\Phi(\lambda I)$ commutes with every rank-1 idempotent. It follows that $\Phi(\lambda I)\in\mathbb{F}I$, that is, $\Phi(\mathbb{F}I)\subseteq\mathbb{F}I$. Then $\Phi(\mathbb{F}x\otimes f)\subseteq\mathbb{F}\Phi(x\otimes f)$ for every rank-1 idempotent $x\otimes f$. This entails that all rank-1 operators are in the range of Φ. Therefore, $\Phi|_{\mathbb{F}I+\mathcal{F}(X)}$ is a ring isomorphism from $\mathbb{F}I+\mathcal{F}(X)$ onto $\mathbb{F}I+\mathcal{F}(Y)$. Then, as is well known, there exist a ring automorphism $\tau:\mathbb{F}\to\mathbb{F}$ and a τ-linear bijective transformation $T:X\to Y$ such that $\Phi(F)=TFT^{-1}$ for all $F\in\mathbb{F}I+\mathcal{F}(X)$. Now, for any $A\in\mathcal{A}$, as

$$\Phi(A)T(x\otimes f)T^{-1}=\Phi(Ax\otimes f)=TA(x\otimes f)T^{-1}=TAT^{-1}T(x\otimes f)T^{-1}$$

holds for all rank-1 operators $x\otimes f$, we see that $\Phi(A)=TAT^{-1}$ for every $A\in\mathcal{A}$. The claims on T are then true by [10] or [11, Theorem 2.4.2]. The proof is completed. □

Recall that a nest $\mathcal{N}$ of a Banach space X is a chain of subspaces of X so that $\{0\},X\in\mathcal{N}$, and, for any subset $\{N_\alpha\}\subseteq\mathcal{N}$, $\cap_\alpha N_\alpha$ and the closed linear span of $\cup_\alpha N_\alpha$ are still in $\mathcal{N}$. The nest algebra Alg$\mathcal{N}$ is the algebra consisting of all operators $T\in\mathcal{B}(X)$ such that $TN\subseteq N$ holds for all $N\in\mathcal{N}$. A subalgebra $\mathcal{A}$ is called a standard subalgebra of nest algebra Alg$\mathcal{N}$ if $\mathcal{A}$ contains the identity I and all finite rank operators in the nest algebra.

If $\mathcal{A}$ is a standard subalgebra of a nest algebra on a Banach space X of dimension at least 3, then, by [12, Theorem 2.1], every multiplicative isomorphism Φ of $\mathcal{A}$ onto an arbitrary ring is additive. Also, a characterization of ring isomorphisms of $\mathcal{A}$ can be found in [13]. So the following result is true.

Theorem 3.3 *Let $\mathcal{A}$ be a standard subalgebra of a nest algebra* Alg$\mathcal{N}$ *on an infinite-dimensional real or complex Banach space X. Let $\Phi:\mathcal{A}\to\mathcal{A}$ be a surjective map. Then the following statements are equivalent:*

(1) Φ is completely fixed points preserving in both directions.

(2) Φ_2 is fixed points preserving in both directions.

(3) There exists a dimension preserving order isomorphism $\theta:\mathcal{N}\to\mathcal{N}$, and an invertible bounded either linear or conjugate linear operator $T:X\to X$ satisfying $T(N)=\theta(N)$ for every $N\in\mathcal{N}$ such that

$$\Phi(A)=TAT^{-1}\quad\text{for all}\quad A\in\mathcal{A}.$$

Denote by $\mathcal{T}(n_1, n_2, \dots, n_k)$ the algebras of upper triangular block matrices over the real or complex field $\mathbb{F}$. The following corollary is the finite dimension version of Theorem 3.3.

Corollary 3.4 *Let $\Phi : \mathcal{T}(n_1, n_2, \dots, n_k) \to \mathcal{T}(n_1, n_2, \dots, n_k)$ be a surjective map. Then the following statements are equivalent:*

(1) Φ is completely fixed points preserving in both directions.

(2) Φ_2 is fixed points preserving in both directions.

(3) There exist an automorphism τ of $\mathbb{F}$ and an invertible matrix $T \in \mathcal{T}(n_1, n_2, \dots, n_k)$ such that $\Phi((a_{ij})) = T(\tau(a_{ij}))T^{-1}$ for all $(a_{ij}) \in \mathcal{T}(n_1, n_2, \dots, n_k)$.

For matrix algebras the condition that all rank-1 idempotents are contained in the range of the map in Theorem 3.2 can be weakened.

Theorem 3.5 *Let $\mathcal{M}_n(\mathbb{F})$ be the algebra of all $n \times n$ matrices over the real or complex field $\mathbb{F}$ and $\Phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ be a nonzero map. If the range of Φ contains $0, I$, then the following statements are equivalent:*

(1) Φ is completely fixed points preserving.

(2) Φ_2 is fixed points preserving.

(3) There exists a ring homomorphism $\tau : \mathbb{F} \to \mathbb{F}$ and an invertible matrix $T \in \mathcal{M}_n(\mathbb{F})$ such that $\Phi(A) = TA_\tau T^{-1}$ for all $A \in \mathcal{M}_n(\mathbb{F})$, where $A_\tau = (\tau(a_{ij}))$ for $A = (a_{ij})$.

(4) There exists a homomorphism $\tau : \mathbb{F} \to \mathbb{F}$ so that Φ is a τ-algebraic homomorphism.

(5) Φ is a ring homomorphism.

Note that, if $\mathbb{F} = \mathbb{R}$, the ring homomorphism τ in (3) is in fact the identity and Φ is an algebraic isomorphism; if $\mathbb{F} = \mathbb{C}$, τ is an injective homomorphism and may not be surjective.

Proof of Theorem 3.5 Note that every nonzero ring homomorphism of $\mathcal{M}_n(\mathbb{F})$ is injective as it is a simple algebra. So, the implications (3)⇒(4)⇒(5)⇒(1)⇒(2) are obvious. Let us check (2)⇒(3). Assume (2) is true; that is, assume that Φ_2 is fixed points preserving. Then, by Theorem 2.5, Φ is a homomorphism with $\Phi(I) = I$.

Now we need the following lemma which is a special case of a result in [14] (also [15, Theorem 2.5] and [11, Theorem 10.1.4]).

Lemma 3.6 *Every multiplicative map $\Psi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ must have one of the following three forms:*

(i) $\Psi(A) = 0$ for all matrices A with rank ≤ 1.

(ii) There exists an invertible matrix $T \in \mathcal{M}_n(\mathbb{F})$, a positive integer $k \leq n$ and a multiplicative map $\Phi_0 : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_{n-k}(\mathbb{F})$ satisfying $\Phi_0(A) = 0$ whenever $\mathrm{rank}(A) \leq 1$, such that $\Psi(A) = T \begin{pmatrix} I_k & 0 \\ 0 & \Phi_0(A) \end{pmatrix} T^{-1}$ for all $A \in \mathcal{M}_n(\mathbb{F})$.

(iii) There exists a ring homomorphism $\tau : \mathbb{F} \to \mathbb{F}$ and an invertible matrix $T \in \mathcal{M}_n(\mathbb{F})$ such that $\Psi(A) = TA_\tau T^{-1}$ for all $A \in \mathcal{M}_n(\mathbb{F})$.

Observe that, by Lemma 3.6, the case (i) implies that $\Phi = 0$ and the case (ii) implies that Φ is not additive. So, Φ must have the form (iii); that is, the statement (3) is true. $\square$

As a corollary of Theorem 3.5, we obtain the following well-known characterization of homomorphisms on $\mathcal{M}_n(\mathbb{F})$.

Corollary 3.7 *Let $\mathcal{M}_n(\mathbb{F})$ be the algebra of all $n \times n$ matrices over the real or complex field $\mathbb{F}$ and $\Phi : \mathcal{M}_n(\mathbb{F}) \to \mathcal{M}_n(\mathbb{F})$ be a nonzero map. Then Φ is a ring homomorphism if and only if there exists a ring homomorphism $\tau : \mathbb{F} \to \mathbb{F}$ and an invertible matrix $T \in \mathcal{M}_n(\mathbb{F})$ such that $\Phi(A) = TA_\tau T^{-1}$ for all $A \in \mathcal{M}_n(\mathbb{F})$.*

To deal with the infinite-dimensional case, we need a result due to An and Hou that can be found in [11, Theorem 10.3.1'], where the result is for the maps from $\mathcal{B}(X)$ into $\mathcal{B}(Y)$, but, by checking the proof there, still valid for maps between closed standard operator algebras. Recall that, for any linear manifold M_1 in X, there exists a linear manifold M_2 in X so that $M_1 \cap M_2 = \{0\}$ and $M_1 + M_2 = X$, and there exists an idempotent linear transformation $P : X \to X$ such that $P(M_1) = M_1$ and $P(M_2) = \{0\}$. M_2 is called an algebraic complementary submanifold of M_1. In this case, every transformation C from X into itself can be represented as a matrix

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix},$$

where C_{ij} is a transformation from M_j into M_i. Note that, P is bounded if and only if both M and N are closed. For any subset L in a Banach space X, we denote by $\mathrm{span}\{L\}$ the closed linear subspace spanned by L.

Lemma 3.8 *Let X and Y be two Banach spaces over the real or complex field $\mathbb{F}$ with $\dim X = \infty$, and let $\mathcal{A}$ and $\mathcal{B}$ be closed standard operator algebras respectively on X and Y. Let $\Psi : \mathcal{A} \to \mathcal{B}$ be a multiplicative map. Assume further that there exists some rank-1 operator $F_0 \in \mathcal{A}$ so that the rank of $\Psi(F_0)$ is not greater than 1 and there exists some rank-k idempotent operator $P_0 \in \mathcal{A}$ so that the rank of $\Psi(P_0)$ is not greater than k, where $2 \le k < \infty$. Then one of the following statements is true:*

(a) $\Psi(F) = 0$ *for every F with rank ≤ 1.*

(b) $\Psi(0)$ *is of rank one and there exists a space decomposition $Y = Y_1 \dot{+} \mathrm{ran}(\Psi(0))$ such that*

$$\Psi(A) = \begin{pmatrix} \Psi_0(A) & 0 \\ 0 & 1 \end{pmatrix},$$

where $\Psi_0 : \mathcal{A} \to P\mathcal{B}P$ is a multiplicative map satisfying $\Psi_0(F) = 0$ for every F with rank ≤ 1 and P is the idempotent operator with $P(Y_1) = Y_1$ and $P(\mathrm{ran}(\Psi(0))) = \{0\}$.

(c) There exists a complementary submanifold N of the closed subspace $M = \mathrm{span}\{\mathrm{range}(\Psi(F)) : \mathrm{rank}(F) = 1\}$ of Y, an invertible bounded linear or conjugate linear operator $T : X \to M$, a multiplicative map $\Psi_0 : \mathcal{A} \to (I-Q)\mathcal{B}(I-Q)$ vanishing on operators with rank ≤ 1 and a map $\Psi_{12} : \mathcal{A} \to Q\mathcal{B}(I-Q)$ which satisfies $\Psi_{12}(AB) = TAT^{-1}\Psi_{12}(B) + \Psi_{12}(A)\Psi_0(B)$ for any $A, B \in \mathcal{A}$ such that

$$\Psi(A) = \begin{pmatrix} TAT^{-1} & \Psi_{12}(A) \\ 0 & \Psi_0(A) \end{pmatrix} \tag{3.1}$$

for all $A \in \mathcal{A}$, where $Q : Y \to Y$ is the idempotent so that $Q(Y) = M$ and $Q(N) = \{0\}$.

Generally speaking, in Lemma 3.8, Q may not belong to $\mathcal{A}$ even if when it is bounded. So, $Q\mathcal{A}Q$ and $(I-Q)\mathcal{A}(I-Q)$ may not be algebras. The point of (c) is that, for each $A \in \mathcal{A}$, $\Psi(A)$ has a matrix representation of the form in Eq. (3.1). We do not know if Ψ_{12} is additive.

Theorem 3.9 *Let X, Y be Banach spaces over the real or complex field $\mathbb{F}$ with $\dim X = \infty$ and $\mathcal{A}, \mathcal{B}$ be closed standard operator algebras on X, Y, respectively. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a map so that $0, I$ are contained in the range of Φ. Assume further that $\mathrm{rank}(\Phi(F_0)) \le 1$ for some rank-1 operator $F_0 \in \mathcal{A}$ and $\mathrm{rank}(\Phi(P_0)) \le k$ for some rank-k idempotent operator $P_0 \in \mathcal{A}$ with $2 \le k < \infty$. Then the following statements are equivalent:*

(1) Φ is completely fixed points preserving in both directions.

(2) Φ is 2-fixed points preserving in both directions.

(3) There exists a complementary submanifold N of the closed subspace $M = \mathrm{span}\{\mathrm{range}(\Phi(F)) : \mathrm{rank}(F) = 1\}$ of Y, an invertible bounded linear or conjugate linear operator $T : X \to M$, an idempotent linear transformation $Q : Y \to Y$ with the range of Q equal to M and $\ker Q = N$, a unital ring homomorphism $\Phi_0 : \mathcal{A} \to \Phi_0(\mathcal{A}) \subset (I-Q)\mathcal{B}(I-Q)$ which vanishes on all finite rank operators, and an additive map $\Phi_{12} : \mathcal{A} \to Q\mathcal{B}(I-Q)$ which satisfies $\Phi_{12}(AB) = TAT^{-1}\Phi_{12}(B) + \Phi_{12}(A)\Phi_0(B)$ for any $A, B \in \mathcal{A}$, such that

$$\Phi(A) = \begin{pmatrix} TAT^{-1} & \Phi_{12}(A) \\ 0 & \Phi_0(A) \end{pmatrix}$$

for all $A \in \mathcal{A}$.

(4) Φ is a unital injective ring homomorphism.

Proof (2) $\Rightarrow$(3). Assume that Φ_2 is fixed points preserving in both directions. As the range of Φ contains $0, I$ and $\mathcal{A}$ is a Banach algebra, by Theorem 3.1, Φ is a unital injective ring homomorphism. Applying Lemma 3.8, we see that Φ can only take the form (c) because the maps of form (a) or (b) are not injective. Thus Φ must have the form

$$\Phi(A) = \begin{pmatrix} TAT^{-1} & \Phi_{12}(A) \\ 0 & \Phi_0(A) \end{pmatrix}$$

for all $A \in \mathcal{A}$, where $A : X \to M$ is a linear or conjugate linear invertible bounded operator, $\Phi_{12} : \mathcal{A} \to Q\mathcal{A}(I-Q)$ is an additive map which satisfies

$$\Phi_{12}(AB) = TAT^{-1}\Phi_{12}(B) + \Phi_{12}(A)\Phi_0(B) \tag{3.2}$$

for any $A, B \in \mathcal{A}$, $\Phi_0(\mathcal{A}) \subset (I-Q)\mathcal{A}(I-Q)$ is a ring and Φ_0 is a ring homomorphism from $\mathcal{A}$ onto $\Phi_0(\mathcal{A})$ which vanishes all finite rank operators, and $Q : X \to X$ is an idempotent linear transformation with range $M = \mathrm{span}\{\mathrm{range}(\Phi(F)) : \mathrm{rank}(F) = 1\}$.

Now, the implications (3)$\Rightarrow$(4)$\Rightarrow$(1)$\Rightarrow$(2) are obvious. $\square$

We do not know the general structure of Φ_{12}. However, in some situations, $\Phi_{12} = 0$. For example, it is clear that, in Theorem 3.9 (3), if $M = Y$, then $\Phi(T) = ATA^{-1}$ for all $T \in \mathcal{A}$. The following corollary is a generalization of the above observation.

Corollary 3.10 *Under the assumption of Theorem 3.9, if* $\dim M^{\perp} < \infty$, *then the following statements are equivalent:*

(1) Φ *is completely fixed points preserving in both directions.*

(2) Φ_2 *is fixed points preserving in both directions.*

(3) $M = Y$ *and there exists an invertible bounded linear or conjugate linear operator* $A : X \to Y$ *such that* $\Phi(T) = ATA^{-1}$ *for all* $T \in \mathcal{A}$.

(4) Φ *is an algebraic isomorphism or conjugate algebraic isomorphism.*

Proof We need only to check (2)⇒(3). By Theorem 3.9, Φ has the form of Theorem 3.9(3). We claim that $\Phi_{12} = 0$ and $\Phi_0 = 0$. In fact, since N is of finite dimension, it is obvious that $\Phi_0 = 0$. Thus, for any $S, T \in \mathcal{A}$, we have $\Phi(TS) = ATA^{-1}\Phi_{12}(S)$. By Eq. (3.2) it is easily checked that $\Phi_{12}(I) = 0$. This entails that

$$\Phi_{12}(T) = \Phi_{12}(T \cdot I) = ATA^{-1}\Phi_{12}(I) = 0$$

for every $T \in \mathcal{A}$; that is, $\Phi_{12} = 0$. However, as $\Phi(I) = I$, we must have $\ker Q = \{0\}$. This entails (3). □

There exist injective unital ring homomorphisms that have the form in Theorem 3.9 (3) with $\Phi_{12} = 0$.

Example 3.11 Let X be an infinite dimensional Banach space and $\mathcal{K}(X)$ be the closed ideal of all compact operators acting on X. Let $\mathcal{C}(X) = \mathcal{B}(X)/\mathcal{K}(X)$ be the Calkin algebra and $\pi : \mathcal{B}(X) \to \mathcal{C}(X)$ be the quotient map. Then, π is a surjective algebraic homomorphism which vanishes at all compact operators. Note that $\mathcal{C}(X) \subset \mathcal{B}(\mathcal{C}(X))$. Let $Y = X \oplus \mathcal{C}(X)$. For an invertible operator $A \in \mathcal{B}(X)$, let $\Phi : \mathcal{B}(X) \to \mathcal{B}(Y)$ be defined by

$$\Phi(T) = \begin{pmatrix} ATA^{-1} & 0 \\ 0 & \pi(T) \end{pmatrix}$$

for all $T \in \mathcal{B}(X)$. Then, Φ is a unital injective algebraic homomorphism.

Acknowledgements The authors wish to give their thanks to the referees for their helpful comments and suggestions that have improved greatly the presentation of this paper.

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