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A New Class AOR Preconditioner for L-Matrices

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Abstract Hadjidimos (1978) proposed a classical accelerated overrelaxation (AOR) iterative method to solve the system of linear equations, and discussed its convergence under the conditions that the coefficient matrices are irreducible diagonal dominant, L-matrices, and consistently orders matrices. Several preconditioned AOR methods have been proposed to solve system of linear equations $Ax = b$, where $A \in \mathbb{R}^{n\times n}$ is an L-matrix. In this work, we introduce a new class preconditioners for solving linear systems and give a comparison result and some convergence result for this class of preconditioners. Numerical results for corresponding preconditioned GMRES methods are given to illustrate the theoretical results.

Keywords AOR iterative method; L-matrix; irreducible matrix; spectral radius; preconditioner; iteration matrix

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1. Introduction

Consider the following linear system

$$Ax = b, \tag{1.1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$ are given and $x \in \mathbb{R}^n$ is unknown. System of form (1.1) appears in many applications such as linear elasticity, fluid dynamics, and constrained quadratic programming [1–4]. When the coefficient matrix of the linear system (1.1) is large and sparse, iterative methods are recommended against direct methods. In order to solve (1.1) more effectively by using the iterative methods, usually, efficient splittings of the coefficient matrix A are required. For example, the classical Jacobi and Gauss-Seidel iterations are obtained by splitting the matrix A into its diagonal and offdiagonal parts. For the numerical solution of (1.1), the accelerated overrelaxation (AOR) method was introduced by Hadjidimos in [5] and is a two-parameter generalization of the successive overrelaxation (SOR) method. In certain cases the AOR method has better convergence rate than Jacobi, JOR, Gauss-Seidel, or SOR method [5, 6]. Sufficient conditions for the convergence of the AOR method have been considered by many authors including [6–14]. One of the techniques to improve the convergence rate of the AOR method are preconditioning AOR (PAOR). These methods have been popular for years as 'standalone' solvers, but nowadays they are most often used as preconditioners for Krylov subspace methods (equivalently, the convergence of these stationary iterations can be accelerated by Krylov subspace methods.)

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To introduce the PAOR method, firstly, a brief review of the classical AOR method is required. For any splitting, $A = M - N$ with $\det(M) \neq 0$, the iterative method for solving Eq. (1.1) is

$$x^{(i+1)} = M^{-1}Nx^{(i)} + M^{-1}b, \quad i = 0, 1, 2, \ldots. \tag{1.2}$$

For simplicity, without loss of generality, we assume throughout this paper that $A = I - L - U$, where I is the identity matrix, and L and U are strictly lower and upper triangular matrices obtained from A, respectively. The AOR iterative method (Hadjidimos, 1978) is defined as follows

$$x^{(i+1)} = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU]x^{(i)} + (I - rL)^{-1}wb \tag{1.3}$$

where $i = 0, 1, 2, \ldots$. Its iteration matrix is

$$L(r, w) = (I - rL)^{-1}[(1 - w)I + (w - r)L + wU] \tag{1.4}$$

where w and r are real parameters with $w \neq 0$. It is well known that, for certain values of the parameters w and r, we obtain the Jacobi, the Gauss-Seidel and the successive overrelaxation (SOR) methods.

We now transform the original system in Eq. (1.1) into the preconditioned form $PAx = Pb$, Then, we can define the basic iterative scheme

$$M_p x^{(i+1)} = N_p x^{(i)} + Pb, \quad i = 0, 1, 2, \ldots \tag{1.5}$$

where $PA = M_p - N_p$ and M_p is nonsingular.

In [16–24] some different preconditioners have been proposed by several authors. In this paper, we propose a new class preconditioned AOR iterative method with a preconditioner $P_{\alpha\beta} = I + S_{\alpha\beta}$ where

$$S_{\alpha\beta} = (s_{ij}), \quad s_{ij} = \begin{cases} -\alpha_{j-1}(a_{i,j-1} + \beta_{j-1}), & i = 1; j = 2, 3, \ldots, n, \\ 0, & \text{otherwise} \end{cases} \tag{1.6}$$

and α, β are real parameters. Let $S_{\alpha\beta}A = E_{\alpha\beta} - F_{\alpha\beta}$ where $E_{\alpha\beta}$ is diagonal matrix and $F_{\alpha\beta}$ is upper triangular matrix, respectively. Assume that

$$\begin{aligned} A_{\alpha\beta} &= P_{\alpha\beta}A = (I + S_{\alpha\beta})A = (I + S_{\alpha\beta})(I - L - U) \\ &= I - L - U + S_{\alpha\beta}A = I - L - U + E_{\alpha\beta} - F_{\alpha\beta} \\ &= (I + E_{\alpha\beta}) - L - (U + F_{\alpha\beta}) = D_{\alpha\beta} - L_{\alpha\beta} - U_{\alpha\beta}, \end{aligned} \tag{1.7}$$

where

$$D_{\alpha\beta} = I + E_{\alpha\beta}, \ L_{\alpha\beta} = L, \ U_{\alpha\beta} = U + F_{\alpha\beta}. \tag{1.8}$$

Here we consider the AOR splitting for $A_{\alpha\beta}$

$$A_{\alpha\beta} = \frac{1}{w}(I + E_{\alpha\beta} - rL) - \frac{1}{w}[(1 - w)(I + E_{\alpha\beta}) + (w - r)L + w(U + F_{\alpha\beta})], \tag{1.9}$$

$$A_{\alpha\beta} = \frac{1}{w}(I - rL) - \frac{1}{w}[(1 - w)I + (w - r)L + w(U_{\alpha\beta} - E_{\alpha\beta})]. \tag{1.10}$$

By considering Eqs. (9) and (10), the AOR iteration matrices associated with $A_{\alpha\beta}$ are

$$\widetilde{L}_{\alpha\beta}(r,w) = (D_{\alpha\beta} - rL_{\alpha\beta})^{-1}[(1-w)D_{\alpha\beta} + (w-r)L_{\alpha\beta} + wU_{\alpha\beta}], \tag{1.11}$$

$$\widehat{L}_{\alpha\beta}(r,w) = (I - rL_{\alpha\beta})^{-1}[(1-w)I + (w-r)L_{\alpha\beta} + w(U_{\alpha\beta} - E_{\alpha\beta})]. \tag{1.12}$$

In the following sections, we will use the above results.

The remainder of this paper is organized as follows: in Section 2, we propose some definitions and lemmas which are essential tools for obtaining our main results. The comparison results are given in Section 3. In Section 4, we employ numerical example to support the theoretical results of this paper.

2. Preliminaries

In this section, we give some definitions and lemmas which are essential tools for describing our main results. A matrix $A \in \mathbb{R}^{n\times n}$ is said to be nonnegative and denoted by $A \geq 0$ if $a_{ij} \geq 0$ for all i and j and A is said to positive and denoted by $A > 0$ if $a_{ij} \geq 0$ for all i and j.

Definition 2.1 ([1]) *A matrix A is a Z-matrix if $a_{ij} \leq 0$ for all $i, j = 1, \ldots, n$ such that $i \neq j$. Also if $a_{ii} > 0$, $i = 1, 2, \ldots, n$ the matrix is called an L-matrix. Furthermore, a Z-matrix is a nonsingular M-matrix, if A is nonsingular and $A^{-1} \geq 0$.*

Definition 2.2 ([2]) *Let A be a real matrix. The representation $A = M - N$ is called a splitting of A if M is a nonsingular matrix. The splitting is called:*

- *convergent if $\rho(M^{-1}N) < 1$;*
- *regular if $M^{-1} \geq 0$ and $N \geq 0$;*
- *nonnegative if $M^{-1}N \geq 0$;*
- *M-splitting if M is a nonsingular M-matrix and $N \geq 0$.*

Definition 2.3 ([2]) *An $n \times n$ matrix $A = (a_{ij})$ is reducible if we may partition $i = 1, \ldots, n$ into two nonempty subsets E, F such that $a_{ij} = 0$ if $i \in E$ and $j \in F$. If A is not a reducible matrix, we call A is an irreducible matrix.*

Lemma 2.4 ([2]) *Let $A \geq 0$ be an irreducible matrix. Then*

- *A has a positive real eigenvalue equal to its spectral radius.*
- *To $\rho(A)$ there corresponds an eigenvector $x > 0$.*
- *$\rho(A)$ is a simple eigenvalue of A.*
- *$\rho(A)$ increases when any entry of A increases.*

Lemma 2.5 ([2]) *Let A be a nonnegative matrix. Then*

(1) If $\alpha x \leq Ax$ for some nonnegative vector x, $x \neq 0$, then $\alpha \leq \rho(A)$.

(2) If $Ax \leq \beta x$ for some positive vector x, then $\rho(A) \leq \beta$. Moreover, if A is irreducible and if $0 \neq \alpha x \leq Ax \leq \beta x$ for some nonnegative vector x, then $\alpha \leq \rho(A) \leq \beta$ and x is a positive vector.

Lemma 2.6 ([2]) *Let $A = M - N$ be an M-spliting of A. Then $\rho(M^{-1}N) < 1$ if and if A is a nonsingular M-matrix.*

Lemma 2.7 ([24]) *Let $\lambda \in (0, 1]$, $y \in (-\infty, 0)$, and $z \in (-\infty, 0)$. Then the set Q*

$$Q = (\frac{\lambda - yz}{y}, -z) \cap (0, -z) \tag{2.1}$$

is nonempty.

Theorem 2.8 *Let $L(r, w)$, $\widetilde{L}_{\alpha\beta}(r, w)$ and $\widehat{L}_{\alpha\beta}(r, w)$ be the iteration matrices of the AOR method given by Eqs. (1.1), (1.11) and (1.12) associated with $P_{\alpha\beta}$. If A is an irreducible L-matrix with $a_{v+1,1}a_{1,v+1} > 0$, $\beta_v \in (\frac{1-a_{v+1,1}a_{1,v+1}}{a_{v+1,1}}, -a_{1,v+1}) \cap (0, -a_{1,v+1})$, $\alpha_v \in (0, 1]$ $(v = 1, 2, \ldots, n-1)$, $\sum_{v=1}^{n-1} \alpha_v \leq 1$ and $0 \leq r < w \leq 1$. Then $L(r, w)$, $\widetilde{L}_{\alpha\beta}(r, w)$, $\widehat{L}_{\alpha\beta}(r, w)$ are nonnegative irreducible matrices.*

Proof Because A is an irreducible L-matrix, L is a nonnegative strictly lower triangular matrix and U is a nonnegative strictly upper triangular matrix. By Eq. (1.4), we have

$$\begin{aligned} L(r, w) &= (I - rL)^{-1}[(1 - w)I + (w - r)L + wU] \\ &= [I + rL + r^2L^2 + \cdots + r^{n-1}L^{n-1}] \times [(1 - w)I + (w - r)L + wU] \\ &= (1 - w)I + (w - r)L + wU + \text{nonnegative terms}. \end{aligned} \tag{2.2}$$

Since $0 \leq r < w \leq 1$, it follows that $L(r, w)$ is nonnegative. We can also get that $(1 - w)I + (w - r)L + wU$ is irreducible for irreducible A, and hence $L(r, w)$ is also irreducible. Now, we show that $D_{\alpha\beta} > 0$, $L_{\alpha\beta} \geq 0$, $U_{\alpha\beta} \geq 0$, and $E_{\alpha\beta} \leq 0$. We obtain

$$\begin{aligned} D_{\alpha\beta} &= \mathrm{diag}(d_{11}, 1, \ldots, 1), \\ d_{11} &= 1 - \alpha_1 a_{21}(a_{12} + \beta_1) - \alpha_2 a_{31}(a_{13} + \beta_2) - \cdots - \alpha_{n-1}a_{n,1}(a_{1,n} + \beta_{n-1}), \\ E_{\alpha\beta} &= \mathrm{diag}(e_{11}, 0, \ldots, 0), \\ e_{11} &= -\alpha_1 a_{21}(a_{12} + \beta_1) - \alpha_2 a_{31}(a_{13} + \beta_2) - \cdots - \alpha_{n-1}a_{n,1}(a_{1,n} + \beta_{n-1}), \end{aligned}$$

$$L_{\alpha\beta} = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ -a_{21} & 0 & \cdots & 0 \\ -a_{31} & -a_{32} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} & \cdots & -a_{n,n-1} & 0 \end{bmatrix}, \quad U_{\alpha\beta} = \begin{bmatrix} 0 & u_{12} & u_{13} & \cdots & u_{1n} \\ 0 & 0 & -a_{23} & \cdots & -a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & -a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

where

$$u_{1j} = -a_{1j} + \alpha_1 a_{2,j}(a_{12} + \beta_1) + \alpha_2 a_{3,j}(a_{13} + \beta_2) + \cdots + \alpha_{n-1}a_{n,j}(a_{1n} + \beta_{n-1}). \tag{2.3}$$

For $\beta_v \in (\frac{1-a_{v+1,1}a_{1,v+1}}{a_{v+1,1}}, -a_{1,v+1}) \cap (0, -a_{1,v+1})$, $\alpha_v \in (0, 1]$ $(v = 1, 2, \ldots, n-1)$, $\sum_{v=1}^{n-1} \alpha_v \leq 1$, we can write

$$1 - \sum_{v=1}^{n-1} \alpha_v a_{v+1,1}(a_{1,v+1} + \beta_v) > 1 - \sum_{v=1}^{n-1} \alpha_v a_{v+1,1}(a_{1,v+1} + \frac{1 - a_{v+1,1}a_{1,v+1}}{a_{1,v+1}})$$

$$= 1 - \sum_{v=1}^{n-1} \alpha_v a_{v+1,1} a_{1,v+1} + \alpha_v (1 - a_{v+1,1} a_{1,v+1})$$
$$= 1 - \sum_{v=1}^{n-1} \alpha_v \geq 0. \tag{2.4}$$

By considering equations (2.4), we get $D_{\alpha\beta} > 0$ and $E_{\alpha\beta} \leq 0$. We can also write

$$-a_{1j} + \sum_{v=1}^{n-1} \alpha_v a_{v,j}(a_{1,v+1} + \beta_v) \geq -a_{1j} + \sum_{v=1}^{n-1} \alpha_v a_{v,j}(a_{1,v+1} - a_{1,v+1})$$
$$= -a_{1j} \geq 0. \tag{2.5}$$

Therefore, $L_{\alpha\beta} \geq 0$ and $U_{\alpha\beta} \geq 0$. Now from equation (1.11), we have

$$\begin{aligned} \widetilde{L}_{\alpha\beta}(r,w) =& (D_{\alpha\beta} - rL_{\alpha\beta})^{-1}[(1-w)D_{\alpha\beta} + (w-r)L_{\alpha\beta} + wU_{\alpha\beta}] \\ =& (I - rD_{\alpha\beta}^{-1}L_{\alpha\beta})^{-1}[(1-w)I + (w-r)D_{\alpha\beta}^{-1}L_{\alpha\beta} + wD_{\alpha\beta}^{-1}U_{\alpha\beta}] \\ =& [I + rD_{\alpha\beta}^{-1}L_{\alpha\beta} + r^2(D_{\alpha\beta}^{-1}L_{\alpha\beta})^2 + \cdots + r^{n-1}(D_{\alpha\beta}^{-1}L_{\alpha\beta})^{n-1}]\times \\ & [(1-w)I + (w-r)D_{\alpha\beta}^{-1}L_{\alpha\beta} + wD_{\alpha\beta}^{-1}U_{\alpha\beta}] \\ =& (1-w)I + (w-r)D_{\alpha\beta}^{-1}L_{\alpha\beta} + wD_{\alpha\beta}^{-1}U_{\alpha\beta} + \text{nonnegative terms}. \end{aligned} \tag{2.6}$$

By the above results, we can see $\widetilde{L}_{\alpha\beta}(r,w)$ are nonnegative irreducible matrix. Similar to the above arguments, we can show that $\widehat{L}_{\alpha\beta}(r,w)$ are nonnegative irreducible matrix. The proof is completed. □

In the next section, applying the above results, we will present the main theorems in this work.

3. Comparison theorems

The spectral radius of the iterative matrix is conclusive for the convergence and stability of the method, and the smaller it is, the faster the method converges when the spectral radius is smaller than 1. In this section, some results for the AOR iterative method with preconditioner $P_{\alpha\beta}$ is given.

Theorem 3.1 *Let $L(r,w)$, $\widetilde{L}_{\alpha\beta}(r,w)$ be the iteration matrices of the AOR method given by equations (1.4) and (1.11) associated with $P_{\alpha\beta}$. If A is an irreducible L-matrix with $a_{v+1,1}a_{1,v+1} > 0$, $\beta_v \in (\frac{1-a_{v+1,1}a_{1,v+1}}{a_{v+1,1}}, -a_{1,v+1}) \cap (0, -a_{1,v+1})$, $\alpha_v \in (0,1]$ $(v = 1,2,\ldots,n-1)$, $\sum_{v=1}^{n-1} \alpha_v \leq 1$ and $0 \leq r < w \leq 1$, then we have*

(1) $\rho(\widetilde{L}_{\alpha\beta}(r,w)) < \rho(L(r,w))$, if $\rho(L(r,w)) < 1$,

(2) $\rho(\widetilde{L}_{\alpha\beta}(r,w)) = \rho(L(r,w))$, if $\rho(L(r,w)) = 1$,

(3) $\rho(\widetilde{L}_{\alpha\beta}(r,w)) > \rho(L(r,w))$, if $\rho(L(r,w)) > 1$.

Proof Theorem 2.8 implies that $L(r,w)$ is a nonnegative irreducible matrix. Hence there exists a positive vector x, such that

$$L(r,w)x = \lambda x \tag{3.1}$$

where $\rho(L(r,w))=\lambda$ or equivalently

$$[(1-w)I+(w-r)L+wU]x=\lambda(I-rL)x. \tag{3.2}$$

By Eq. (3.2) we can obtain

$$\begin{aligned}
&\widetilde{L}_{\alpha\beta}(r,w)x-\lambda x\\
&\quad=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[(1-w)D_{\alpha\beta}+(w-r)L_{\alpha\beta}+wU_{\alpha\beta}]x-\lambda x\\
&\quad=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[(1-w)D_{\alpha\beta}+(w-r)L_{\alpha\beta}+wU_{\alpha\beta}-\lambda(D_{\alpha\beta}-rL_{\alpha\beta})]x\\
&\quad=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[(1-w)(I+E_{\alpha\beta})+(w-r)L+w(U+F_{\alpha\beta})-\lambda(I+E_{\alpha\beta}-rL)]x\\
&\quad=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[(1-w)(I+E_{\alpha\beta})+(w-r)L+w(U+F_{\alpha\beta})-\lambda(I-rL)-\lambda E_{\alpha\beta}]x\\
&\quad=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[(1-w)(I+E_{\alpha\beta})+(w-r)L+w(U+F_{\alpha\beta})-\\
&\qquad(1-w)I-(w-r)L-wU-\lambda E_{\alpha\beta}]x\\
&\quad=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[(1-w)E_{\alpha\beta}+wF_{\alpha\beta}-\lambda E_{\alpha\beta}]x\\
&\quad=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[(1-\lambda)E_{\alpha\beta}+w(F_{\alpha\beta}-E_{\alpha\beta})]x\\
&\quad=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[(1-\lambda)E_{\alpha\beta}-wS_{\alpha\beta}A]x\\
&\quad=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[(1-\lambda)E_{\alpha\beta}+S_{\alpha\beta}(-wA)]x\\
&\quad=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[(1-\lambda)E_{\alpha\beta}+S_{\alpha\beta}((\lambda-1)(I-rL))]x\\
&\quad=(\frac{\lambda-1}{\lambda})(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[-\lambda E_{\alpha\beta}+S_{\alpha\beta}(\lambda(I-rL))]x\\
&\quad=(\frac{\lambda-1}{\lambda})(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[-\lambda E_{\alpha\beta}+(1-w)S_{\alpha\beta}+(w-r)S_{\alpha\beta}L+wS_{\alpha\beta}U]x. \qquad (3.3)
\end{aligned}$$

Now let

$$Q=(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}[-\lambda E_{\alpha\beta}+(1-w)S_{\alpha\beta}+(w-r)S_{\alpha\beta}L+wS_{\alpha\beta}U]. \tag{3.4}$$

From $S_{\alpha\beta}\geq 0$, $S_{\alpha\beta}L\geq 0$, $E_{\alpha\beta}\leq 0$, $S_{\alpha\beta}U\geq 0$ we have

$$[-\lambda E_{\alpha\beta}+(1-w)S_{\alpha\beta}+(w-r)S_{\alpha\beta}L+wS_{\alpha\beta}U]\geq 0.$$

From Definition 2.2, we have the splitting $R=D_{\alpha\beta}-rL_{\alpha\beta}$ as an M-splitting of R. Since $rD_{\alpha\beta}^{-1}L_{\alpha\beta}$ is a strictly lower triangular matrix so that $\rho(rD_{\alpha\beta}^{-1}L_{\alpha\beta})=0<1$. By considering Lemma 2.6, we have R is a nonsingular M-matrix. Therefore, $(D_{\alpha\beta}-rL_{\alpha\beta})^{-1}\geq 0$, and so $Q\geq 0$.

(1) If $\lambda<1$, then $\widetilde{L}_{\alpha\beta}(r,w)x-\lambda x\leq 0$. Therefore, $\widetilde{L}_{\alpha\beta}(r,w)x\leq\lambda x$. By using Lemma 2.5, we get $\rho(\widetilde{L}_{\alpha\beta}(r,w))<\lambda=\rho(L(r,w))$;

(2) If $\lambda=1$, then $\widetilde{L}_{\alpha\beta}(r,w)x-\lambda x=0$. Therefore, $\widetilde{L}_{\alpha\beta}(r,w)x=\lambda x$. By using Lemma 2.5, we get $\rho(\widetilde{L}_{\alpha\beta}(r,w))=\lambda=\rho(L(r,w))$;

(3) If $\lambda>1$, then $\widetilde{L}_{\alpha\beta}(r,w)x-\lambda x\geq 0$. Therefore, $\widetilde{L}_{\alpha\beta}(r,w)x\geq\lambda x$. By using Lemma 2.5, we get $\rho(\widetilde{L}_{\alpha\beta}(r,w))\geq\lambda=\rho(L(r,w))$. $\square$

Theorem 3.2 *Let $L(r,w)$, $\widehat{L}_{\alpha\beta}(r,w)$ be the iteration matrices of the AOR method given by equations (1.4) and (1.12) associated with $P_{\alpha\beta}$. If A is an irreducible L-matrix with*

$a_{v+1,1}a_{1,v+1} > 0$, $\beta_v \in (\frac{1-a_{v+1,1}a_{1,v+1}}{a_{v+1,1}}, -a_{1,v+1}) \cap (0, -a_{1,v+1})$, $\alpha_v \in (0,1]$ $(v = 1,2,\ldots,n-1)$, $\sum_{v=1}^{n-1} \alpha_v \le 1$, *and* $0 \le r < w \le 1$, *then we have*

(1) $\rho(\widehat{L}_{\alpha\beta}(r,w)) < \rho(L(r,w))$, *if* $\rho(L(r,w)) < 1$;

(2) $\rho(\widehat{L}_{\alpha\beta}(r,w)) = \rho(L(r,w))$, *if* $\rho(L(r,w)) = 1$;

(3) $\rho(\widehat{L}_{\alpha\beta}(r,w)) > \rho(L(r,w))$, *if* $\rho(L(r,w)) > 1$.

Proof *By Eq. (3.2) we can obtain*

$$\begin{aligned}
&\widehat{L}_{\alpha\beta}(r,w)x - \lambda x = (I - rL)^{-1}[(1-w)I + (w-r)L + w(U_{\alpha\beta} - E_{\alpha\beta})]x - \lambda x\\
&\quad = (I - rL)^{-1}[(1-w)I + (w-r)L + w(U_{\alpha\beta} - E_{\alpha\beta}) - \lambda(I - rL)]x\\
&\quad = (I - rL)^{-1}[(1-w)I + (w-r)L + w(U_{\alpha\beta} - E_{\alpha\beta}) - (1-w)I - (w-r)L - wU]x\\
&\quad = (I - rL)^{-1}[w(F_{\alpha\beta} - E_{\alpha\beta})]x = (I - rL)^{-1}[w(-S_{\alpha\beta}A)]x\\
&\quad = (I - rL)^{-1}[S_{\alpha\beta}(-wA)]x = (I - rL)^{-1}[S_{\alpha\beta}((\lambda - 1)(I - rL))]x\\
&\quad = (\frac{\lambda - 1}{\lambda})(I - rL)^{-1}[(1-w)S_{\alpha\beta} + (w-r)S_{\alpha\beta}L + wS_{\alpha\beta}U]. \qquad (3.5)
\end{aligned}$$

Now let

$$\overline{Q} = (I - rL)^{-1}[(1-w)S_{\alpha\beta} + (w-r)S_{\alpha\beta}L + wS_{\alpha\beta}U]. \qquad (3.6)$$

Similarly we have $\overline{Q} \ge 0$.

(1) *If* $\lambda < 1$, *then* $\widehat{L}_{\alpha\beta}(r,w)x - \lambda x \le 0$. *Therefore,* $\widehat{L}_{\alpha\beta}(r,w)x \le \lambda x$. *By using Lemma 2.5, we get* $\rho(\widehat{L}_{\alpha\beta}(r,w)) < \lambda = \rho(L(r,w))$;

(2) *If* $\lambda = 1$, *then* $\widehat{L}_{\alpha\beta}(r,w)x - \lambda x = 0$. *Therefore,* $\widehat{L}_{\alpha\beta}(r,w)x = \lambda x$. *By using Lemma 2.5, we get* $\rho(\widehat{L}_{\alpha\beta}(r,w)) = \lambda = \rho(L(r,w))$;

(3) *If* $\lambda > 1$, *then* $\widehat{L}_{\alpha\beta}(r,w)x - \lambda x \ge 0$. *Therefore,* $\widehat{L}_{\alpha\beta}(r,w)x \ge \lambda x$. *By using Lemma 2.5, we get* $\rho(\widehat{L}_{\alpha\beta}(r,w)) \ge \lambda = \rho(L(r,w))$.

We know, when $w = r$ the AOR method reduces to the SOR method. For $w = r$, $L(r,w)$, $\widetilde{L}_{\alpha\beta}(r,w)$ and $\widehat{L}_{\alpha\beta}(r,w)$, $T(w)$, $\widetilde{T}_{\alpha\beta}(w)$ and $\widehat{T}_{\alpha\beta}(w)$ are presented as follows

$$T(w) = (I - wL)^{-1}[(1-w)I + wU], \qquad (3.7)$$

$$\widetilde{T}_{\alpha\beta}(w) = (D_{\alpha\beta} - wL_{\alpha\beta})^{-1}[(1-w)D_{\alpha\beta} + wU_{\alpha\beta}], \qquad (3.8)$$

$$\widehat{T}_{\alpha\beta}(w) = (I - wL_{\alpha\beta})^{-1}[(1-w)I + w(U_{\alpha\beta} - E_{\alpha\beta})]. \qquad (3.9)$$

Using the similar arguments of Theorems 3.1 and 3.2, we can obtain the following results.

Corollary 3.3 ([2]) *Let* $T(w)$, $\widetilde{T}_{\alpha\beta}(w)$ *be defined by Eqs. (3.7) and (3.9) associated with* $P_{\alpha\beta}$. *If* A *is an irreducible L-matrix with* $a_{v+1,1}a_{1,v+1} > 0$, $\beta_v \in (\frac{1-a_{v+1,1}a_{1,v+1}}{a_{v+1,1}}, -a_{1,v+1}) \cap (0, -a_{1,v+1})$, $\alpha_v \in (0,1]$ $(v = 1,2,\ldots,n-1)$, $\sum_{v=1}^{n-1} \alpha_v \le 1$, *and* $0 < w < 1$, *then we have*

(1) $\rho(\widetilde{T}_{\alpha\beta}(w)) < \rho(T(w))$, *if* $\rho(T(w)) < 1$;

(2) $\rho(\widetilde{T}_{\alpha\beta}(w)) = \rho(T(w))$, *if* $\rho(T(w)) = 1$;

(3) $\rho(\widetilde{T}_{\alpha\beta}(w)) > \rho(T(w))$, *if* $\rho(T(w)) > 1$.

Corollary 3.4 ([2]) *Let* $T(w)$, $\widehat{T}_{\alpha\beta}(w)$ *be defined by Eqs. (3.7) and (3.9) associated with* $P_{\alpha\beta}$. *If*

A is an irreducible L-matrix with $a_{v+1,1}a_{1,v+1} > 0$, $\beta_v \in (\frac{1-a_{v+1,1}a_{1,v+1}}{a_{v+1,1}}, -a_{1,v+1}) \cap (0, -a_{1,v+1})$, $\alpha_v \in (0,1]$ $(v = 1, 2, \dots, n-1)$, $\sum_{v=1}^{n-1} \alpha_v \le 1$, *and* $0 < w < 1$, *then we have*

(1) $\rho(\widehat{T}_{\alpha\beta}(w)) < \rho(T(w))$, *if* $\rho(T(w)) < 1$;

(2) $\rho(\widehat{T}_{\alpha\beta}(w)) = \rho(T(w))$, *if* $\rho(T(w)) = 1$;

(3) $\rho(\widehat{T}_{\alpha\beta}(w)) > \rho(T(w))$, *if* $\rho(T(w)) > 1$.

4. Numerical experiments

The numerical experiments presented in this section were computed in double precision with some MATLAB 8.3 (R2014a) codes on a Corei5 PC, with a 2.53 2.53 GHz CPU and 4.00GB of RAM.

N	(r, w)	P_0	P_1	P_2	P_3
5	(0.6,0.8)	0.8423	0.7964	0.7674	0.7121
10	(0.6,1)	0.7739	0.7697	0.6541	0.6002
20	(0.6,0.8)	0.9474	0.8854	0.8657	0.7458
30	(0.6,1)	0.9289	0.9000	0.8745	0.8223

Table 1 The comparison of the spectral radius for Example 4.1

Example 4.1 We consider the two dimensional convection-diffusion equation [26]

$$-(u_{xx} + u_{yy}) + u_x + 2u_y = f(x, y), \quad \text{in} \quad \Omega = (0,1) \times (0,1) \tag{4.1}$$

with the homogeneous Dirichlet boundary conditions. Discretization of this equation on a uniform grid with $N \times N$ interior nodes ($n = N^2$), by using the second order centered differences for the second and first order differentials gives a linear system of equations of order n with n unknowns. The coefficient matrix of the obtained system is of the form

$$A = I \otimes P + Q \otimes I, \tag{4.2}$$

where $\otimes$ denotes the Kronecker product,

$$P = \text{tridiag}(-\frac{2+h}{8}, 1, -\frac{2-8}{8}), \quad Q = \text{tridiag}(-\frac{1+h}{4}, 1, -\frac{1-h}{4}) \tag{4.3}$$

are $N \times N$ tridiagonal matrices, and the step size is $h = \frac{1}{N}$. We consider three preconditioners of the form $P_0 = I$, P_1, P_2 and P_3. where for the preconditioner P_k, $k = 1, 2, 3$, α_i's and β_i's are random numbers uniformly distributed in the corresponding interval. We mention that $P_0 = I$ means that no preconditioner is used. In Table 1, the spectral radius of the AOR iterative method applied to the preconditioned systems $P_iAx = P_ib$, $i = 0, \dots, 3$ for different values of r, w and n are given.

For more investigation, we apply the GMRES(m) method [27] with $m = 10$ to solve $P_iAx = P_ib$, $i = 0, \dots, 3$. In all the experiments, vector $b = A(1, 1, \dots, 1)^T$ was taken to be the right-hand side of the linear system and a null vector as an initial guess. The stopping criterion used was always

$$\frac{||b - Ax_k||_2}{||b||_2} < 10^{-10}. \tag{4.4}$$

In Table 2, we report the number of iterations and the CPU time (in parenthesis) for the convergence.

N	(r, w)	P_0	P_1	P_2	P_3
50	(0.8,1)	90(0.43)	81 (0.31)	50(0.51)	31(0.16)
100	(0.6,1)	340(5.33)	120(3.83)	122(3.31)	70 (1.43)
150	(0.8,0.8)	750(31.08)	374(18.73)	234(16.56)	185(11.08)

Table 2 Number of iterations and the CPU time for the convergence of the GMRES(10) for Example 4.1

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References

[1] D. M. YOUNG. *Iterative Solution of Large Linear Systems*. Academic Press, New York, NY, USA, 1971.

[2] R. S. VARGA. *Matrix Iterative Analysis*. Springer, Berlin, Germany, 2000.

[3] A. BERMAN, R. J. PLEMONS. *Nonnegative Matrices in the Mathematics Sciences*. SIAM, Philadelphia, Pa, USA, 1994.

[4] Y. SAAD. *Iterative Methods for Sparse Linear Systems*. PWS Publishing, Boston, Mass, USA, 1996.

[5] A. HADJIDIMOS. *Accelerated overrelaxation method*. Math. Comp., 1978, **32**(141): 149–157.

[6] G. AVDELAS, A. HADIDIMOS. *Some theoretical and computational results concerning the accelerated overrelaxation (AOR) method*. Anal. Numér. Théor. Approx, 1980, **9**(1): 5–10.

[7] A. K. YEYIOS. *A necessary condition for the convergence of the accelerated overrelaxation (AOR) method*. J. Comput. Appl. Math., 1989, **26**(3): 371–373.

[8] Zhongzhi BAI. *The monotone convergence rate of the parallel nonlinear AOR method*. Comput. Math. Appl., 1996, **31**(7): 1–8.

[9] Zhongzhi BAI. *Parallel nonlinear AOR method and its convergence*. Comput. Math. Appl., 1996, **31**(2): 21–31.

[10] Zhongzhi BAI. *Asynchronous multisplitting AOR methods for a class of systems of weakly nonlinear equations*. Appl. Math. Comput., 1999, **98**(1): 49–59.

[11] L. CVETKOVIĆ, V. KOSTIĆ. *A note on the convergence of the AOR method*. Appl. Math. Comput., 2007, **194**(2): 394–399.

[12] Zhongxi GAO, Tingzhu HUANG. *Convergence of AOR method*. Appl. Math. Comput., 2006, **176**(1): 134–140.

[13] Jinyun YUAN, Xiaoqing JIN. *Convergence of the generalized AOR method*. Appl. Math. Comput., 1999, **99**(1): 35–46.

[14] Wen LI, Weiwei SUN. *Comparison results for parallel multisplitting methods with applications to AOR methods*. Linear Algebra Appl., 2001, **331**(1-3): 131–144.

[15] Zhongzhi BAI. *Sharp error bounds of some Krylov subspace methods for non-Hermitian linear systems*. Appl. Math. Comput., 2000, **109**(2-3): 273–285.

[16] A. D. GUNAWARDENA, S. K. JAIN, L. SNYDER. *Modified iterative methods for consistent linear systems*. Linear Algebra Appl., 1991, **154/156**: 123–143.

[17] D. J. EVANS, M. M. MARTINS, M. E. TRIGO. *The AOR iterative method for new preconditioned linear systems*. J. Comput. Appl. Math., 2001, **132**(2): 461–466.

[18] H. KOTAKEMORI, K. HARADA, M. MORIMOTO, et al. *A comparison theorem for the iterative method with the preconditioner* $(I + S_{\max})$, J. Comput. Appl. Math., 2002, **145**(2): 373–378.

[19] A. HADJIDIMOS, D. NOUTSOS, M. TZOUMAS. *More on modifications and improvements of classical iterative schemes for M-matrices*. Linear Algebra Appl., 2003, **364**: 253–279.

[20] H. NIKI, K. HARADA, M. MORIMOTO, et al. *The survey of preconditioners used for accelerating the rate of convergence in the Gauss-Seidel method*. J. Comput. Appl. Math., 2004, **164/165**: 587–600.

[21] Liying SUN. *A comparison theorem for the SOR iterative method*. J. Comput. Appl. Math., 2005, **181**(2): 336–341.

[22] Chengyi ZHANG, Chengxian XU, Shuanghua LUO. *Covergence of block iterative methods for linear systems with generalized H-matrices*. Comput. Appl. Math., 2009, **229**(1): 70–84.

[23] Aijuan LI. *Improving AOR iterative methods for irreducible L-matrices*. Engineering Letters, 2011, **19**(1): 46–49.

[24] M. DEHGHAN, M. HAJARIAN. *Modified AOR Iterative Methods to Solve Linear Systems*. Journal of Vibration and Control, 2012.

[25] Yaotang LI, Cuixia LI, Shiliang WU. *Improvements of preconditioned AOR iterative method for L-matrices*. J. Comput. Appl. Math., 2007, **206**(2): 656–665.

[26] Meijun WU, Li WANG, Yongzhong SONG. *Preconditioned AOR iterative method for linear systems*. Appl. Numer. Math., 2007, **57**(5-7): 672–685.

[27] Y. SAAD, M. H. SCHULTZ. *GMRES: A generalized minimal residual method for solving nonsymmetric linear systems*. SIAM J. Sci. Statist. Comput., 1986, **7**: 856–869.