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QDB-Tensors and $SQDB$-Tensors

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Abstract In this paper, we propose four new classes of structured tensors: $QDB(QDB_0)$-tensors and $SQDB(SQDB_0)$-tensors, and prove that even order symmetric QDB-tensors and $SQDB$-tensors are positive definite, even order symmetric QDB_0-tensors and $SQDB_0$-tensors are positive semi-definite.

Keywords B-tensors; QDB-tensors; $SQDB$-tensors; positive definite; P-tensors

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1. Introduction

A real order m dimension n tensor $\mathcal{A} = (a_{i_1\cdots i_m})$, denoted by $\mathcal{A} \in R^{[m,n]}$, consists of n^m real entries:

$$a_{i_1\cdots i_m} \in R,$$

where $i_j \in N = \{1,\dots,n\}$ for $j = 1,\dots,m$. Obviously, a vector is a tensor of order 1 and a matrix is a tensor of order 2. A real tensor $\mathcal{A} = (a_{i_1\cdots i_m})$ is called symmetric [1] if

$$a_{i_1,\dots,i_m} = a_{\pi(i_1,\dots,i_m)}, \quad \forall \pi \in \Pi_m,$$

where Π_m is the permutation group of m indices. Furthermore, a real tensor of order m dimension n is called the unit tensor, if its entries are $\delta_{i_1\cdots i_m}$ for $i_1,\dots,i_m \in N$ (see [2,3]), where

$$\delta_{i_1\cdots i_m} = \begin{cases} 1, & \text{if } i_1 = \cdots = i_m, \\ 0, & \text{otherwise.} \end{cases}$$

For a tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, if there are a number $\lambda \in R$ and a nonzero vector $x = (x_1, x_2, \dots, x_n)^T \in R^n$ that are solutions of the following homogeneous polynomial equations:

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

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where

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m\in N} a_{ii_2\cdots i_m} x_{i_2}\cdots x_{i_m},$$

and $(x^{[m-1]})_i = x_i^{m-1}$, then λ is called an H-eigenvalue of $\mathcal{A}$ and x is called a corresponding H-eigenvector of $\mathcal{A}$ associated with λ (see [1]). As shown in [1], Qi used H-eigenvalues of real symmetric tensors to determine positive (semi-)definite tensors, that is, an even order real symmetric tensor is positive (semi-)definite if and only if all its H-eigenvalues are positive (non-negative). Here a tensor $\mathcal{A}$ is called positive (semi-)definite [4,5] if for any nonzero vector $x \in R^n$, such that

$$\mathcal{A}x^m > (\geq)\ 0,$$

where $\mathcal{A}x^m = \sum_{i_1,i_2,\ldots,i_m\in N} a_{i_1i_2\cdots i_m} x_{i_1}\cdots x_{i_m}$.

Positive definiteness and semi-definiteness of real symmetric tensors have important applications in automatical control, polynomial problems, magnetic resonance imaging and spectral hypergraph theory [4, 5, 7–18].

It is not effective by using H-eigenvalues in some cases to determine that a real symmetric tensor $\mathcal{A}$ is positive (semi-)definite because it is not easy to compute the smallest H-eigenvalue of that tensor when its order and dimension are large. Hence one tries to give some checkable sufficient conditions [4–6,18–20]. In [4], Song and Qi introduced the class of $B(B_0)$-tensors, which is a natural extension of B-matrices, to provide a checkable sufficient condition for positive (semi-) definite tensors.

Definition 1.1 ([4]) *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in R^{[m,n]}$. $\mathcal{A}$ is called a $B(B_0)$-tensor if for all $i \in N$*

$$\sum_{i_2,\ldots,i_m\in N} a_{ii_2\cdots i_m} > (\geq)\ 0,$$

and

$$\frac{1}{n^{m-1}}\Big(\sum_{i_2,\ldots,i_m\in N} a_{ii_2\cdots i_m}\Big) > (\geq)\ a_{ij_2\cdots j_m},\ \text{ for } j_2\cdots j_m \in N,\ \delta_{ij_2\cdots j_m} = 0.$$

By Definition 1.1, Song and Qi [4] gave the following property of $B(B_0)$-tensors.

Proposition 1.2 ([4]) *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ be a real tensor of order m dimension n. Then $\mathcal{A}$ is a $B(B_0)$-tensor if and only if for each $i \in N$,*

$$\sum_{i_2,\ldots,i_m\in N} a_{ii_2\cdots i_m} > (\geq)\ n^{m-1}\beta_i(\mathcal{A}),$$

where

$$\beta_i(\mathcal{A}) = \max_{\substack{j_2,\ldots,j_m\in N,\\ \delta_{ij_2\cdots j_m}=0}} \{0, a_{ij_2\cdots j_m}\}.$$

In [4], Song and Qi gave the definition of $P(P_0)$-tensor as an extension of $P(P_0)$-matrices.

Definition 1.3 ([4]) *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in R^{[m,n]}$. We say that $\mathcal{A}$ is*

(1) A P_0 tensor if for any nonzero vector $x \in R^n$, there exists $i \in N$ such that $x_i \neq 0$ and

$$x_i(\mathcal{A}x^{m-1})_i \geq 0;$$

(2) A P tensor if for any nonzero vector $x \in R^n$, $\max_{i\in N} x_i(\mathcal{A}x^{m-1})_i > 0$.

In [6], Li and Li provided the following necessary and sufficient conditions for $B(B_0)$-tensors.

Proposition 1.4 ([6]) *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in R^{[m,n]}$. Then*

(1) $\mathcal{A}$ is a B-tensor if and only if for each $i \in N$,

$$a_{ii\cdots i} - \beta_i(\mathcal{A}) > \Phi_i(\mathcal{A}),$$

where $\beta_i(\mathcal{A}) = \max_{\substack{j_2\cdots j_m \in N,\\ \delta_{ij_2\cdots j_m}=0}} \{0, a_{ij_2\cdots j_m}\}$ and $\Phi_i(\mathcal{A}) = \sum_{\substack{i_2\cdots i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} (\beta_i(\mathcal{A}) - a_{ii_2\cdots i_m})$.

(2) $\mathcal{A}$ is a B_0-tensor if and only if for each $i \in N$, $a_{ii\cdots i} - \beta_i(\mathcal{A}) \geq \Phi_i(\mathcal{A})$.

In [8], Ding et al. gave the definition of Z-tensors as an extension of Z-matrices.

Definition 1.5 ([8]) *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ with $n \geq 2$. If all of the off-diagonal entries of $\mathcal{A}$ are non-positive, that is $a_{i_1 i_2\cdots i_m} \leq 0$, for $i_j \in N$, $j = 1, 2, \ldots, m$, and $\delta_{i_1 i_2\cdots i_m} = 0$, then $\mathcal{A}$ is called a Z-tensor.*

In this paper, we continue to focus on the positive (semi-) definiteness identification problem of real tensors. By introducing four new classes of structured tensors: QDB-tensors, QDB_0-tensors, $SQDB$-tensors and $SQDB_0$-tensors, we give two checkable sufficient condition for the positive definiteness of tensors and two checkable sufficient condition for the positive semi-definiteness of tensors. Also the relationships between these four classes of tensors and some existing classes are given.

2. $QDB(QDB_0)$-tensors and $SQDB(SQDB_0)$-tensors

We begin with some notation. Given a tensor $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, let

$$\Delta_i = \{(i_2, i_3, \ldots, i_m):\ i_j = i \text{ for some } j \in \{2, \ldots, m\},\ \text{where } i, i_2, \ldots, i_m \in N\},$$

$$\bar{\Delta}_i = \{(i_2, i_3, \ldots, i_m) : i_j \neq i \text{ for any } j \in \{2, \ldots, m\},\ \text{where } i, i_2, \ldots, i_m \in N\}.$$

$$r_i^{\Delta_i}(\mathcal{A}) = \sum_{\substack{(i_2\cdots i_m)\in\Delta_i,\\ \delta_{ii_2\cdots i_m}=0}} |a_{ii_2\cdots i_m}|,\quad r_i^{\bar{\Delta}_i}(\mathcal{A}) = \sum_{(i_2\cdots i_m)\in\bar{\Delta}_i} |a_{ii_2\cdots i_m}|.$$

Obviously, $r_i(\mathcal{A}) = r_i^{\Delta_i}(\mathcal{A}) + r_i^{\bar{\Delta}_i}(\mathcal{A})$, $r_i^j(\mathcal{A}) = r_i^{\Delta_i}(\mathcal{A}) + r_i^{\bar{\Delta}_i}(\mathcal{A}) - |a_{ij\cdots j}|$.

Given a nonempty proper subset S of N, we denote

$$\Delta^N = \{(i_2, i_3, \ldots, i_m):\ \text{each } i_j \in N \text{ for } j = 2, \ldots, m\},$$

$$\Delta^S = \{(i_2, i_3, \ldots, i_m):\ \text{each } i_j \in S \text{ for } j = 2, \ldots, m\},$$

and then $\overline{\Delta^S} = \Delta^N \backslash \Delta^S$.

For a tensor $\mathcal{A} = (a_{i_1 i_2\cdots i_m}) \in R^{[m,n]}$, we can write it as

$$\mathcal{A} = \mathcal{B}^+ + \mathcal{C},$$

where $\mathcal{B}^+ = (b_{i_1 i_2\cdots i_m}) \in R^{[m,n]}$, $\mathcal{C} = (c_{i_1 i_2\cdots i_m}) \in R^{[m,n]}$,

$$b_{ii_2\cdots i_m} = a_{ii_2\cdots i_m} - \beta_i(\mathcal{A}) \text{ for } i \in N,$$

and

$$c_{ii_2\cdots i_m} = \beta_i(\mathcal{A}) \text{ for } i \in N.$$

Obviously, $b_{ii_2\cdots i_m} = a_{ii_2\cdots i_m} - \beta_i(\mathcal{A}) \le 0$ for $i, i_2, \ldots, i_m \in N$ and $\delta_{ii_2\cdots i_m} = 0$. It is easy to get that $\mathcal{B}^+$ is a Z-tensor.

Definition 2.1 ([21]) *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ with $n \ge 2$. $\mathcal{A}$ is called a $QSDD(QSDD_0)$-tensor, if the following two statements hold:*

(1) For any $i, j \in N, i \ne j,\ |a_{i\cdots i}| > (\ge) r_i^j(\mathcal{A})$;

(2) For any $i, j \in N, i \ne j$,

$$\big(a_{i\cdots i} - r_i^j(\mathcal{A})\big)\big(a_{j\cdots j} - r_j^{\bar{\Delta}_i}(\mathcal{A})\big) > (\ge)|a_{ij\cdots j}| r_j^{\Delta_i}(\mathcal{A}),\ \text{ or } |a_{i\cdots i}| > (\ge) r_i(\mathcal{A}).$$

By Definition 2.1 and Proposition 1.4, we can get the following definition.

Definition 2.2 *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in R^{[m,n]}$ with $a_{ii\cdots i} > \beta_i(\mathcal{A})$ for all $i \in N$. $\mathcal{A}$ is called a $QDB(QDB_0)$-tensor if the following two inequalities hold:*

(1) For any $i, j \in N, i \ne j,\ a_{i\cdots i} - \beta_i(\mathcal{A}) > (\ge) \Phi_i^j(\mathcal{A})$;

(2) For any $i, j \in N, i \ne j$,

$$\big(a_{i\cdots i} - \beta_i(\mathcal{A}) - \Phi_i^j(\mathcal{A})\big)\big(a_{j\cdots j} - \beta_j(\mathcal{A}) - \Phi_j^{\bar{\Delta}_i}(\mathcal{A})\big) > (\ge)(\beta_i(\mathcal{A}) - a_{ij\cdots j})\Phi_j^{\Delta_i}(\mathcal{A}),$$

or

$$a_{i\cdots i} - \beta_i(\mathcal{A}) > (\ge) \Phi_i(\mathcal{A})$$

where

$$\Phi_i^j(\mathcal{A}) = \Phi_i(\mathcal{A}) - (\beta_i(\mathcal{A}) - a_{ij\cdots j}) = \sum_{\delta_{ij_2\cdots j_m}=0} (\beta_i(\mathcal{A}) - a_{ij_2\cdots j_m}),$$

$$\Phi_j^{\Delta_i} = \sum_{\substack{(i_2,\ldots,i_m)\in\Delta_i,\\ \delta_j i_2\cdots i_m=0}} (\beta_j - a_{ji_2\cdots i_m}),\quad \Phi_j^{\bar{\Delta}_i}(\mathcal{A}) = \Phi_j(\mathcal{A}) - \Phi_j^{\Delta_i}(\mathcal{A}).$$

Proposition 2.3 *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m}) \in R^{[m,n]}$ with $n \ge 2$. Then $\mathcal{A}$ is a $QDB(QDB_0)$-tensor if and only if $\mathcal{B}^+$ is a $QSDD(QSDD_0)$-tensor, that is, for any $i, j \in N, i \ne j$,*

$$b_{i\cdots i} > (\ge)\ r_i^j(\mathcal{B}^+),$$

and

$$\big(b_{i\cdots i} - r_i^j(\mathcal{B}^+)\big)\big(b_{j\cdots j} - r_j^{\bar{\Delta}_i}(\mathcal{B}^+)\big) > (\ge)|b_{ij\cdots j}| r_j^{\Delta_i}(\mathcal{B}^+),\ \text{ or } b_{i\cdots i} > (\ge) r_i(\mathcal{B}^+).$$

Definition 2.4 ([22]) *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$ with $n \ge 2$, S be a nonempty proper subset of N. $\mathcal{A}$ is called an $S - QSDD(S - QSDD_0)$-tensor if the following four statements hold:*

(1) For each $i \in S,\ j \in \bar{S},\ |a_{i\cdots i}| > (\ge) r_i^j(\mathcal{A})$;

(2) For each $i \in \bar{S},\ j \in S,\ |a_{i\cdots i}| > (\ge) r_i^j(\mathcal{A})$;

(3) For each $i \in S,\ j \in \bar{S}$,

$$(a_{i\cdots i} - r_i^j(\mathcal{A}))(a_{j\cdots j} - r_j^{\Delta^{\bar{S}}}(\mathcal{A})) > (\ge) r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})|a_{ij\cdots j}|, \quad \text{or } |a_{i\cdots i}| > r_i(\mathcal{A});$$

(4) For each $i \in \bar{S},\ j \in S$,

$$(a_{i\cdots i} - r_i^j(\mathcal{A}))(a_{j\cdots j} - r_j^{\Delta^S}(\mathcal{A})) > (\geq) r_j^{\overline{\Delta^S}}(\mathcal{A})|a_{ij\cdots j}|, \quad \text{or } |a_{i\cdots i}| > r_i(\mathcal{A}).$$

By Definition 2.4 and Proposition 1.4, we can get the following definition.

Definition 2.5 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in R^{[m,n]}$ with $a_{ii\cdots i} > \beta_i(\mathcal{A})$ for all $i \in N$, and S be a nonempty proper subset of N. Then $\mathcal{A}$ is called an $SQDB(SQDB_0)$ tensor if for each $i \in S$ and each $j \in \bar{S}$, the following four inequalities hold:*

(1) For each $i \in S,\ j \in \bar{S},\ a_{i\cdots i} - \beta_i(\mathcal{A}) > (\geq)\Phi_i^j(\mathcal{A})$;

(2) For each $i \in \bar{S},\ j \in S,\ a_{i\cdots i} - \beta_i(\mathcal{A}) > (\geq)\Phi_i^j(\mathcal{A})$;

(3) For each $i \in S,\ j \in \bar{S}$,

$$(a_{i\cdots i}\beta_i(\mathcal{A}) - \Phi_i^j(\mathcal{A}))(a_{j\cdots j} - \beta_j(\mathcal{A}) - \Phi_j^{\Delta^{\bar{S}}}(\mathcal{A})) > (\geq)\Phi_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{A})(\beta_i(\mathcal{A}) - a_{ij\cdots j}),$$

$$\text{or } a_{i\cdots i} - \beta_i(\mathcal{A}) > \Phi_i(\mathcal{A}).$$

(4) For each $i \in \bar{S},\ j \in S$,

$$(a_{i\cdots i} - \beta_i(\mathcal{A}) - \Phi_i^j(\mathcal{A}))(a_{j\cdots j} - \beta_j(\mathcal{A}) - \Phi_j^{\Delta^S}(\mathcal{A})) > (\geq)\Phi_j^{\overline{\Delta^S}}(\mathcal{A})(\beta_i(\mathcal{A}) - a_{ij\cdots j}),$$

$$\text{or } a_{i\cdots i} - \beta_i(\mathcal{A}) > \Phi_i(\mathcal{A})$$

where

$$\Phi_i^j(\mathcal{A}) = \Phi_i(\mathcal{A}) - (\beta_i(\mathcal{A}) - a_{ij\cdots j}) = \sum_{\delta_{ij_2\cdots j_m}=0} (\beta_i(\mathcal{A}) - a_{ij_2\cdots j_m}).$$

$$\Phi_j^{\Delta^S}(\mathcal{A}) = \sum_{\substack{(j_2\cdots j_m)\in S,\\ \delta_{jj_2\cdots j_m}=0}} (\beta_j(\mathcal{A}) - a_{jj_2\cdots j_m}), \quad \Phi_j^{\Delta^{\bar{S}}}(\mathcal{A}) = \sum_{(j_2\cdots j_m)\in\bar{S}} (\beta_j(\mathcal{A}) - a_{jj_2\cdots j_m}).$$

By Definition 2.5 we can get the following proposition.

Proposition 2.6 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in R^{[m,n]}$ and S be a nonempty proper subset of N. Then $\mathcal{A}$ is an $SQDB(SQDB_0)$-tensors if and only if $\mathcal{B}^+$ is an $S-QSDD(S-QSDD_0)$-tensors, that is for each*

(1) $i \in S, j \in \bar{S}, b_{i\cdots i} > (\geq) r_i^j(\mathcal{B}^+)$;

(2) $i \in \bar{S}, j \in S, b_{i\cdots i} > (\geq) r_i^j(\mathcal{B}^+)$;

(3) $i \in S, j \in \bar{S}$,

$$(b_{i\cdots i} - r_i^j(\mathcal{B}^+))(b_{j\cdots j} - r_j^{\Delta^{\bar{S}}}(\mathcal{B}^+)) > (\geq) r_j^{\overline{\Delta^{\bar{S}}}}(\mathcal{B}^+)|b_{ij\cdots j}|,$$

or $b_{i\cdots i} > r_i(\mathcal{B}^+)$;

(4) $i \in \bar{S}, j \in S$,

$$(b_{i\cdots i} - r_i^j(\mathcal{B}^+))(b_{j\cdots j} - r_j^{\Delta^S}(\mathcal{B}^+)) > (\geq) r_j^{\overline{\Delta^S}}(\mathcal{B}^+)|b_{ij\cdots j}|,$$

or $b_{i\cdots i} > r_i(\mathcal{B}^+)$.

3. Positive definiteness

Now, we discuss the positive (semi-)definiteness of $QDB(QDB_0)$-tensors and $SQDB(SQDB_0)$ -tensors. In [21,22], Jiao et al. gave sufficient conditions for positive (semi-)definiteness of $QSDD$ $(QSDD_0)$-tensors and S-$QSDD(S$-$QSDD_0)$-tensors.

Lemma 3.1 ([21, Theorem 5]) *An even order $QSDD$ symmetric tensor with all positive diagonal entries is positive definite. And an even order $QSDD_0$ symmetric tensor with all nonnegative diagonal entries is positive semi-definite.*

Lemma 3.2 ([22, Theorem 6]) *Let $\mathcal{A} = (a_{i_1\cdots i_m}) \in R^{[m,n]}$, with $n \geq 2$ and S be a nonempty proper subset of N. If $\mathcal{A}$ is an even order $S - QSDD(S - QSDD_0)$ symmetric tensor with $a_{k\cdots k} > (\geq)\ 0$ for all $k \in N$, then $\mathcal{A}$ is positive (semi-)definite.*

Now according to Lemmas 3.1 and 3.2, we study the positive (semi-)definiteness of symmetric $QDB(QDB_0)$-tensors and symmetric $SQDB(SQDB_0)$-tensors. Before that we give the definition of partially all one tensors, proposed by Qi and Song [5]. Suppose that $\mathcal{A}$ is a symmetric tensor of order m dimension n, and has a principal sub-tensor $\mathcal{A}_r^J$ with $J \in N$ and $|J| = r$ $(1 \leq r \leq n)$ such that all the entries of $\mathcal{A}_r^J$ are one, and all the other entries of $\mathcal{A}$ are zero, then $\mathcal{A}$ is called a partially all one tensor, and denoted by ε^J. If $J = N$, then we denote ε^J simply by ε and call it an all one tensor. And an even order partially all one tensor is positive semi-definite, see [5] for details.

Theorem 3.3 *Let $\mathcal{A} = (a_{i_1 i_2\cdots i_m})$ be a real symmetric QDB-tensor of order m dimension n. Then either $\mathcal{A}$ is a $QSDD$ symmetric Z-tensor itself, or*

$$\mathcal{A} = \mathcal{M} + \sum_{k=1}^{s} h_k \varepsilon^{\hat{J}_k}, \tag{3.1}$$

where $\mathcal{M}$ is a $QSDD$ symmetric Z-tensor, s is a positive integer, $h_k > 0$ and $\hat{J}_k \subseteq N$, for $k = 1, 2, \ldots, s$. Furthermore, if m is even, then $\mathcal{A}$ is positive definite, consequently, $\mathcal{A}$ is a P-tensor.

Proof Let $\hat{J}(\mathcal{A}) = \{i \in N :$ there is at least one positive off-diagonal entry in the ith row of $\mathcal{A}\}$. Obviously, $\hat{J}(\mathcal{A}) \subseteq N$. If $\hat{J}(\mathcal{A}) = \phi$, then $\mathcal{A}$ is a Z-tensor. The conclusion follows in the case.

Now we suppose that $\hat{J}(\mathcal{A}) \neq \phi$, let $\mathcal{A}_1 = \mathcal{A} = (a^{(1)}_{i_1\cdots i_m})$, and let $d_i^{(1)}$ be the value of the largest off-diagonal entry in the ith row of $\mathcal{A}_1$, that is,

$$d_i^{(1)} = \max_{\substack{i_2\cdots i_m \in N,\\ \delta_{ii_2\cdots i_m}=0}} a^{(1)}_{ii_2\cdots i_m}.$$

Furthermore, let $\hat{J}_1 = \hat{J}(\mathcal{A}_1)$, $h_1 = \min_{i\in \hat{J}_1} d_i^{(1)}$ and

$$J_1 = \{i \in \hat{J}_1 : d_i^{(1)} = h_1\}.$$

Then $J_1 \subseteq \hat{J}_1$ and $h_1 > 0$.

Consider $\mathcal{A}_2 = \mathcal{A}_1 - h_1\varepsilon^{\hat{J}_1} = (a^{(2)}_{i_1\cdots i_m})$. Obviously, $\mathcal{A}_2$ is also symmetric by the definition of $\varepsilon^{\hat{J}_1}$. Note that

$$a^{(2)}_{i_1\cdots i_m} = \begin{cases} a^{(1)}_{i_1\cdots i_m} - h_1, & i_1, i_2, \ldots, i_m \in \hat{J}_1; \\ a^{(1)}_{i_1\cdots i_m}, & \text{otherwise}, \end{cases} \tag{3.2}$$

for $i \in J_1$,

$$\beta_i(\mathcal{A}_2) = \beta_i(\mathcal{A}_1) - h_1 = 0, \tag{3.3}$$

and that for $i \in \hat{J}_1 \backslash J_1$,

$$\beta_i(\mathcal{A}_2) = \beta_i(\mathcal{A}_1) - h_1 > 0. \tag{3.4}$$

Combining (3.2)–(3.4) with the fact that for each $j \notin \hat{J}_1$, $\beta_i(\mathcal{A}_2) = \beta_i(\mathcal{A}_1)$, we easily obtain by Definition 2.2 that $\mathcal{A}_2$ is still a symmetric QDB-tensor.

Now replace $\mathcal{A}_1$ by $\mathcal{A}_2$, and repeat this process. Let $\hat{J}(\mathcal{A}_2) = \{i \in N$: there is at least one positive off-diagonal entry in the ith row of $\mathcal{A}_2\}$. Then $\hat{J}(\mathcal{A}_2) = \hat{J}_1 \backslash J_1$. Repeat this process until $\hat{J}(\mathcal{A}_{s+1}) = \phi$. Let $\mathcal{M} = \mathcal{A}_{s+1}$. Then (3.1) holds.

Furthermore, if m is even, then $\mathcal{A}$ is a symmetric QDB-tensor of even order. If $\mathcal{A}$ itself is a $QSDD$ symmetric Z-tensor, then it is positive definite by Lemma 3.1. Otherwise, (3.1) holds with $s > 0$. Let $x \in R^n$. Then by (3.1) and the fact that $\mathcal{M}$ is positive definite, we have

$$\mathcal{A}x^m = \mathcal{M}x^m + \sum_{k=1}^{s} h_k \varepsilon^{\hat{J}_k} x^m = \mathcal{M}x^m + \sum_{k=1}^{s} h_k ||x_{\hat{J}_k}||_m^m \geq \mathcal{M}x^m > 0.$$

This implies that $\mathcal{A}$ is positive definite. Note that a symmetric tensor is a P-tensor if and only if it is positive definite [4], therefore $\mathcal{A}$ is a P-tensor. $\Box$.

Theorem 3.4 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a real symmetric QDB_0 tensor of order m dimension n. Then either $\mathcal{A}$ is a $QSDD_0$ symmetric Z-tensor itself, or*

$$\mathcal{A} = \mathcal{M} + \sum_{k=1}^{s} h_k \varepsilon^{\hat{J}_k},$$

where $\mathcal{M}$ is a $QSDD_0$ symmetric Z-tensor, s is a positive integer, $h_k > 0$ and $\hat{J}_k \subseteq N$, for $k = 1, 2, \ldots, s$. Furthermore, if m is even, then $\mathcal{A}$ is positive semi-definite, consequently, $\mathcal{A}$ is a P_0-tensor.

Similar to the proof of Theorem 3.3, by Lemma 3.2 we easily get the following results.

Theorem 3.5 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a real symmetric $SQDB$-tensor of order m dimension n. Then either $\mathcal{A}$ is an $S - QSDD$ symmetric Z-tensor itself, or*

$$\mathcal{A} = \mathcal{M} + \sum_{k=1}^{s} h_k \varepsilon^{\hat{J}_k},$$

where $\mathcal{M}$ is an $S - QSDD$ symmetric Z-tensor, s is a positive integer, $h_k > 0$ and $\hat{J}_k \subseteq N$, for $k = 1, 2, \ldots, s$. Furthermore, if m is even, then $\mathcal{A}$ is positive definite, consequently, $\mathcal{A}$ is a P-tensor.

Theorem 3.6 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be a real symmetric $SQDB_0$-tensor of order m dimension n. Then either $\mathcal{A}$ is an $S - QSDD_0$ symmetric Z-tensor itself, or*

$$\mathcal{A} = \mathcal{M} + \sum_{k=1}^{s} h_k \varepsilon^{\hat{J}_k},$$

where $\mathcal{M}$ is an $S-QSDD_0$ symmetric Z-tensor, s is a positive integer, $h_k > 0$ and $\hat{J}_k \subseteq N$, for $k = 1, 2, \ldots, s$. Furthermore, if m is even, then $\mathcal{A}$ is positive semi-definite, consequently, $\mathcal{A}$ is a P_0-tensor.

Since an even order real symmetric tensor is positive (semi-)definite if and only if all of its H-eigenvalues are positive (non-negative) [1], by Theorems 3.3–3.6 we have the following results.

Corollary 3.7 *(1) All the H-eigenvalues of an even order symmetric QDB-tensor are positive.*

(2) All the H-eigenvalues of an even order symmetric QDB_0-tensor are nonnegative.

(3) All the H-eigenvalues of an even order symmetric $SQDB$-tensor are positive.

(4) All the H-eigenvalues of an even order symmetric $SQDB_0$-tensor are nonnegative.

4. Relationships between QDB-tensors and $SQDB$-tensors

In this section, we discuss the relationships between B-tensors, QDB-tensors and $SQDB$-tensors.

Proposition 4.1 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in R^{[m,n]}, n \geq 2$ and S be a nonempty proper subset of N. If $\mathcal{A}$ is a B-tensor, then $\mathcal{A}$ is a QDB-tensor. If $\mathcal{A}$ is a QDB-tensor, then $\mathcal{A}$ is an $SQDB$-tensor.*

Proof First, let $\mathcal{A}$ be a B-tensor. It is easy to see from Proposition 1.4 that for any $i \in N$,

$$a_{ii\cdots i} - \beta_i(\mathcal{A}) > \Phi_i(\mathcal{A}),$$

then

$$a_{ii\cdots i} - \beta_i(\mathcal{A}) > \Phi_i(\mathcal{A}) \geq \Phi_i^j(\mathcal{A}).$$

So, by Definition 2.2, $\mathcal{A}$ is QDB-tensors.

Secondly, let $\mathcal{A}$ be a QDB-tensor. It is easy to see from Proposition 2.6 that for any $i, j \in N, i \neq j$,

(1) $b_{i\cdots i} > r_i^j(\mathcal{B}^+)$, if $i \in S, j \in \bar{S}$, then (3.1) holds; if $i \in \bar{S}, j \in S$, then (3.2) holds.

(2) $(b_{i\cdots i} - r_i^j(\mathcal{B}^+))(b_{j\cdots j} - r_j^{\bar{\Delta}_i}(\mathcal{B}^+)) > (\geq)|b_{ij\cdots j}| r_j^{\Delta_i}(\mathcal{B}^+)$, or $b_{i\cdots i} > r_i(\mathcal{B}^+)$. If $i \in S, j \in \bar{S}$, then (3.3) holds; if $i \in \bar{S}, j \in S$, then (3.4) holds. So, by Proposition 4.1, $\mathcal{A}$ is $SQDB$-tensors. □

By Proposition 4.1, we easily obtain the following proposition.

Proposition 4.2 *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m}) \in R^{[m,n]}, n \geq 2$ and S be a nonempty proper subset of N. If $\mathcal{A}$ is a B_0-tensor, then $\mathcal{A}$ is a QDB_0-tensor. If $\mathcal{A}$ is a QDB_0-tensor, then $\mathcal{A}$ is an $SQDB_0$-tensor.*

By Propositions 4.1 and 4.2, we can get the following result.

Corollary 4.3 *The relationships of $B(B_0)$-tensors, $QDB(QDB_0)$-tensors and $SQDB(SQDB_0)$-tensors are as follows:*

(1) $\{B$-tensors$\} \subset \{QDB$-tensors$\} \subset \{SQDB$-tensors$\}$.

(2) $\{B_0$-tensors$\} \subset \{QDB_0$-tensors$\} \subset \{SQDB_0$-tensors$\}$.

5. Conclusions

In this paper, we define four classes of structured tensors: $QDB(QDB_0)$-tensors and $SQDB$ ($SQDB_0$)-tensors. We prove that even order symmetric QDB-tensors and $SQDB$-tensors are positive definite and they are subclasses of P-tensors, even order symmetric QDB_0-tensors and $SQDB_0$-tensors are positive semi-definite and they are subclasses of P_0-tensors.

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