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The Application of Differential Characteristic Set Method to Pseudo Differential Operator and Lax Representation

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Abstract Differential characteristic set method is applied to the calculation of pseudo differential operators and Lax representation of nonlinear evolution equations. Firstly, differential characteristic set method and differential division with remainder are used for the calculation of inverse and extraction root of pseudo differential operator, such that the process is simplified since it is unnecessary to solve ordinary differential equation systems and substitute the solutions. Secondly, using differential characteristic set method, the nonlinear partial differential equation systems derived from the generalized Lax equation and Zakharov-Shabat equation, are reduced, and the corresponding nonlinear evolution equation is obtained. The related programs are compiled in Mathematica, a computer-based computer algebra system, and Lax representation of some nonlinear evolution equations can be calculated with the aid of the computer.

Keywords differential characteristic set; differential division with remainder; pseudo differential operator; Lax representation; Zakharov-Shabat equation

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1. Introduction

In integrable systems, constructing Lax representations of nonlinear evolution equations, i.e., finding a pair of differential operators L and B, is a key problem, and L and B are called Lax Pair. However, there is no general method to construct Lax representations, so it is significant and interesting to find the Lax representations of nonlinear evolution equations. Using pseudo differential operators (abbr. PDO), we can get a series of Lax representations in a simple way [1].

Originally, PDOs were only used in integrable systems in $1+1$ dimensions, and the nonlinear evolution equation is obtained from the generalized Lax equation. According to Sato theory, the approach is extended to higher dimensions, and through the "Dressing operator", Zakharov-Shabat equation is introduced. Further, PDOs are widely applied to τ-function [2], Darboux transformation [3], Hamilton structure [1] and so on, and developed into a standard technique in the theory of integrable systems.

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To construct Lax representations via PDOs, there are many differential polynomials to be solved. For a PDO L, such as the calculation of the inverse L^{-1} or the extraction of the nth root $L^{\frac{1}{n}}$ of L, it is necessary to solve the ordinary differential equation systems firstly, and then L^{-1} and $L^{\frac{1}{n}}$ are obtained by substituting the solutions. From L^{-1} and $L^{\frac{1}{n}}$, the differential operator B is derived. Although the calculation formula of $L^{-1}, L^{\frac{1}{n}}$ and B are straightforward, the calculation of all differential polynomials still requires a lot of work. Nonlinear evolution equation is obtained from the generalized Lax equation or Zakharov-Shabat equation, but it is not direct. From the generalized Lax equation or Zakharov-Shabat equation, a nonlinear partial differential equation system is derived. Via observation, we eliminate the auxiliary fields from the nonlinear partial differential equation system by using compatibility conditions, and obtain the nonlinear evolution equation. Thereby, it is complicated to calculate the construction of Lax representations of nonlinear evolution equation with PDOs.

The main aim of this paper is to apply differential characteristic set method [4–7] to the construction of Lax representations of nonlinear evolution equation with PDOs. For L^{-1} and $L^{\frac{1}{n}}$, using differential characteristic set method, the triangular form differential characteristic set of corresponding ordinary differential equation system is obtained, and then, $L^{-1}, L^{\frac{1}{n}}$ and B_m are calculated by differential division with remainder directly, and it is unnecessary to solve the ordinary differential equations and substitute the solutions into the formula of $L^{-1}, L^{\frac{1}{n}}$ and B_m. Thereby, the calculations are simplified. On the other hand, the nonlinear partial differential equation system generated from the generalized Lax equation or Zakharov-Shabat equation is reduced by differential characteristic set method, and for the given suitable order, the first equation of the differential characteristic set is just the desired nonlinear evolution equation.

In the process of calculation, the infinite series are implemented in truncated form, and the truncation order is determined by B or Res.

The programs of differential characteristic set method, PDOs and Lax representations calculation are all provided in MATHEMATICA language. Some Lax representation of nonlinear evolution equations are calculated using the programs.

2. Differential characteristic set method

The differential characteristic set method is based on the order, so the order of differential polynomials should be given first.

2.1. The order of differential polynomials

Let $\mathbb{F}$ be a field of characteristic zero. Then $\mathbb{F}$ is endowed n operators $D_1, \dots, D_n$, and the following conditions are satisfied:

(1) $D_i(f+g) = D_i(f) + D_i(g)$;

(2) $D_i(fg) = gD_i(f) + fD_i(g)$;

(3) $D_i(D_j(f)) = D_j(D_i(f))$,

where $f, g \in \mathbb{F}$. Then $\mathbb{F}$ is called an n-element differential field, and $D_1, \dots, D_n$ are basic differential operators. c is a constant in $\mathbb{F}$ if $D_i(c) = 0$, $i = 1, \ldots, n$.

An arbitrary differential operator D in $\mathbb{F}$ is an element of $\mathbb{F}$ of the form

$$D = D_1^{\alpha_1} \cdots D_n^{\alpha_n}, \tag{2.1}$$

where α_i, $i = 1, \ldots, n$ are nonnegative integers, and $D_i^{\alpha_i}$ means that D_i implements α_i times. From (2.1), the mapping

$$D = D_1^{\alpha_1} \cdots D_n^{\alpha_n} \to (\alpha_1, \ldots, \alpha_n) = \alpha. \tag{2.2}$$

is $1-1$. $\alpha \in \mathbb{N}^n$ is called multiple index (abbr. multi-index), meanwhile, $D_1^{\alpha_1} \cdots D_n^{\alpha_n}$ is denoted by D_α, and $|\alpha| = \sum_{i=1}^n \alpha_i$ is the order of D_α. Any finite set of multi-indices is a subset of $\mathbb{N}^n$, and denoted by N^n. In this section, the multi-indices are all n-dimensional. For $\forall \alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n) \in \mathbb{N}^n$, we have $\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n)$, and

$$D_\alpha D_\beta = D_\beta D_\alpha = D_{\alpha+\beta} = D_1^{\alpha_1+\beta_1} \cdots D_n^{\alpha_n+\beta_n}.$$

For a given n-element differential field $\mathbb{F}$, $u^1, \ldots, u^m$ are independent indeterminate elements, and can be treated as elements of a certain extended field of $\mathbb{F}$. We take the notation

$$Du = \{D_\sigma u^p | \sigma \in N^n,\ p = 1, 2, \ldots, m\},$$

where p is called the class of $D_\sigma u^p$. $D_\sigma u^p$ is called formal derivative, and denoted by $u_{(p,\sigma)}$ sometimes. $u_{(p,\sigma)}$ is a proper derivative as $|\sigma| > 0$, and $u_{(p,\sigma)}$ is u^p as σ is the zero vector.

Definition 2.1 (diff-graded reverse lex order) *$D_\alpha u^p$ is larger than $D_\beta u^q$, and denoted by $D_\alpha u^p > D_\beta u^q$, if*

(1) $p > q$; or

(2) $p = q$, and $|\alpha| < |\beta|$; or

(3) $p = q$, and $|\alpha| = |\beta|$, there exists i, s.t., $\alpha_i < \beta_i$, $\alpha_{i+1} < \beta_{i+1}, \ldots, \alpha_n < \beta_n$ $(1 \le i \le n)$.

The differential monomial on $\mathbb{F}$ is the finite multiplication of elements in Du,

$$c \prod u_{(p,\sigma)}^{k_{(p,\sigma)}},$$

where $c \in \mathbb{F}$, and $k_{(p,\sigma)}$ is a positive integer. Differential polynomial (abbr. d-pol) is a finite sum of differential monomials. f is a d-pol, according to the order in Definition 2.1, the largest derivative occurring in f is called leading derivative, and denoted by $\mathrm{ld}(f)$, and f can be written as

$$f = f_d u_{(p,\sigma)}^d + f_{d-1} u_{(p,\sigma)}^{d-1} + \cdots + f_0,$$

where $u_{(p,\sigma)}$ is the leading derivative of f, and $f_d \ne 0$. p is called the class of f, and denoted by $\mathrm{cl}(f)$; d is called the degree of f, and denoted by $\deg(f)$; f_d is called the initial of f, and denoted by $\mathrm{Ini}(f)$.

$$\frac{\partial f}{\partial u_{(p,\sigma)}} = d f_d u_{(p,\sigma)}^{d-1} + (d-1) f_{d-1} u_{(p,\sigma)}^{d-2} + \cdots$$

is called the separant of f and denoted by $\mathrm{Sep}(f)$. If $\gamma \in \mathbb{N}^n$ is a nonzero vector,

$$D_\gamma(f) = \frac{\partial f}{\partial u_{(p,\sigma)}} u_{(p,\gamma+\sigma)} + \cdots. \tag{2.3}$$

Then $\mathrm{ld}(D_\gamma(f)) = u_{(p,\gamma+\sigma)}$, and $\deg(D_\gamma(f)) = 1$. $\mathrm{Sep}(f)$ is the initial of $D_\gamma(f)$ essentially. $u_{(p,\gamma+\sigma)}$ is the proper derivative of $\mathrm{ld}(f)$, and is called the principal derivative of f.

(p, σ, d) is called the rank of f. f is called non-trivial if formal derivative occurs in f, or is trivial when the rank is $(0, \sigma^{(0)}, 0)$, where $\sigma^{(0)}$ is the zero vector. In addition, if q is not the class of f, (σ, q) is called the rank of $u_{(\sigma,d)}$ in f, where σ is the largest multi-index about u^q, and d is the highest power of $D_\sigma u^p$ in f.

All d-pols make up a ring that is closed to derivative, which is called the differential polynomial ring, and denoted by $\mathbb{F}\{Du\}$. The order is introduced in $\mathbb{F}\{Du\}$ as follows.

Definition 2.2 *Let us assume that f and g are two non-trivial differential polynomials (abbr. d-pols), and (p, α, d_f), (q, β, d_g) are ranks, respectively. Then f is larger than g, denoted by $f > g$, if*

(1) $p > q$; or

(2) $p = q$, and $\alpha > \beta$; or

(3) $p = q$, and $\alpha = \beta$, and $d_f > d_g$.

If $(p_f, \sigma_f, d_f) = (p_g, \sigma_g, d_g)$, then f and g are incomparable, denoted by $f \sim g$.

Lemma 2.3 *The order $>$ is a well order on $\mathbb{F}\{Du\}$. This means that any subset of $\mathbb{F}\{Du\}$ has smallest elements under $>$.*

From the order for $\mathbb{F}\{Du\}$, a sequence of d-pols is called an ascent (a decreasing) sequence if it does not decrease (ascend) according to ranks.

Lemma 2.4 *Every strictly decreasing sequence of d-pols is finite.*

Definition 2.5 *Let f and g be d-plos, and g be non-trivial. f is said to be reduced w.r.t. g if*

(1) no proper derivative of $\mathrm{ld}(g)$ occurs in f; and

(2) the degree of $\mathrm{ld}(g)$ in f is lower than $\deg(g)$.

Let *dps* be a nonempty differential polynomial set (system) (abbr. d-pol-set), and f be a d-pol. f is said to be reduced w.r.t. *dps*, if f is reduced w.r.t. every d-pol in *dps*. d-pol-set *dqs* is reduced w.r.t. *dps*, if every d-pol in *dqs* is reduced w.r.t. *dps*.

Definition 2.6 *The ascent sequence of d-pols*

$$f_1 < \cdots < f_k$$

is differential ascent set (abbr. d-asc-set) if either

(1) $k = 1$, and f_1 is trivial, and is then said to be trivial; or

(2) $k > 1$, $f_1, \ldots, f_k$ are all non-trivial, and for $j < i$, f_i is reduced w.r.t. all f_j.

Obviously, the number of any d-asc-set of d-pol-set is finite.

Definition 2.7 *Assuming that there are two d-asc-sets*

$$G: \ g_1 < \cdots < g_k, \ \text{and } H: \ h_1 < \cdots < h_l,$$

G is higher than H, and denoted by $G > H$ if

(1) H is trivial, and G is non-trivial; or

(2) G and H are both non-trivial, and there exists i, s.t., $g_1 \sim h_1, \dots, g_{i-1} \sim h_{i-1}$, and $g_i > h_i$; or

(3) $l < k$, and $g_1 \sim h_1, \dots, g_l \sim h_l$.

G and H are two d-asc-sets. G is equivalent H, denoted by $G \sim H$, if G and H are both trivial, or $k = l$ and $g_i \sim h_i$.

Lemma 2.8 *Any strictly decreasing sequence of d-asc-set*

$$d\text{-asc-set}_1 > d\text{-asc-set}_2 > \cdots$$

is finite.

Let *dps* be a set of d-pol-set. All of d-asc-sets contained in *dps* form a set, and from Lemma 2.8, the set has the lowest elements. The lowest d-asc-sets of *dps* are called differential basic ascent set (abbr. d-bas-asc-set) of *dps*, and any two d-bas-asc-sets of *dps* are equivalent.

Theorem 2.9 *The differential ascent set G is the d-bas-asc-set of d-pol-set dps, iff $dps \setminus G$ does not contain nonzero d-pol which reduced w.r.t G.*

Theorem 2.9 also provides an algorithm to construct the d-bas-asc-set of d-pol-set. Let *dps* be a d-pol-set. If *dps* contains trivial nonzero d-pol f, the f is d-bas-asc-set, or a d-pol is taken from the lowest polynomials of *dps*, and denoted by g_1. If $dps\{g_1\}$ contains the d-pols reduced w.r.t. g_1, one of the lowest ranks can be taken from $dps\{g_1\}$, denoted by g_2. Obviously, $g_1 < g_2$ is a d-asc-set. If $dps \setminus \{g_1, g_2\}$ contains the d-pols reduced w.r.t. g_1, g_2, one of the lowest rank is taken from them, denoted by g_3. Then, $g_1 < g_2 < g_3$ is a d-asc-set. Continuing, the d-asc-set

$$G: \ g_1 < g_2 < \cdots < g_k$$

is obtained, and $dps \setminus G$ does not contain the d-pol reduced w.r.t. G. So G is a d-bas-asc-set of *dps*.

Definition 2.10 *Let dps and dqs be two differential polynomial sets. dps is larger than dqs, denoted by $dps > dqs$, if the d-bas-asc-set of dps is larger than d-bas-asc-set of dqs.*

From Lemma 2.8, it is easy to prove the following lemma.

Lemma 2.11 *Any strictly decreasing sequence of d-pol-sets in order*

$$dps_1 > dps_2 > \cdots$$

is finite.

Definition 2.7 also provides a way for lowering the order of a d-pol-set:

Theorem 2.12 *Let G be a d-bas-asc-set of d-pol-set dps, and d-pol-set dqs be reduced w.r.t. G. Then $dps \cup dqs < dps$.*

2.2. Differential division with remainder

Differential division with remainder (also called differential pseudo division with remainder) is the foundation of reduction of d-pol-set, including ordinary differential equation system (abbr. ODEs), and partial differential equation system (abbr. PDEs).

Theorem 2.13 *Let f and g be two d-pols, g be non-trivial. Then the following formula can be obtained in finite steps*

$$Jf = \sum_{\alpha} h_\alpha D_\alpha g + r, \tag{2.4}$$

where h_α and r are all d-pols, and J is the power product of $\mathrm{Ini}(g)$ *and* $\mathrm{Sep}(g)$. *r is zero polynomial or reduced w.r.t. g. r is called the differential remainder (abbr. d-remainder) of f w.r.t. g, denoted by* $\mathrm{Red}(f, g)$, *and (2.4) is called d-remainder formula.*

Proof Let $\mathrm{ld}(g) = D_\sigma u^p$, and $I = \mathrm{Ini}(g)$, $S = \mathrm{Sep}(g)$, and

$$M_\sigma^p(f) = \{\beta | D_\beta u^p \text{ occurs in } f\},$$

where $D_\beta u^p$ is the principal derivative of g, i.e., the proper derivative of $D_\sigma u^p$. It is discussed in three cases:

(1) If $M_\sigma^p(f) = \emptyset$, and $\deg(g)$ is higher than the degree of $D_\sigma u^p$ in f, f is reduced w.r.t. g;

(2) If $M_\sigma^p(f) \neq \emptyset$, we might as well let γ be the largest multi-index, and $\gamma - \sigma \in \mathbb{N}^n$. Thus $D_\gamma u^p$ is the leading derivative of $D_{\gamma-\sigma}(g)$, and S is the initial of $D_{\gamma-\sigma}(g)$, i.e., separant of g. f can be treated as a polynomial in indeterminate $D_\gamma u^p$, according to Euclidean algorithm, in finite steps, the following reduction can be reached

$$S^r \cdot f = h_1 D_{\gamma-\sigma}(g) + f_1. \tag{2.5}$$

From (2.3), $D_\gamma u^p$ does not occur in f_1, and the rank of u^p in f_1 is lower than in f. If $M_\sigma^p(f_1) \neq \emptyset$, repeating the above process to f_1. Continuing the process, until $M_\sigma^p(f_k) = \emptyset$, and one can get

$$S^c \cdot f = h_k D_{\gamma-\sigma}(g) + f_k. \tag{2.6}$$

(3) If $\deg(g)$ is lower than the degree of $D_\sigma u^p$ in f_k, f_k can be likewise treated as a polynomial in indeterminate $D_\sigma u^p$. Using Euclidean algorithm again, the following formula

$$I^d \cdot f_k = hg + r \tag{2.7}$$

can be derived, and the degree of $D_\sigma u^p$ in r is lower than $\deg(g)$.

From (2.6) and (2.7), formula (2.4) can be derived, meanwhile, the algorithm is given out. □

Let $G : g_1 < \cdots < g_{k-1} < g_k$ be a d-asc-set. Successively using d-remainder formula, then

$$r_k = \mathrm{Red}(f, g_k), r_{k-1} = \mathrm{Red}(r_k, g_{k-1}), \ldots, r_1 = \mathrm{Red}(r_2, g_1),$$

and

$$J_1 \cdots J_k f = \sum_{i,\alpha} h_\alpha^i D_\alpha(g_i) + r, \tag{2.8}$$

where J_i are the power product of $\mathrm{Ini}(g_i)$ and $\mathrm{Sep}(g_i)$, and r is zero polynomial or reduced w.r.t. G. r is called the d-remainder of f w.r.t. G, denoted by $\mathrm{Red}(f, G)$.

Let $H = \{h_1, \ldots, h_l\}$ be a d-pol-set, and $G : g_1 < \cdots < g_k$ be a d-asc-set. The set $\{\mathrm{Red}(h_1, G), \ldots, \mathrm{Red}(h_k, G)\} \backslash \{0\}$ is called the d-remainder of H w.r.t. G, denoted by $\mathrm{Red}(H, G)$.

Definition 2.14 *Let G be the d-bas-asc-set of d-pol-set dps. G is said to be the d-bas-set of dps if* $\mathrm{Red}(dps, G) = \emptyset$.

2.3. Integrability conditions

Integrability conditions include minimal integrability conditions and supplementary integrability conditions. Minimal integrability conditions are the relations among d-pol-set, and can be found out directly. Supplementary integrability conditions are the potential relations among d-pol-set, and can be obtained through the completion of the d-pol-set.

Minimal integrability conditions are the derivation operation must be compatible, i.e., the property (3) of basic derivation operators: the order of derivation operations can be interchanged. For example, $F_x = f$, $F_y = g$, then

$$F_{xy} = F_{yx} \Rightarrow f_y = g_x.$$

The equation $f_y = g_x$ is the integrable condition, and also called compatible condition. Let f and g be two same class d-pols, and the leading derivatives be $D_\alpha u^p$ and $D_\beta u^p$, respectively, i.e.,

$$f = f_d D_\alpha u^p + f_1, \quad g = g_d D_\beta u^p + g_1.$$

Then

$$S(f,g) = \frac{g_d D_{\sigma-\alpha} f - f_d D_{\sigma-\beta} g}{GCD(f_d, g_d)} \tag{2.9}$$

is the compatible condition of f and g, where $\mathrm{GCD}(f_d, g_d)$ is the greatest common divisor of f_d and g_d, and $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i = \max\{\alpha_i, \beta_i\}$. (2.9) is called S-polynomial of f and g.

Let G be a d-asc-set and all be S-polynomials of G, denoted by $S_{\min}(G)$. Then $\mathrm{Red}(S_{\min}(G), G)$ is the minimal integrability conditions of G, denoted by $I_{\min}(G)$.

Supplementary integrability conditions come from the completion of d-pol set, which exercises through the completion of the corresponding multi-indices set of leading derivatives. Let Γ be a set of multi-indices and $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$ be two multi-indices. The following notations are used:

- $\gamma = (\gamma_1, \ldots, \gamma_n)$ is the bounded element of Γ, where $\gamma_i = \max\{\alpha_i | \alpha \in \Gamma\}$.
- $\mathbf{b}$ is an operator on Γ as $\mathbf{b}(\Gamma) = \gamma$, and the i-th component of $\mathbf{b}(\Gamma)$ is $\mathbf{b}_i(\Gamma) = \gamma_i$.
- $\delta^{(i)}$ is the vector whose i-th component is 1 and the other components are all zeros.
- The Hadamard's product of α and β is

$$\alpha \circ \beta = (\alpha_1\beta_1, \ldots, \alpha_n\beta_n) \tag{2.10}$$

- $\alpha + \beta$ is called prolongation (multiple) of β. All of prolongations of α are denoted by

$$M(\alpha) = \{\alpha + \beta | \beta \in \mathbb{N}^n\} \tag{2.11}$$

The prolongation along δ direction is

$$M(\alpha, \delta) = \{\alpha + \beta \circ \delta | \beta \in \mathbb{N}^n\}$$

- $M(\Gamma) = \bigcup_{\alpha \in \Gamma} M(\alpha)$ is called the prolongation (multiple) space of Γ.

In the differential characteristic set method, completion is indispensable, and there are several completion methods, but it is very difficult to find out the minimal complete set of multi-indices set. A less complete set can be obtained through the following method.

Definition 2.15 *For* $\forall \alpha \in \Gamma$, $\delta(\alpha)$ *is the multiplicative direction of* α *defined as*

$$\delta_n(\alpha) = \begin{cases} 1, & \alpha_n = \mathbf{b}_n(\Gamma) \\ 0, & otherwise \end{cases} \qquad \delta_k(\alpha) = \begin{cases} 1, & \alpha_k = \mathbf{b}_k(\Gamma_{k\gamma_{k+1}\cdots\gamma_n}) \\ 0, & otherwise \end{cases} \tag{2.12}$$

where $\Gamma_{k\gamma_{k+1}\cdots\gamma_n} = \{\beta \in \Gamma | \beta_k = k+1, \ldots, \beta_n = n\}$. δ *is the multiplicative direction of* Γ.

Lemma 2.16 *For* $\forall \alpha, \beta \in \Gamma$, $M(\alpha, \delta(\alpha)) \cap M(\beta, \delta(\beta)) = \emptyset$.

Definition 2.17 Γ *is called the completion set if* $M(\Gamma) = \bigcup_{\alpha\in\Gamma} M(\alpha, \delta(\alpha))$.

Theorem 2.18 Γ *is completion set iff*

$$\forall \alpha \in \Gamma, \delta_k(\alpha) = 0 \Rightarrow \alpha + \delta^{(k)} \in \bigcup_{\alpha\in\Gamma} M(\alpha, \delta(\alpha)) \tag{2.13}$$

Definition 2.19 Γ' *is complete, and called the completion set of* Γ, *if* $\Gamma' \supseteq \Gamma$ *and* $M(\Gamma') \supseteq M(\Gamma)$.

For a finite multi-indices set Γ, the completion set of Γ is not unique, and the intersection of two completion sets may not be completion set. In order to make up for the detective, a special completion set, closed set will be introduced.

Definition 2.20 *Finite multi-indices set* Γ *is called closed set if*

$$\alpha \in \Gamma, \delta_k(\alpha) = 0 \Rightarrow \alpha + \delta^{(k)} \in \Gamma,$$

where δ *is the multiplicative direction of* Γ.

From Theorem 2.18, closed set must be complete set.

Theorem 2.21 *The intersection of any two closed sets is also closed set.*

Definition 2.22 *The minimal closed set containing* Γ *is called the closure of* Γ, *denoted by* $\overline{\Gamma}$.

Obviously, the closure of the Γ exists and is unique, and is the intersection of all closed sets containing Γ. The closure of Γ can be obtained from the following algorithm.

Completion Algorithm:

(1) Let $b_n = \mathbf{b}_n(\Gamma)$, and step by step, we determine the following sets:

$$\begin{aligned} &\overline{\Gamma}_0 = \{\alpha \in \Gamma | \alpha_n = 0\} \\ &\overline{\Gamma}_1 = \{\alpha \in \Gamma | \alpha_n = 1\} \cup \{\alpha + \delta^{(n)} | \alpha \in \overline{\Gamma}_0\} \\ &\cdots \\ &\overline{\Gamma}_n = \{\alpha \in \Gamma | \alpha_n = b_n\} \cup \{\alpha + \delta^{(n)} | \alpha \in \overline{\Gamma}_{b_n-1}\} \end{aligned}$$

(2) Assuming that $\overline{\Gamma}_{j_{k+1}\cdots j_n}$ are all determined, and $b_{kj_{k+1}\cdots j_n} = \mathbf{b}_{j_{k+1}\cdots j_n}(\overline{\Gamma}_{j_{k+1}\cdots j_n})$, and

step by step, the following sets are determined

$$\begin{aligned}
\overline{\Gamma}_{0j_{k+1}\cdots j_n} &= \{\alpha \in \overline{\Gamma}_{j_{k+1}\cdots j_n} | \alpha_k = 0\} \\
\overline{\Gamma}_{1j_{k+1}\cdots j_n} &= \{\alpha \in \overline{\Gamma}_{j_{k+1}\cdots j_n} | \alpha_k = 1\} \cup \{\alpha + \delta^{(k)} | \alpha \in \overline{\Gamma}_{0j_{k+1}\cdots j_n}\} \\
&\cdots \\
\overline{\Gamma}_{lj_{k+1}\cdots j_n} &= \{\alpha \in \overline{\Gamma}_{j_{k+1}\cdots j_n} | \alpha_k = l\} \cup \{\alpha + \delta^{(k)} | \alpha \in \overline{\Gamma}_{l-1,j_{k+1}\cdots j_n}\}
\end{aligned}$$

where $l = b_{kj_{k+1}\cdots j_n}$.

(3) $\overline{\Gamma} = \bigcup_{j_1 j_2 \cdots j_n} \overline{\Gamma}_{j_1 j_2 \cdots j_n}$.

Theorem 2.23 *The result $\overline{\Gamma}$ of the above algorithm is the closure of Γ.*

Proof Notice the following facts:

$$(\overline{\Gamma})_{j_k \cdots j_n} = \overline{\Gamma}_{j_k \cdots j_n}, \quad k = 1, 2, \ldots, n.$$

Let $\overline{\delta}$ be the multiplicative direction of $\overline{\Gamma}$. For $\alpha \in \Gamma$, $\alpha_k < \mathbf{b}_k(\overline{\Gamma}_{\alpha_{k+1}\cdots\alpha_n})$ if $\overline{\delta}(\alpha_k) = 0$. From the algorithm, $\alpha + \delta^k \in \overline{\Gamma}_{\alpha_k+1,\alpha_{k+1}\cdots\alpha_n} \subseteq \overline{\Gamma}$. So, $\overline{\Gamma}$ is closed set.

Assuming that Γ' is another closed set, and $\Gamma' \supseteq \Gamma$. Obviously, $\mathbf{b}_n(\Gamma') \geq \mathbf{b}_n(\Gamma)$, and $\overline{\Gamma}_0 \subseteq \Gamma'_0$, $\{\alpha \in \Gamma | \gamma_1 = 1\} \subseteq \Gamma'_1$. Since Γ' is closed set, it means $\overline{\Gamma}_1 \subseteq \Gamma'$. Similarly, it shows that $\overline{\Gamma}_{j_n} \subseteq \Gamma'_{j_n}$ step by step. Further, using mathematical induction, $\overline{\Gamma}_{j_k\cdots j_n} \subseteq \Gamma'_{j_k\cdots j_n}$ can be proved. These relations indicate $\overline{\Gamma} \subseteq \Gamma'$. Therefore, $\overline{\Gamma}$ is the closure of $\subseteq \Gamma$. $\Box$

$\overline{\Gamma}$ may not be the minimal completion set of Γ, but it is easy to calculate the multiplicative direction, and so is $\overline{\Gamma}$. In addition, based on $\overline{\Gamma}$, a completion set which is smaller than $\overline{\Gamma}$ in size can be obtained from the following algorithm.

Contraction Algorithm:

For every $\overline{\Gamma}_{j_2\cdots j_n}$, set $\overline{\Gamma}_{j_2\cdots j_n} \setminus \{\alpha\}$, if the following conditions are satisfied:

(1) $\overline{\Gamma}_{j_2\cdots j_n}$ has two elements at least;

(2) $\alpha = (b_{1j_2\cdots j_n}) \notin \Gamma$.

Repeating the above steps until at least one condition is not satisfied. $\Gamma^* = \bigcup_{j_2\cdots j_n} \overline{\Gamma}_{j_2\cdots j_n}$ is also a completion set of Γ, called modified completion set. Generally, the number of elements in Γ^* is less than Γ.

The main aim of completion of multi-indices is to complete the corresponding d-pol set, and supplementary integrability conditions can be offered. In the following algorithm, supplementary integrability conditions of d-pol-set set G are determined.

Supplementary integrability conditions Algorithm:

(1) G is separated into a disjoint union of $G_1, \ldots, G_m$, where $G_p = \{g \in G | \mathrm{cl}(g) = p\}$, then $G = \bigcup_{p=1}^m G_p$ is a classification of G.

(2) Let $G_p = \{g_1, \ldots, g_k\}$, and the corresponding multi-indices set be $\Gamma_p = \{\alpha^1, \ldots, \alpha^k\}$. Γ_p is completed through the above Completion Algorithm and Contraction Algorithm, while $\overline{\alpha}^i$ is the all prolongations of α^i, then $\overline{\Gamma}^*_p = \bigcup_{i=1}^k \overline{\alpha}^i$ is the completion set of Γ_p, and $\overline{\alpha}^i \cap \overline{\alpha}^j = \emptyset$ if $i \neq j$, meanwhile it forms the d-pol-set $\overline{g}_i = \{D_{\beta - \alpha^i} | \beta \in \overline{\alpha}^i\}$, and $\overline{G}_p = \bigcup_{i=1}^k \overline{g}_i$.

(3) $\{S(f,g)|\forall f \in \overline{g}^i, g \in \overline{g}^j, i \neq j\}$ is the collection of S-polynomials of $\overline{G}_p$, and $S(\overline{G})$ is the collection of S-polynomials of $\overline{G}$.

(4) $I_{\text{supp}} = \text{Red}(S(\overline{G}), G)$ is the supplementary conditions of G.

Definition 2.24 *A non-trivial d-asc-set G is said to be passive, if $I_{\min}(G)$ and $I_{\text{supp}}(G)$ are both empty.*

2.4. Differential characteristic set

Based on the above preparing, the main result is given.

Theorem 2.25 *Let dps be a finite d-pol system. Then there exists a passive d-bas-set dcs of dps, called differential characteristic set (abbr. d-char-set) of dps, s.t.,*

$$d\text{-Zero}(dcs/J) \subset d\text{-Zero}(dps) \subset d\text{-Zero}(dcs), \tag{2.14}$$

$$d\text{-Zero}(dps) = d\text{-Zero}(dcs/J) + \sum_i d\text{-Zero}(dps'_i) + \sum_i d\text{-Zero}(dps''_i), \tag{2.15}$$

where d-Zero(dps) are all Zero points of dps in field $\mathbb{F}$. J is the product of initials and separants of dcs. d-Zero(dcs/J) are all Zero points of dps which are not Zero points of J. dps'_i and dps''_i are the enlarged d-pol-set obtained from the initials and separants of dcs.

Proof Let $dps_0 = dps$. A d-bas-asc-set dbs_0 of dps_0 is selected, and let $rs_0 = \text{Red}((dps_0 \setminus dbs_0), dbs_0)$. If $rs_0 = \emptyset$, dbs_0 is the d-bas-set of dps, otherwise, let $dps_0 = dps_0 \cup rs_0$. Selecting dbs_0 of dps_0, and calculating $rs_0 = \text{Red}((dps_0 \setminus dbs_0), dbs_0)$ again. Repeating the above procedure until $rs_0 = \emptyset$.

Let $I^0_{\min} = I_{\min}(dbs_0)$ be the minimal integrability conditions of dbs_0. If $I^0_{\min} \neq \emptyset$, let $dps_0 = dps_0 \cup I^0_{\min}$, and selecting dbs_0 of dps_0 and calculating $I_{\min}(dbs_0)$ of dbs_0 again, and repeating the above procedure until $I^0_{\min} = \emptyset$, otherwise, calculating supplementary integrability conditions $I^0_{\text{supp}} = I_{\text{supp}}(dbs_0)$ of dbs_0.

If $I^0_{\text{supp}} = \emptyset$, $dbs_0 = dcs$ is the d-char-set of dps, otherwise, let $dps_1 = dps_0 \cup I^0_{\text{supp}}$. Selecting the d-bas-asc-set dbs_1 of dps_1, and calculating minimal integrability conditions $I^1_{\min}$ and supplementary integrability conditions I^1_{supp} of dbs_1. If $I^1_{\text{supp}} = \emptyset$, $dbs_1 = dcs$, otherwise, let $dps_2 = dps_1 \cup I^1_{\text{supp}}$. The series d-pol-sets: dps_i, dbs_i, and $rs_i, I^i_{\min}, I^i_{\text{supp}}$ are gained. Let $RIS_i = rs_i \cup I^i_{\min} \cup I^i_{\text{supp}}$. Then, there exist following relations:

$$dps_i = dps_{i-1} \cup RIS_{i-1} \tag{2.16}$$

$$dbs_0 > dbs_1 > \cdots > dbs_{i-1} > dbs_i > \cdots \tag{2.17}$$

From Lemma 2.8, the series in (2.17) should be ended at a certain stage k, that is in finite steps $dbs_k = dcs$ is a passive d-bas-set, and $RIS_k = \emptyset$.

According to the above-mentioned construction process of d-char-set, we see that

$$d\text{-Zero}(dps) = d\text{-Zero}(dps_0) = \cdots = d\text{-Zero}(dps_k) \tag{2.18}$$

From the d-remainder formula, and the emptiness of RIS_k, we have

$$d\text{-Zero}(dps) = d\text{-Zero}(dbs_k/J) + \sum_i d\text{-Zero}(dps'_{ki}) + \sum_i d\text{-Zero}(dps''_{ki}), \tag{2.19}$$

where each dps'_{ki} and dps''_{ki} is the enlarged d-plo-set derived from dps_k by adjoining to its initials and separants of dcs. For each i, it is clear from (2.18) that we have

$$d\text{-Zero}(dps'_{ki}) = d\text{-Zero}(dps'_k), \quad d\text{-Zero}(dps''_{ki}) = d\text{-Zero}(dps''_k). \quad \Box \tag{2.20}$$

The formula (2.15) can be obtained from (2.18)–(2.20). The formula (2.14) is also obtained from the construction immediately.

In the above deducing procedure, d-char-set dcs can be formed in the following scheme:

$$\begin{array}{ccccccc} dps = dps_0 & \subset & dps_1 & \subset & \cdots & \subset & dps_k \\ \downarrow & & \downarrow & & \cdots & & \downarrow \\ dbs_0 & > & dbs_1 & > & \cdots & > & dbs_k = dcs \\ \downarrow & & \downarrow & & \cdots & & \downarrow \\ RIS_0 \uparrow & & RIS_1 \uparrow & & \cdots & & RIS_k = \emptyset \end{array} \tag{W}$$

(W) is called Wu-Ritt process. The d-pol-sets dps'_i and dps''_i in (2.15) can be further split into the sum of d-char-set as in (2.15). The procedure needs to be terminated in finite steps since all dps'_i and dps''_i are lower than dps. So the following theorem can be obtained finally.

Theorem 2.26 (zero decomposition theorem) *There exists an algorithmic procedure which permits us to give for any finite d-plo-set dps a decomposition of the form:*

$$d\text{-Zero}(dps) = \sum_i d\text{-Zero}(dcs_i/J_i),$$

where dcs_i is passive d-char-set, and J_i is the multiplication of initials and separants of dcs_i.

The d-char-set is in triangular or trapezoid form. It is important for many problems which contain d-pols. For ODEs, there are no integrability conditions, then d-bas-set is d-char-set.

3. Pseudo differential operators

In the subsequent sections, D denotes $\frac{\partial}{\partial x}$, and prime denotes the differential in x. For any function $f(x)$, Df is regarded as an operator as follows:

$$Df = f' + fD, \tag{3.1}$$

and for function $g(x)$,

$$Df(g) = (f' + fD)g = f'g + fg'. \tag{3.2}$$

So,

$$D^n f = \sum_{i=0}^{n} \frac{n(n-1)\cdots(n-i+1)}{i!} f^{(i)} D^{n-i}. \tag{3.3}$$

D^{-1} is the inverse of D and defined as

$$DD^{-1} = D^{-1}D = 1. \tag{3.4}$$

D^{-1} can be understood as a formal integration symbol, and for a function $f(x)$,

$$D^{-1}f = fD^{-1} - f'D^{-2} + f''D^{-3} + \cdots , \tag{3.5}$$

and

$$D^{-n}f = \sum_{i=0}^{\infty} \frac{(-n)(-n-1)\cdots(-n-i+1)}{i!} f^{(i)}D^{-n-i}, \quad n > 0. \tag{3.6}$$

From (3.3) and (3.6), the Leibniz rule is employed as

$$D^{n}f = \sum_{i=0}^{\infty} \frac{n(n-1)\cdots(n-i+1)}{i!} f^{(i)}D^{n-i}, \quad n \in \mathcal{Z}. \tag{3.7}$$

A PDO of order n is defined as an infinite series

$$U = \sum_{i=-\infty}^{n} u_i D^i, \tag{3.8}$$

where coefficients u_i are differential polynomials. According to the operations of series and (3.7), one can define the operations between PDO and PDO, between differential operators and PDO, and between PDO and functions. The multiplication of PDOs U and V can be treated as a composition of operators, i.e., for any function f,

$$(UV)f = U(Vf) = (U \circ V)f. \tag{3.9}$$

The multiplication is non-commutative, but for arbitrary integers m and n,

$$D^n \circ D^m = D^n D^m = D^{n+m} = D^m \circ D^n. \tag{3.10}$$

Let $V = \sum_{j=-\infty}^{m} v_j D^j$. Then the multiplication is

$$UV = \sum_{k=0}^{\infty} \Big[\sum_{i=0}^{k} \sum_{j=0}^{k-i} C_{n-j}^{k-i-j} u_j v_i^{(k-i-j)} \Big] D^{m+n-k}. \tag{3.11}$$

Based on multiplication, the power, the inverse, and the extraction of a root of PDOs can be defined, and composed of non-commutative algebra [1], and play a very important role in integrable systems. The concrete calculation of the inverse and the extraction of a root of PDOs is presented utilizing differential characteristic set method in Section 4.

Given a PDO, its positive part and negative part are shown as follows

$$(U)_+ = \sum_{i=0}^{n} u_i D^i, \tag{3.12}$$

$$(U)_- = \sum_{i=-\infty}^{-1} u_i D^i, \tag{3.13}$$

and its residue is defined as the coefficient of D^{-1},

$$\mathrm{Res}(U) = u_{-1}. \tag{3.14}$$

$(U)_+$ is a normal linear differential operator. From residue, the properties of first integrals, conservation laws, etc., of a nonlinear evolution equation can be deduced.

4. Lax representation

In integrable systems, according to Lax scheme [1,8], if evolution equation

$$\frac{\partial u}{\partial t} = K(u), \tag{4.1}$$

can be represented as the compatibility conditions of equations

$$L\psi = \lambda\psi, \tag{4.2a}$$

$$\frac{\partial \psi}{\partial t} = B\psi, \tag{4.2b}$$

$$\frac{\partial \lambda}{\partial t} = 0, \tag{4.2c}$$

then the equation (4.1) is integrable, where L and B are the differential operators with coefficients being differential polynomials about u. Differentiating (4.2a) w.r.t. t, and substituting into (4.2b), the famous Lax equation

$$\frac{\partial L}{\partial t} = [B, L], \tag{4.3}$$

is yielded. (4.3) is called the Lax representation and L and B are called Lax pair of the evolution equation (4.1). Using the inverse scattering method, the explicit solution can be derived.

For a given equation, however, unfortunately, there exists no general method to construct its Lax pair. Starting with a linear differential operator L, PDOs provides a simple way to construct a whole hierarchy of integrable evolution equations. Firstly, $1+1$ dimensional systems are presented.

Let L be a PDO of order n

$$L = D^n + \sum_{i=0}^{n-2} u_i(x,t)D^i, \tag{4.4}$$

where $t = (t_1, t_2, \ldots)$. Then, the unique n-th root and the inverse of L exist in following forms

$$L^{\frac{1}{n}} = D + \sum_{i}^{\infty} v_i(x,t)D^{-i}, \tag{4.5}$$

$$L^{-1} = D^{-n} + \sum_{i}^{\infty} v_i(x,t)D^{-n-i}. \tag{4.6}$$

$L^{\frac{1}{n}}$ and L^{-1} commute with L. Linear differential operators $B_m = (L^{\frac{m}{n}})_+$, $m = 1, 2, \ldots$ are the partners of L in Lax pair. From Lax equations

$$\frac{\partial L}{\partial t_m} = [B_m, L], \tag{4.7}$$

a hierarchy is defined, and called Gelfand-Dickey hierarchy. (4.7) is called the generalized Lax equation. Furthermore,

$$\frac{\partial L^{\frac{k}{n}}}{\partial t_m} = [B_m, L^{\frac{k}{n}}], \quad k = 1, 2, \ldots \tag{4.8}$$

can be derived from (4.7). Then,

$$J_k = \int \operatorname{Res} L^{\frac{k}{n}} \mathrm{d}x, \quad k = 1, 2, \ldots \tag{4.9}$$

are all first integrals of the n-th hierarchy, and

$$\frac{\partial J_k}{\partial t_m} = \int \mathrm{Res}[B_m, L^{\frac{k}{n}}]\mathrm{d}x = 0, \quad k = 1, 2, \ldots. \tag{4.10}$$

Using Sato theory [2, 9, 10], this method can be extended to more than $1+1$ dimensional systems. Let

$$W = 1 + w_1 D^{-1} + w_2 D^{-2} + w_3 D^{-3} + \cdots \tag{4.11}$$

where the coefficients w_i, $i = 1, 2, \ldots$ are the function of $x, t = (t_1, t_2, \ldots)$, which are the fields of the theory satisfying nonlinear evolution equations. W is called "dressing" operator [1, 11, 12]. From the multiplication of PDOs, the inverse of W exists and is also a PDO, denoted by W^{-1}, i.e.,

$$WW^{-1} = W^{-1}W = 1. \tag{4.12}$$

Based on Cramer Rule and the theory of ordinary differential equation, one can yield the famous Sato equation

$$\frac{\partial W}{\partial t_n} = B_n W - W B_n, \tag{4.13a}$$

$$B_n = (W D^n W^{-1})_+. \tag{4.13b}$$

The operator L is defined by

$$L = W D W^{-1}, \tag{4.14}$$

and

$$\frac{\partial L}{\partial t_n} = \frac{\partial W}{\partial t_n} D W^{-1} + W D \frac{\partial W^{-1}}{\partial t_n}. \tag{4.15}$$

Since $WW^{-1} = 1$,

$$0 = \frac{\partial 1}{\partial t_n} = \frac{\partial (WW^{-1})}{\partial t_n} = \frac{\partial W}{\partial t_n} W^{-1} + W \frac{\partial W^{-1}}{\partial t_n}, \tag{4.16}$$

we have

$$\frac{\partial W^{-1}}{\partial t_n} = -W^{-1} \frac{\partial W}{\partial t_n} W^{-1}. \tag{4.17}$$

Substituting (4.17) into (4.13a), the generalized Lax equation can be derived

$$\frac{\partial L}{\partial t_n} = [B_n, L]. \tag{4.18}$$

For an arbitrary positive integer m,

$$L^m = (WDW^{-1})(WDW^{-1}) \cdots (WDW^{-1}) = W D^m W^{-1}, \tag{4.19}$$

therefore, a general form of the generalized Lax equation can be obtained by repeating the above analysis with L^m instead of L

$$\frac{\partial L^m}{\partial t_n} = [B_n, L^m], \tag{4.20}$$

and the following equation holds too

$$\frac{\partial L^n}{\partial t_m} = [B_m, L^n]. \tag{4.21}$$

The Zakharov-Shabat equation can be derived from (4.20) and (4.21) directly

$$\frac{\partial B_m}{\partial t_n} - \frac{\partial B_n}{\partial t_m} = [B_n, B_m]. \tag{4.22}$$

PDO L and linear differential operator B have more general forms

$$L = u_n D^n + u_{n-1} D^{n-1} + u_1 D + u_0 + u_{-1} D^{-1} + \cdots, \quad B = (L^{\frac{m}{n}})_{\geqslant k}, \tag{4.23}$$

and are called "nonstandard cases" [9].

5. Differential characteristic set method for Lax representation

In the construction of Lax representations, differential characteristic set method is mainly applied in two aspects: calculating the inverse or the extraction of a root of PDOs, and reducing the nonlinear PDEs which are yielded from the generalized Lax equation or Zakharov-Shabat equation.

For simplicity, we start with the detailed calculation procedure of the inverse of dress operator W. Let

$$W = 1 + w_1 D^{-1} + w_2 D^{-2} + w_3 D^{-3} + \cdots, \tag{5.1}$$

and the inverse of W be

$$W^{-1} = 1 + v_1 D^{-1} + v_2 D^{-2} + v_3 D^{-3} + \cdots. \tag{5.2}$$

$WW^{-1} \equiv 1$ gives

$$\begin{aligned} 1 &\equiv (1 + w_1 D^{-1} + w_2 D^{-2} + w_3 D^{-3} + \cdots) \circ (1 + v_1 D^{-1} + v_2 D^{-2} + v_3 D^{-3} + \cdots) \\ &= 1 + (w_1 + v_1) D^{-1} + (w_3 + w_2 v_1 + w_1 v_2 + v_3 - w_1 v_1') D^{-2} + \cdots. \end{aligned} \tag{5.3}$$

From (5.3), the ODEs on v_i, $i = 1, 2, \ldots$, denoted by *dos*, is given

$$\begin{aligned} & w_1 + v_1 = 0, \\ & w_2 + w_1 v_1 + v_2 = 0, \\ & w_3 + w_2 v_1 + w_1 v_2 + v_3 - w_1 v_1' = 0, \\ & w_4 + w_3 v_1 + w_2 v_2 + w_1 v_3 + v_4 - 2 w_2 v_1' - w_1 v_2' = 0, \\ & \cdots, \end{aligned} \tag{5.4}$$

where w_i, $i = 1, 2, \ldots$ can be treated as known variables. Solving *dos* and substituting v_i into (5.2), W^{-1} can be obtained. However, the number of equations in *dos* is infinite, so W^{-1} is not obtained completely. From the last section, W and W^{-1} are used to construct the differential operator $B_m = (WD^m W^{-1})_+$. For a given m, B_m is confirmed, and has finite terms. So after a certain term, $(WD^m W^{-1})_+$ is not changed. Therefore, this term can be treated as the order of truncation of W and W^{-1}, thus *dos* has finite equations. Actually, in the calculation process, the minimum order of W and W^{-1} decrease step by step from m, until

$$(WD^m W^{-1})_+ = [(1 + w_1 D^{-1} + w_2 D^{-2} + \cdots) \circ D^m \circ (1 + v_1 D^{-1} + v_2 D^{-2} + \cdots)]_+ \tag{5.5}$$

does not change, i.e., and truncation order is the maximum order such that $(WDW^{-1})_+$ does not change.

The process to solve *dos* is quite tedious and complicated, though *dos* is composed of linear ordinary differential equations. Since the equations in *dos* are all d-pols, differential characteristic set method is used to solve *dos*. Let $v_1 < v_2 < \cdots$. According to (W), the *d*-char-set *dcs* of *dos* can be obtained. Since *dos* is a system of linear ordinary differential equations, integrability conditions do not exist, and the d-bas-set *dbs* of *dos* is *dcs*. We see that *dcs* is a system of linear ordinary differential equations. In fact, since $dos_0 = dos$ is a linear system, the d-bas-asc-set dbs_0 of dos_0 is also a linear system, then v_i and its derivatives do not occur in the initials and separants of dbs_0. From formula (2.4), the degree of $rs_0 = \mathrm{Red}(dos_0 \setminus dbs_0, dbs_0)$ is not higher than 1. Therefore $dos_1 = dos_0 \cup rs_0$ is also a linear system, and throughout the process, the dos_i, $i = 1, \ldots, k$ are always linear systems. So the *d*-char-set *dcs* of *dos* is linear system of linear ordinary differential equations and in triangle form, and then the solution of v_i, $i = 1, 2, \ldots$ can be solved directly. W^{-1} is calculated through substituting v_i into (5.2).

From (5.2), even though W^{-1} can be obtained directly, the process of calculation is fairly tedious. Here, differential division with remainder is utilized to calculate W^{-1}, since it is a kind of substitution. In (5.2), W^{-1} can be seen as *d*-plo, and $v_1 < v_2 < \cdots$ as order. From Theorem 2.13, the *d*-remainder $\mathrm{Red}(W^{-1}, dcs)$ is reduced w.r.t. *dcs*, and *dcs* is in triangle form, and the degree of v_i is 1 in *dcs*, then v_i does not occur in $\mathrm{Red}(W^{-1}, dcs)$. Therefore, $\mathrm{Red}(W^{-1}, dcs)$ is the desired result, denoted by W^{-1}, i.e.,

$$W^{-1} = \mathrm{Red}(W^{-1}, dcs).$$

In the process of W^{-1} calculation, D does not participate as a parameter variable.

With the same argument, B_m can be calculated through

$$B_m = \mathrm{Red}((WD^m W^{-1})_+, dcs),$$

where *dcs* is also the *d*-char-set of (5.4).

Analogously, differential characteristic set method can be applied to the calculations of extraction root of PDOs. Let

$$L^{\frac{1}{n}} = D + v_1 D^{-1} + v_2 D^{-2} + \cdots \tag{5.6}$$

be the n-th root of L in (4.4). For a given m, the differential operator $B_m = (L^{\frac{m}{n}})_+$ is determined, so the truncation order of $L^{\frac{1}{n}}$ is the maximum order s.t.,

$$(L^{\frac{m}{n}})_+ = [(D + v_1 D^{-1} + v_2 D^{-2} + \cdots)^m]_+ \tag{5.7}$$

does not change. From

$$L = (L^{\frac{1}{n}})^n, \tag{5.8}$$

an ODEs *dos* on v_i, $i = 1, 2, \ldots$ can be given. Utilizing differential characteristic set method, the *d*-char-set *dcs* of *dos* is obtained, then $\mathrm{Red}(L^{\frac{1}{n}}, dcs)$ is the nth root of L, denoted by $L^{\frac{1}{n}}$, i.e.,

$$L^{\frac{1}{n}} = \mathrm{Red}(L^{\frac{1}{n}}, dcs) \text{ and } B_m = \mathrm{Red}((L^{\frac{m}{n}})_+, dcs)$$

is the desired differential operator.

In the above calculations, using characteristic set method and differential division with remainder, it is not necessary to solve the ODEs and substitute the solutions into $W^{-1}, L^{\frac{1}{n}}$ and B_m, so the computational procedure is simplified, and the desired result can be obtained rapidly with the help of computer.

The nonlinear evolution equation is yielded from the generalized Lax equation or Zakharov-Shabat equation. However, in general, it is embarrassed that the desired result cannot be obtained directly. From generalized Lax equation or Zakharov-Shabat equation, nonlinear PDEs are derived. The PDEs are composed of d-pols and contain some auxiliary fields, even overdetermined for some cases. To get the evolution equation, one needs to eliminate the auxiliary fields and superfluous equations from the PDEs. The usual way is to make use of compatible conditions via observing, so the process not only is complicated, but also requires certain skills, and the workload of calculation is heavy as the number of equations is large.

Since it is a powerful technique to deal with PDEs, differential characteristic set method aims for reducing the nonlinear PDEs derived from the generalized Lax equation or Zakharov-Shabat equation. Based on the given order, constructing the d-char-set of the PDEs, superfluous equations are eliminated. Since the d-char-set is in triangular form, choosing a suitable order, including dependent variables order and independent variables order, the first d-pol of the results is the desired nonlinear evolution equation. Differential characteristic set method provides a general algorithm to handle the PDEs derived from the generalized Lax equation or Zakharov-Shabat equation, and the whole procedure can be performed in computer.

In the computer algebra system MATHEMATICA, programs of differential characteristic set method, PDOs and Lax representation have been compiled. Considering differential characteristic set method based on order, in the program designing, the order is given by macro definition and called by parameter mode, which makes it convenient to choose order according to the requirements.

6. Examples

In this section, some illustrative examples are presented to show that differential characteristic set method is used for Lax representation of nonlinear evolution equations.

Example 6.1 Boussinesq Equation.

Given differential operator

$$L = D^3 + uD + v, \tag{6.1}$$

where $u = u(x,t)$, $v = v(x,t)$ and linear differential operator $B_2 = (L^{\frac{2}{3}})_+$. The maximum order D^{-1} is obtained when B_2 does not change, so assuming the 3-rd root of L is

$$L^{\frac{1}{3}} = D + w_1 D^{-1}, \tag{6.2}$$

and

$$B_2 = (L^{\frac{2}{3}})_+ = D^2 + 2w_1. \tag{6.3}$$

From

$$L = (L^{\frac{1}{3}})^3, \tag{6.4}$$

we have

$$3w_1 - u = 0. \tag{6.5}$$

The solution of (6.5) is

$$w_1 = \frac{1}{3}u.$$

Then

$$L^{\frac{1}{3}} = D + \frac{1}{3}uD^{-1}, \tag{6.6}$$

and

$$B_2 = (L^{\frac{2}{3}})_+ = D^2 + \frac{2}{3}u. \tag{6.7}$$

Substituting L and B_2 into the generalized Lax equations (4.3), we have

$$\begin{aligned} &u_t - 2v_x + u_{xx} = 0, \\ &v_t + \frac{2}{3}uu_x - v_{xx} + \frac{2}{3}u_{xxx} = 0. \end{aligned} \tag{6.8}$$

Under the order of dependent variables $u < v$, and the order of independents variables $x < t$, the d-char-set of (6.8) is

$$4(u_x)^2 + 3u_{tt} + 4uu_{xx} + u_{xxxx} = 0, \tag{6.9a}$$

$$u_t - 2v_t + u_{xx} = 0, \tag{6.9b}$$

$$6v_t + 4uu_x - 3u_{xt} + u_{xxx} = 0, \tag{6.9c}$$

(6.9a) is Boussinesq Equation, and can be written as

$$u_{tt} = -\frac{1}{3}u^{(4)} - \frac{4}{3}(uu')'. \tag{6.10}$$

Example 6.2 Kaup-Kupershmidt Equation.

Given differential operator

$$L = D^3 + 2uD + u', \tag{6.11}$$

where $u = u(x, t_1, t_2, t_3, t_4, t_5)$ and linear differential operator $B_5 = (L^{\frac{5}{3}})_+$. The maximum order D^{-4} is obtained when B_5 does not change, so let the 3-rd root of L be

$$L^{\frac{1}{3}} = D + w_1D^{-1} + w_2D^{-2} + w_3D^{-3} + w_4D^{-4}, \tag{6.12}$$

and

$$\begin{aligned} (L^{\frac{5}{3}})_+ = {} & D^5 + 5w_1D^3 + (5w_2 + 10w_1')D^2 + (10w_1^2 + 5w_3 + 10w_2' + 10w_1'')D \\ & + 20w_1w_2 + 5w_4 + 20w_1w_1' + 10w_3' + 10w_2'' + 5w_1'''. \end{aligned} \tag{6.13}$$

From

$$L = (L^{\frac{1}{5}})^5, \tag{6.14}$$

the following ODEs is derived

$$\begin{aligned} &3w_1 - 2u = 0, \\ &3w_2 - u' + 3w_1' = 0, \\ &3w_1^2 + 3w_3 + 3w_2' + w_1'' = 0, \\ &6w_1w_2 + 3w_4 + 3w_3' + w_2'' = 0. \end{aligned} \tag{6.15}$$

Under the order $u < w_1 < w_2 < w_3 < w_4$, the d-char-set dcs of (6.15) is

$$\begin{aligned} &3w_1 - 2u = 0, \\ &3w_2 + u' = 0, \\ &4u^2 + 9w_3 - u'' = 0, \\ &3w_4 - 4uu' = 0. \end{aligned} \tag{6.16}$$

According to formula (2.8),

$$B_5 = \mathrm{Red}((L^{\frac{5}{3}})_+, dcs) = D^5 + \frac{10}{3}uD^3 + 5u'D^2 + (\frac{20}{9}u^2 + \frac{35}{9}u'')D + \frac{20}{9}uu' + \frac{10}{9}u'''. \tag{6.17}$$

From the generalized Lax equations (4.3), the following overdetermined PDEs is derived,

$$\begin{aligned} &40u_xu^2 + 20u_{xxx}u + 2u_{t_5} + 50u_xu_{xx} + 2u_{xxxxx} = 0, \\ &20u_{xx}u^2 + 40u_x^2u + 10u_{xxxx}u + 25u_{xx}^2 + u_{xt_5} + 35u_xu_{xxx} + u_{xxxxxx} = 0. \end{aligned} \tag{6.18}$$

There are no integrability conditions and dependent order for (6.18) that contains only one dependent variable u. Under the order of independent variables $x < t_5$, the d-char-set of (6.18) is

$$u_{t_5} + 20u^2u_x + 10uu_{xxx} + 25u_xu_{xx} + u_{xxxxx} = 0. \tag{6.19}$$

(6.19) is Kaup-Kupershmidt Equation. The 3-rd root of L can be obtained via d-remainder of (6.12) w.r.t. (6.16), if it is used in subsequent calculations.

The above examples are both $1+1$ dimensional systems, as a $2+1$ dimensional example, KP equation can be considered firstly.

Example 6.3 KP Equation.

KP equation is a $2+1$ dimensional system, and is the simplest non-trivial equation of KP hierarchy. The KP hierarchy is generated by the generalized Lax equation (4.18) or Zakharov-Shabat equation (4.22). Now, utilizing differential characteristic set method, the way to get KP is shown concretely. Let PDO

$$W = 1 + w_1D^{-1} + w_2D^{-2} + w_3D^{-3} + \cdots, \tag{6.20}$$

and

$$W^{-1} = 1 + v_1D^{-1} + v_2D^{-2} + v_3D^{-3} + \cdots. \tag{6.21}$$

According to the Sato theory, $L = WDW^{-1}$, and $B_m = (L^m)_+ = (WD^mW^{-1})_+$. As evaluating

$m = 2$ and $m = 3$

$$\begin{aligned}(WD^2W^{-1})_+ =& D^2 + (v_1 + w_1)D + v_2 + v_1w_1 + w_2 + 2v_1',\\ (WD^3W^{-1})_+ =& D^3 + (v_1 + w_1)D^2 + (v_2 + v_1w_1 + w_2 + 3v_1')D+ \\ & v_3 + v_2w_1 + v_1w_2 + w_3 + 2w_1v_1' + 3v_2' + 3v_1'',\end{aligned} \tag{6.22}$$

where the maximum order D^{-3} is obtained when B_3 does not change, then the truncation order of W and W^{-1} are both D^{-3}, meanwhile B_2 does not change too, and

$$\begin{aligned}L =& D + v_1 + w_1 + (v_2 + v_1w_1 + w_2 + v_1')D^{-1} + (v_3 + v_2w_1 + v_1w_2 + w_3 + v_2')D^{-2}+ \\ & (v_3w_1 - v_1''w_1 + v_2w_2 + v_1w_3 - w_2v_1' + v_3')D^{-3}.\end{aligned} \tag{6.23}$$

From (5.3), i.e., $WW^{-1} = 1$, the following ODEs is yielded

$$\begin{aligned}& w_1 + v_1 = 0,\\ & w_2 + w_1v_1 + v_2 = 0,\\ & w_3 + w_2v_1 + w_1v_2 + v_3 - v_1'w_1 = 0.\end{aligned} \tag{6.24}$$

Under the order $w_1 < w_2 < w_3 < v_1 < v_2 < v_3$, the d-char-set dcs_1 of (6.24) is

$$\begin{aligned}& v_1 + w_1 = 0,\\ & v_2 - w_1^2 + w_2 = 0,\\ & v_3 + w_1^3 - 2w_2w_1 + w_1'w_1 + w_3 = 0.\end{aligned} \tag{6.25}$$

Then from (2.8)

$$B_2 = \mathrm{Red}((WD^2W^{-1})_+, dcs_1) = D^2 - 2w', \tag{6.26a}$$

$$B_3 = \mathrm{Red}((WD^3W^{-1})_+, dcs_1) = D^3 - 3w_1'D + 3w_1w_1' - 3w_2' - 3w_1'', \tag{6.26b}$$

$$\begin{aligned}L =& \mathrm{Red}(L, dcs_1) = D - w_1'D^{-1} + [w_1w_1' - w_2']D^{-2}- \\ & [w_1'w_1^2 - (w_1w_2)' + (w_1')^2 + w_3']D^{-3}.\end{aligned} \tag{6.27}$$

Let the L in (6.27) have the form

$$L = D + u_1D^{-1} + u_2D^{-2} + u_3D^{-3}. \tag{6.28}$$

The KP equation is derived in terms of u's instead of w's. Comparing (6.27) with (6.28), the relation between u's and w's is

$$\begin{aligned}& u_1 = -w_1',\\ & u_2 = w_1w_1' - w_2',\\ & u_3 = -w_1'w_1^2 + w_2'w_1 - (w_1')^2 + w_2w_1' - w_3'.\end{aligned} \tag{6.29}$$

Under the order $u_1 < u_2 < u_3 < w_1 < w_2 < w_3$, d-char-set dcs_2 of (6.29) is

$$\begin{aligned}& u_1 + w_1' = 0,\\ & u_2 + u_1w_1 + w_2' = 0,\\ & u_1^2 + w_2u_1 + u_3 + u_2w_1 + w_3' = 0.\end{aligned} \tag{6.30}$$

Using differential division with remainder, the d-remainder of (6.26a) and (6.26b) w.r.t. dcs_2 are B_2 and B_3, respectively, then B_2 and B_3 are written in terms of u's, i.e.,

$$\begin{aligned} B_2 &= \mathrm{Red}(B_2, dcs_2) = D^2 + 2u_1', \\ B_3 &= \mathrm{Red}(B_3, dcs_2) = D^3 + 3u_1'D + 3u_2 + 3u_1'. \end{aligned} \tag{6.31}$$

Let $u_1 = \frac{u}{2}, u_2 = v$, $t_1 = y, t_2 = t$. Zakharov-Shabat equation (4.22) leads to

$$\begin{aligned} &-\tfrac{3}{2}u_y - 6v_x - \tfrac{3}{2}u_{xx} = 0, \\ &u_t - 3v_y + \tfrac{3}{2}uu_x - \tfrac{3}{2}u_{xy} - 3v_{xx} - \tfrac{1}{2}u_{xxx} = 0. \end{aligned} \tag{6.32}$$

Under the order $u < v, x < y < t$, the d-char-set of (6.32) is

$$\begin{aligned} &6u_x^2 + 3u_{yy} + 4u_{xt} + 6uu_{xx} + u_{xxxx} = 0, \\ &-\frac{3}{2}u_y - 6v_x - \frac{3}{2}u_{xx} = 0, \\ &4u_t - 12v_y + 6uu_x - 3u_{xy} + u_{xxx} = 0. \end{aligned} \tag{6.33}$$

The first equation in (6.33) is KP equation, and it can be written as

$$3u_{yy} = (4u_t - 6uu_x - u_{xxx})_x.$$

or

$$u_t = \frac{1}{4}u_{xxx} + 3uu_x + \frac{3}{4}D^{-1}u_{yy}. \tag{6.34}$$

KP equation can also be yielded by the generalized Lax equation with L and B_2, B_3, but it requires more complicated calculations.

In the generalized Lax equation (4.7), if L is an infinite series, L also adopts a truncation form. The coefficient of D^{-1}, i.e., Res plays an important part in integrable systems and PDOs, and has the property (4.10). Thus, the truncation order of L in (4.7) reaches the maximum when the coefficient of D^{-1} of Possion bracket $[B_m, L]$ does not change. The next two examples are the cases.

Example 6.4 mKP Equation.

Let

$$L = D + u_0 + u_1D^{-1} + u_2D^{-2} + u_3D^{-3} + \cdots, \tag{6.35}$$

and

$$B_m = (L^m)_{\geqslant 1}, \quad m = 1, 2, 3, \ldots. \tag{6.36}$$

L and B_m are nonstandard cases. mKP hierarchy can be derived from the generalized Lax equation (4.7) or Zakharov-Shabat equation (4.22).

In (4.7), the truncation order of L is obtained when the D^{-1} entry of $[B_m, L]$ does not change. As $m = 2, 3$, let $u_0 = u, u_1 = v, u_2 = w, u_3 = r$ and $t_1 = y, t_2 = t$

$$\begin{aligned} B_2 &= D^2 + 2uD, \\ B_3 &= D^3 + 3uD^2 + (3u^2 + 3v + 3u')D. \end{aligned} \tag{6.37}$$

L and B_2, and L and B_3 are substituted into the generalized Lax equation (4.7), respectively, we have

$$u_y - 2uu_x - 2v_x - u_{xx} = 0, \tag{6.38a}$$
$$v_y - 2vu_x - 2uv_x - 2w_x - v_{xx} = 0, \tag{6.38b}$$
$$w_y - 4wu_x - 2uw_x - w_{xx} = 0, \tag{6.38c}$$

and

$$u_t - 3u_xu^2 - 6v_xu - 3u_{xx}u - 3u_x^2 - 6vu_x - 3w_x - 3v_{xx} - u_{xxx} = 0, \tag{6.39a}$$
$$\begin{aligned} &v_t - 6(vu_xu + w_xu + wu_x + vv_x) - 3(v_{xx}u + v_xu^2 + u_xv_x+ \\ &\quad r_x + vu_{xx} + w_{xx}) - v_{xxx} = 0, \end{aligned} \tag{6.39b}$$
$$\begin{aligned} &w_t - 12wu_xu - 9ru_x - 6(r_xu + wv_x + wu_{xx}) - 3(w_xu^2 + w_{xx}u + vw_x+ \\ &\quad u_xw_x + r_{xx}) - w_{xxx} = 0, \end{aligned} \tag{6.39c}$$
$$r_t - 3r_xu^2 - 18ru_xu - 3r_{xx}u - 9rv_x - 3vr_x - 3u_xr_x - 9ru_{xx} - r_{xxx} = 0. \tag{6.39d}$$

(6.38a), (6.38b), (6.39a), (6.39b) compose new PDEs *dps*. Under the order $u < v < w < r$ and $x < y < t$, the d-char-set of *dps* is

$$\begin{aligned}
&12(u_{xx}u_x^3 - uu_{xxx}u_x^2) + 6(6uu_{xx}^2 - u_{xy}u_{xx} + u_yu_{xxx})u_x - 12u_yu_{xx}^2 - 3u_{xyy}u_{xx}+ \\
&\quad 4u_{xx}u_{xxt} + (3u_{yy} - 4u_{xt})u_{xxx} + u_{xxx}u_{xxxx} - u_{xx}u_{xxxxx} = 0, \\
&24uu_{xx}u_x^3 - 12uu_{xxy}u_x^2 - (12u_{yy}u_{xx} + 8u_{xt}u_{xx} + 24uu_{xy}u_{xx} + 6u_yu_{xxy}- \\
&\quad 2u_{xx}u_{xxxx})u_x + (12uu_y + 8u_t)u_{xx}^2 - (3u_{yyy} - 6u_yu_{xy} + 4u_{xyt})u_{xx}+ \\
&\quad (3u_{yy} + 4u_{xt})u_{xxy} + 2u_{xx}^2u_{xxx} + u_{xxy}u_{xxxx} - u_{xx}u_{xxxxy} = 0, \\
&6u_yu_x - 12uu_x^2 + 6u_{xx}u_x + 3u_{yy} - 4u_{xt} + 12vu_{xx} + u_{xxxx} = 0, \\
&6(u_y + u_{xx})u_x^2 - 12uu_x^3 + (3u_{yy} - 4u_{xt} + 6u^2u_{xx} + u_{xxxx})u_x + 6uu_{xx}^2+ \\
&\quad (2u_t - 6uu_y - 6w_x - 3u_{xy} + u_{xxx})u_{xx} = 0, \\
&36u_xu_{xx}^2u^3 - 18(8u_x^3u_{xx} + 3u_{xx}^3 - 3u_yu_{xx}^2)u^2 - (54u_{xy} + 72u_x^2 + 24u_t)u_{xx}^2u+ \\
&\quad 36(3u_yu_x^2 + u_{yy}u_x - 2u_xu_{xt})u_{xx}u + 6(2u_x^2u_{xxt} + 3u_{xx}^2u_{xxx} + 2u_xu_{xx}u_{xxxx})u+ \\
&\quad 54u_xu_{xx}^3 + 6(3u_{yy} + 12wu_x - 6u_yu_x + 6r_x - u_{xt})u_{xx}^2 - (12u_tu_x^2 + 9u_yu_{yy}- \\
&\quad 3u_{yyt}u_x - 6u_{yt}u_x - 18u_yu_{xt} + 4u_{xtt})u_{xx} + 18u_y^2 - (3u_{yy} - 6u_yu_x + 4u_{xt})u_{xxt}- \\
&\quad 12u_{xx}^2u_{xxy} + (6u_{xx}^2 - 3u_yu_{xx} - u_{xxt})u_{xxxx} + u_{xx}u_{xxxxt} = 0.
\end{aligned} \tag{6.40}$$

The first equation in (6.40) is mKP equation. In [9, 13], the form of mKP is

$$u_t = \frac{1}{4}u_{xxx} - \frac{3}{2}u^2u_x + \frac{3}{2}u_xD^{-1}u_y + \frac{3}{4}D^{-1}u_{yy} = 0. \tag{6.41}$$

Using $DD^{-1} = 1$, (6.41) can be transformed into the first equation (6.40). mKP equation can also be yielded from Zakharov-Shabatequation (4.22) with $m = 2, n = 3$.

The applications of PDO to KP hierarchy [14] and mKP hierarchy [15] have been discussed. In [14] and [15], graded Lie algebra is exhibited for KP hierarchy and mKP hierarchy. The results are the same as (6.34) and (6.41), respectively in form, except for slightly different coefficients. Furthermore, in [14], if

$$A_m = P(\underbrace{0, \dots, 0}_{m}, \frac{1}{3}(-\sqrt{3})^{m-1}),$$

as $m = 3$, the result is exactly (6.34).

Example 6.5 $2 + 1$ dimensional Dym equation.

$2 + 1$ dimensional Dym hierarchy is also a nonstandard case, and is associated with the following PDO

$$L = uD + u_0 + u_1 D^{-1} + u_2 D^{-2} + u_3 D^{-3} + u_4 D^{-4} + \cdots . \tag{6.42}$$

The differential operators B_m are taken as

$$B_m = (L^m)_{\geqslant 2}. \tag{6.43}$$

As $m = 2, 3$, let $u_0 = v, u_1 = w, u_2 = r$ and $t_1 = y, t_2 = t$. To calculate $2 + 1$ dimensional Dym equation, the truncation order of L is obtained when B_3 does not change, and

$$\begin{aligned} B_2 &= (L^2)_{\geqslant 2} = u^2 D^2, \\ B_3 &= (L^3)_{\geqslant 2} = u^3 D^3 + 3u^2(v + u_x)D^2. \end{aligned} \tag{6.44}$$

The following equation system is obtained from Zakharov-Shabat equation

$$\begin{aligned} &(6r_x + 3w_{xx})u^4 + [12ru_x + 6(wv)_x + 3u_x w_x - 3wu_{xx}]u^3 - (3wu_x^2 + 3w_y - 6vwu_x + \\ &\quad 3v_{xy} + u_{xxy})u^2 + [2v_t - 6wu_y - 6vv_y - 3(vu_x)_y - 6u_y v_x + u_{xt} - 2(u_x u_y)_x]u - \\ &\quad u_y u_x^2 + 2vu_t - 3v^2 u_y + (u_t - 3vu_y)u_x = 0, \\ &(3r_{xx} + w_{xxx})u^4 - 6(w - v_x)w_x u^3 + (6v + 15u_x)r_x u^3 + (6r - 2w_x)u_{xx}u^3 + \\ &\quad ![3wv_{xx} + (6v + 3u_x)w_{xx} - 2wu_{xxx}]u^3 + [12ru_x^2 - 3r_y - 6w^2 u_x + 12vru_x + \\ &\quad (6v + 9u_x)wv_x + (6v^2 - 2u_x^2 + 12vu_x)w_x - 3w_{xy} - 8wu_x u_{xx} - v_{xxy}]u^2 \\ &\quad (-2wu_x^3 + 2w_t - 6ru_y - 6wv_y - 6vw_y + 6v^2 wu_x - 3v_y v_x - 6u_y w_x + v_{xt} - \\ &\quad v_x u_{xy} - 3vv_{xy} - u_x v_{xy} - 2u_y v_{xx})u + 2wu_t + 2vv_t - 6vwu_y - 3v^2 v_y + \\ &\quad u_t v_x - 3vu_y v_x - u_y u_x v_x = 0. \end{aligned} \tag{6.45}$$

Under the order $u < v < w < r$ and $x < y < t$, one can calculate d-char-set of (6.45). The d-char-set of (6.45) is so complicated that it is not presented wholly, and only the first equation, i.e., $2 + 1$ dimensional Dym equation, is presented

$$\begin{aligned} &(u_{xy}u_{xxxxx} - u_{xxy}u_{xxxx})u^7 + (2u_{xy}u_{xx}u_{xxx} - 2u_x u_{xxy}u_{xxx} + 6u_x u_{xy}u_{xxxx} + u_y u_{xx}u_{xxxx} - \\ &\quad u_y u_x u_{xxxxx})u^6 + 6u_x^2(u_{xy}u_{xxx} - u_y u_{xxxx})u^5 + (-6u_y u_{xxx}u_x^3 - 4u_{xy}u_{xxt} + 4u_{xt}u_{xxy})u^4 + \\ &\quad (3u_{xy}u_{xyy} - 4u_y u_{xt}u_{xx} + 4u_t u_{xy}u_{xx} + 4u_y u_x u_{xxt} - 3u_{xyy}u_{xxy} - 4u_t u_x u_{xxy})u^3 + \\ &\quad (6u_y u_{xy}^2 - 3u_y u_x u_{xyy} + 3u_y u_{xyy}u_{xx})u^2 - 12u_y^2 u_x u_{xy}u + 6u_y^3 u_x^2 = 0. \end{aligned} \tag{6.46}$$

Equation (6.46) can also be derived from the generalized Lax equation. Dym equation has been discussed in [9, 16]. The results of B_2 and B_3 are all the same as (6.44), and it can be verified that (6.46) is identical with $2+1$ dimensional Dym equation

$$4u_t = u^3 u_{xxx} - 3\frac{1}{u}(u^2 D^{-1}(\frac{1}{u})_y)_y, \tag{6.47}$$

which is presented in [9].

In (6.41) and (6.47), $D^{-1} = \int f(x)\mathrm{d}x$ is the integration, and can be transformed into derivative form through $DD^{-1} = 1$. So using differential characteristic set method, the Lax representation of nonlinear evolution equations with integration can be obtained too. Although the form of the final result is more complicated, differential characteristic set method is a general and effective way for PDEs, especially for nonlinear PDEs.

7. Conclusion and discussion

Differential characteristic set method can be applied to Lax representation of nonlinear evolution equations. It not only decreases the steps and the burden of calculation, then simplifies the calculations, but also provides a complete procedure for PDOs and the construction of the Lax representation of nonlinear evolution on computer algebra system. Further, based on the procedure, differential characteristic set method can be applied to the research of bi-Hamilton structure, tan-functions, Darboux transformations, and other sides of the integrable systems.

Utilizing differential characteristic set method reduces the nonlinear partial equation systems derived from the generalized Lax equation or Zakharov-Shabat equation. There is a shortage that the results are complicated sometimes. How to simplify the results should be revisited.

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