A subdivision vertex-edge corona \( G_1^S \circ (G_2^V \cup G_3^E) \) is a graph that consists of \( S(G_1) \), \(|V(G_1)|\) copies of \( G_2 \) and \(|I(G_1)|\) copies of \( G_3 \) by joining the \( i \)-th vertex in \( V(G_1) \) to each vertex in the \( i \)-th copy of \( G_2 \) and \( i \)-th vertex of \( I(G_1) \) to each vertex in the \( i \)-th copy of \( G_3 \). In this paper, we determine the normalized Laplacian spectrum of \( G_1^S \circ (G_2^V \cup G_3^E) \) in terms of the corresponding normalized Laplacian spectra of three connected regular graphs \( G_1, G_2 \) and \( G_3 \). As applications, we construct some non-regular normalized Laplacian cospectral graphs. In addition, we also give the multiplicative degree-Kirchhoff index, the Kemeny’s constant and the number of the spanning trees of \( G_1^S \circ (G_2^V \cup G_3^E) \) on three regular graphs.

**Keywords** normalized Laplacian spectrum; cospectral graphs; spanning trees; subdivision vertex-edge corona

**MR(2010) Subject Classification** 05C50

1. Introduction

Throughout this paper, all graphs considered are simple undirected and connected. Let \( G = (V(G), E(G)) \) be a graph with vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \) and edge set \( E(G) = \{e_1, e_2, \ldots, e_m\} \), where \(|V(G)| = n\) and \(|E(G)| = m\). Let \( d_G(v_i) \) be the degree of the vertex \( v_i \) in \( G \). The incidence matrix of \( G \), denoted by \( R(G) \), is the \( n \times m \) matrix whose \((i, j)\)-entry is 1 if \( v_i \) and \( e_j \) are adjacent in \( G \) and 0 otherwise. As usual, we denote by \( A(G) \) and \( D(G) \) the adjacency matrix and the degree diagonal matrix of \( G \), respectively. The Laplacian matrix of \( G \) is \( L(G) = D(G) - A(G) \) and the signless Laplacian matrix of \( G \) is \( Q(G) = D(G) + A(G) \). Chung [1] introduced the normalized Laplacian matrix of \( G \), denoted by \( \mathcal{L}(G) = D^{-1/2}(G)(D(G) - A(G))D^{-1/2}(G) = I - D^{-1/2}(G)A(G)D^{-1/2}(G) \), which is a square matrix with rows and columns being indexed by vertices of \( G \). The \( \mathcal{L} \)-characteristic polynomial of \( G \) is defined as \( \Phi_{\mathcal{L}(G)}(\lambda) = \det(\lambda I - \mathcal{L}(G)) \). Since \( \mathcal{L}(G) \) is real symmetric, their eigenvalues are real number. The multiset of eigenvalues of \( \mathcal{L}(G) \) is called the \( \mathcal{L} \)-spectrum of \( G \) and the \( \mathcal{L} \)-eigenvalues are arranged as \( 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq 2 \). Graphs \( G \) and \( H \) are said to be \( A \)-cospectral (resp., \( \mathcal{L} \)-cospectral) if they share the same \( A \)-spectrum (resp., \( \mathcal{L} \)-spectrum). Furthermore, \( K_n \) and \( P_n \) denote, respectively, the complete graph and the path on \( n \) vertices.
Graph operations are becoming increasingly useful mathematical models for a broad range of applications, such as complex systems theory, computer security [2], and so on. Recently, many graph operations such as the disjoint union, the corona, the edge corona and the neighborhood corona have been introduced, and their adjacency, Laplacian and signless Laplacian spectra are computed in [3-8], respectively. Banerjee [9] investigated how the normalized Laplacian spectrum is affected by operations like joining. For the aspect of the \( L \)-cospectral spectrum, Butler [10] produced large families of non-bipartite, non-regular graphs which are mutually \( L \)-cospectral. In 2016, Song [11] obtained the \( A \)-spectrum and \( L \)-spectrum by graph operation of the subdivision vertex-edge corona \( G^S \circ (G^V \cup G^E) \), which is the graph described below.

For a graph \( G_1 \), let \( S(G_1) \) be the subdividing graph of \( G_1 \) whose vertex set has two parts: one the original vertices \( V(G_1) \), another, denoted by \( I(G_1) \), the inserting vertices corresponding to the edges of \( G_1 \). Let \( G_2 \) and \( G_3 \) be other two disjoint graphs.

**Definition 1.1** ([11]) The subdivision vertex-edge corona (briefly \( SVE \)-corona) of \( G_1 \) with \( G_2 \) and \( G_3 \), denoted by \( G^S_1 \circ (G^V_2 \cup G^E_3) \), is the graph consisting of \( S(G_1) \), \(|V(G_1)|\) copies of \( G_2 \) and \(|I(G_1)|\) copies of \( G_3 \) by joining the \( i \)-th vertex in \( V(G_1) \) to each vertex in the \( i \)-th copy of \( G_2 \) and \( i \)-th vertex of \( I(G_1) \) to each vertex in the \( i \)-th copy of \( G_3 \). (for example, see \( P^S_4 \circ (P^V_3 \cup P^E_2) \) in Figure 1)

\[
\begin{align*}
P_4 & \quad P_5 \quad P_2 \\
\begin{array}{c}
\begin{array}{c}
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\end{array}
\end{array}
\end{array}
\end{array}
\end{align*}
\]

Figure 1 \( P^S_4 \circ (P^V_3 \cup P^E_2) \)

One can easily check that \( G^S_1 \circ (G^V_2 \cup G^E_3) \) has \( n = n_1 + m_1 + n_1n_2 + m_1n_3 \) vertices and \( m = 2m_1 + n_1n_2 + m_1n_3 + n_1m_2 + m_1m_3 \) edges, where \( n_i \) and \( m_i \) are the number of vertices and edges of \( G_i \) for \( i = 1, 2, 3 \). We see that \( G^S_1 \circ (G^V_2 \cup G^E_3) \) will be a subdivision-vertex corona if \( G_3 \) is null, and will be a subdivision-edge corona if \( G_2 \) is null. Thus subdivision vertex-edge corona can be viewed as the generalizations of both subdivision-vertex corona (denoted by \( G_1 \circ G_2 \)) (see [12]) and subdivision-edge corona (denoted by \( G_1 \circ G_2 \)).

Calculating the spectra of graphs as well as formulating the characteristic polynomials of graphs is a fundamental and very meaningful work in spectral graph theory. In this paper, we determine the normalized Laplacian spectrum of \( G^S_1 \circ (G^V_2 \cup G^E_3) \) in terms of the corresponding normalized Laplacian spectra of three connected regular graphs \( G_1, G_2 \) and \( G_3 \). As applications, we construct some non-regular normalized Laplacian cospectral graphs. In addition, we also give the multiplicative degree-Kirchhoff index, the Kemeny’s constant and the number of the spanning trees of \( G^S_1 \circ (G^V_2 \cup G^E_3) \) for three regular graphs.
2. Preliminaries

In this section we give some useful established results which are required in the proof of the main result.

**Lemma 2.1** ([3]) For a graph \( G \), let \( R(G) \) be the incidence matrix of \( G \). Then
\[
R(G)R(G)^T = D(G) + A(G) = Q(G).
\]

**Lemma 2.2** ([13]) Let \( M_1, M_2, M_3, M_4 \) be respectively \( p \times p, p \times q, q \times p, q \times q \) matrices with \( M_1 \) and \( M_4 \) invertible. Then
\[
\det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_4) \cdot \det(M_1 - M_2M_4^{-1}M_3)
\]

where \( M_1 - M_2M_4^{-1}M_3 \) and \( M_4 - M_3M_1^{-1}M_2 \) are called the Schur complements of \( M_4 \) and \( M_1 \).

**Lemma 2.3** ([14]) The Kronecker product \( A \otimes B \) of two matrices \( A = (a_{ij})_{m \times n} \) and \( B = (b_{ij})_{p \times q} \) is the \( mp \times nq \) matrix obtained from \( A \) by replacing each element \( a_{ij} \) by \( a_{ij}B \). It is known that:

(a) \( (M \otimes P)(N \otimes Q) = MN \otimes PQ \), for matrices \( M, N, P, Q \) of suitable sizes;

(b) \( (M \otimes N)^{-1} = M^{-1} \otimes N^{-1} \), for non-singular matrices \( M \) and \( N \);

(c) \( \det(M \otimes N) = (\det M)^s(\det N)^k \), where \( M \) is a matrix of order \( k \) and \( N \) is a matrix of order \( s \);

(d) \( (M \otimes N)^T = M^T \otimes N^T \), for any two matrices \( M \) and \( N \).

The reader is referred to [14] for other properties of the Kronecker product not mentioned here.

**Definition 2.4** ([14]) For two matrices \( A = (a_{ij})_{m \times n} \) and \( B = (b_{ij})_{m \times n} \), the Hadamard product \( A \bullet B \) is a matrix of size \( m \times n \) with entries given by
\[
(A \bullet B)_{ij} = a_{ij} \cdot b_{ij}.
\]

**Definition 2.5** ([15]) Let matrix \( B = cJ_n - (c - 1)I_n \) where \( c \) is a constant and \( J_n \) denotes the matrix of size \( n \) whose entry equal to one, and \( C \) denotes the column vector of dimension \( n \), respectively. For the regular graph \( G \) with \( n \) vertices and parameter \( \lambda \), we have
\[
\chi_G(B, C, \lambda) = C^T(\lambda I_n - (\mathcal{L}(G) \bullet B(G)))^{-1}C,
\]
where the notion \( \chi_G(B, C, \lambda) \) is similar to the notion ‘coronal’ in [13].

**Definition 2.6** ([16]) The multiplicative degree-Kirchhoff index of \( G \) is defined as:
\[
Kf^+(G) = \sum_{i<j} d_id_j r_{ij},
\]
where \( r_{ij} \) denotes the resistance distance between \( v_i \) and \( v_j \). It has been proved [16] that \( Kf^+(G) \) can be expressed by the edge number \( m \) and the normalized Laplacian spectrum \( \text{Spec}_\mathcal{L}(G) = \)
\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ below:

$$Kf^*(G) = 2m \sum_{k=2}^{n} \frac{1}{\lambda_k}, \text{ where } 0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_n \leq 2.$$ 

For a graph $G$, Kemeny’s constant $K(G)$, is the expected number of steps required for the transition from a starting vertex $i$ to a destination vertex, which is independent of the selection of starting vertex $i$ (see [17]). Moreover, Kemeny’s [18] constant can be computed from the normalized Laplacian spectrum as follows:

$$K(G) = \sum_{k=2}^{n} \frac{1}{\lambda_k}.$$

3. Main results

In this section, we present the normalized Laplacian matrix, $\mathcal{L}$-spectra and some applications of subdivision vertex-edge corona for three regular graphs. For convenience, let $\eta_i, \mu_j$ and $\theta_k$ be an eigenvalue of $\mathcal{L}(G_1), \mathcal{L}(G_2)$ and $\mathcal{L}(G_3)$, respectively.

For $i = 1, 2, 3$, let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges. First we label the vertices of $G = G_1^r \circ (G_2^v \cup G_3^F)$: $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$, $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$ and $V(G_3) = \{w_1, w_2, \ldots, w_{n_3}\}$; for $i = 1, 2, \ldots, n_1$, let $U_i = \{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ denote the vertices of the $i$-th copy of $G_2$ in $G$, and $W_j = \{w_1^j, w_2^j, \ldots, w_{n_3}^j\}$ ($j = 1, 2, \ldots, m_1$) the $j$-th copy of $G_3$ in $G$. Then the vertices of $G$ are partitioned by

$$V(G_1) \cup I(G_1) \cup (U_1 \cup U_2 \cup \cdots \cup U_{n_1}) \cup (W_1 \cup W_2 \cup \cdots \cup W_{m_1}).$$

Clearly, the degrees of the vertices of $G = G_1^r \circ (G_2^v \cup G_3^F)$ are:

$$d_G(v_i) = d_{G_1}(v_i) + n_2, \quad i = 1, 2, \ldots, n_1;$$

$$d_G(e_i) = n_3 + 2, \quad i = 1, 2, \ldots, m_1;$$

$$d_G(u_j^i) = d_{G_2}(u_j) + 1, \quad j = 1, 2, \ldots, n_2, \quad i = 1, 2, \ldots, n_1;$$

$$d_G(w_j^i) = d_{G_3}(w_j) + 1, \quad j = 1, 2, \ldots, n_3, \quad i = 1, 2, \ldots, m_1.$$ 

**Theorem 3.1** Let $G = G_1^r \circ (G_2^v \cup G_3^F)$. If $G_i$ is an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges ($i = 1, 2, 3$), then

$$\mathcal{L}(G) = \begin{pmatrix}
I_{n_1} & -aR(G_1) & -I_{n_1} \otimes b_{n_2}^T & O_{n_1 \times n_1; n_2} & O_{n_1 \times n_1; n_3} \\
-aR(G_1)^T & I_{n_1} & O_{n_1 \times n_2} & -I_{n_1} \otimes c_{n_3}^T & O_{n_1 \times n_1; n_3} \\
-I_{n_1} \otimes b_{n_2} & O_{n_1; n_2 \times n_1} & I_{n_1} \otimes (\mathcal{L}(G_2) \cdot B(G_2)) & O_{n_1; n_2 \times n_3} & O_{n_1 \times n_1; n_3} \\
O_{n_2; n_2 \times n_1} & -I_{n_1} \otimes c_{n_3} & O_{n_2; n_3 \times n_1} & I_{n_1} \otimes (\mathcal{L}(G_3) \cdot B(G_3)) & O_{n_1 \times n_2; n_3}
\end{pmatrix},$$

where $b_{n_2}$ and $c_{n_3}$ are the column vector of size $n_2$ and $n_3$ with all entries equal to \(\frac{1}{\sqrt{(r_1+n_2)(r_2+1)}}\) and \(\frac{1}{\sqrt{(n_3+2)(r_3+1)}}\), respectively. $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and
The normalized Laplacian spectrum of subdivision vertex-edge corona for graphs

off-diagonal entries are $\frac{r}{r_2+1}$, $B(G_3)$ is the $n_3 \times n_3$ matrix whose all diagonal entries are 1 and off-
diagonal entries are $\frac{r}{r_2+1}$. $O$ is zero matrix and $a$ is the constant whose value is $\frac{1}{\sqrt{(r_1+n_2)(n_3+2)}}$.

**Proof** The adjacency matrix and the degree diagonal matrix of $G_3^L \circ (G_1^V \cup G_3^E)$ can be represented
in the form of block-matrix according to the ordering of $V(G_1)$, $I(G_1)$, $U_1$, $\ldots$, $U_{n_1}$, $W_1$, $\ldots$, $W_{m_1}$
as follows:

$$
A(G) = \begin{pmatrix}
O_{n_1 \times n_1} & R(G_1) & I_{n_1} \otimes 1_{n_2} & O_{n_1 \times n_1 n_3} \\
R(G_1)^T & O_{m_1 \times m_1} & O_{m_1 \times n_2 n_1} & I_{m_1} \otimes 1_{n_3} \\
I_{n_1} \otimes 1_{n_2} & O_{n_1 n_2 \times m_1} & I_{n_1} \otimes A(G_2) & O_{n_1 n_2 \times m_1 n_3} \\
O_{m_1 n_3 \times n_1} & I_{m_1} \otimes 1_{n_3} & O_{m_1 n_3 \times n_1 n_2} & I_{m_1} \otimes A(G_3)
\end{pmatrix},
$$

where $1_{n_2}$ is the column vector of size $n_2$ with all entries equal to 1.

$$
D(G) = \begin{pmatrix}
(n_1 + n_2)I_{n_1} \\
(n_3 + 2)I_{m_1} \\
(r_2 + 1)I_{n_1 n_2} \\
(r_3 + 1)I_{m_1 n_3}
\end{pmatrix}.
$$

Since $G_2$ is an $r_2$-regular graph, we have $L(G_2) = I_{n_2} - \frac{1}{r_2} A(G_2)$. So

$$
L(G_2) \bullet B(G_2) = (I_{n_2} - \frac{1}{r_2} A(G_2)) \bullet B(G_2) = I_{n_2} - \frac{1}{r_2 + 1} A(G_2).
$$

Thus

$$
I_{n_1 n_2} = \frac{1}{r_2 + 1} I_{n_1} \otimes A(G_2) = I_{n_1} \otimes (L(G_2) \bullet B(G_2)).
$$

Furthermore, we can obtain that

$$
I_{m_1 n_3} = \frac{1}{r_3 + 1} I_{m_1} \otimes A(G_3) = I_{m_1} \otimes (L(G_3) \bullet B(G_3)).
$$

By $L(G) = I - D(G)^{-1/2} A(G) D(G)^{-1/2}$, the required normalized Laplacian matrix is given below:

$$
L(G) = \begin{pmatrix}
I_{n_1} & -aR(G_1) & -I_{n_1} \otimes b_{n_2}^T & O_{n_1 \times n_1 n_3} \\
-aR(G_1)^T & I_{m_1} & O_{m_1 \times n_1 n_2} & -I_{m_1} \otimes c_{n_3}^T \\
-I_{n_1} \otimes b_{n_2} & O_{n_1 n_2 \times m_1} & I_{n_1} \otimes (L(G_2) \bullet B(G_2)) & O_{n_1 n_2 \times m_1 n_3} \\
O_{m_1 n_3 \times n_1} & -I_{m_1} \otimes c_{n_3} & O_{m_1 n_3 \times n_1 n_2} & I_{m_1} \otimes (L(G_3) \bullet B(G_3))
\end{pmatrix}.
$$

**Theorem 3.2** Let $G = G_3^L \circ (G_1^V \cup G_3^E)$. If $G_i$ is an $r_i$-regular graph with $n_i$ vertices and $m_i$
edges ($i = 1, 2, 3$), then the normalized Laplacian spectrum of $G$ consists of:

(a) $\frac{1+2r_1}{r_2+1}$ repeated $n_1$ times for each eigenvalue $\mu_j$ of $L(G_2)$, $j = 2, 3, \ldots, n_2$;

(b) $\frac{1+2r_2}{r_3+1}$ repeated $m_1$ times for each eigenvalue $\theta_k$ of $L(G_3)$, $k = 2, 3, \ldots, n_3$;

(c) two roots of the equation $(n_3 r_3 + n_3 + 2r_3 + 2)\lambda^2 - (n_3 r_3 + 2n_3 + 2r_3 + 4)\lambda + 2 = 0$,

where each root repeats $m_1 - n_1$ times;

(d) four roots of the equation

$$(r_1+n_2)(n_3+2)((n_3+2)(n_3+1)\lambda^2 - (n_3+2)(r_3+2)\lambda + 2)((r_1+n_2)(r_2+1)\lambda^2 -$$
From Lemma 2.1, we can obtain that
\[(r_1 + n_2)(r_2 + 2)\lambda + r_1) - r_1(2 - \eta_1)((n_3 + 2)(r_3 + 1)\lambda -
\[= (n_3 + 2)((r_1 + n_2)(r_2 + 1)\lambda - (r_1 + n_2)) = 0,
\]
where each eigenvalue $\eta_i$ of $\mathcal{L}(G_1)$, $i = 1, 2, \ldots, n_1$.

**Proof** According to Theorem 3.1, the normalized Laplacian characteristic polynomial of $G^S_1 \circ (G^Y_2 \cup G^F_3)$ is
\[
\Phi_{\mathcal{L}(G)}(\lambda) = \det(\lambda I_n - \mathcal{L}(G)) = \det(B_0),
\]
where
\[
B_0 = \begin{pmatrix}
(\lambda - 1) I_{n_1} & aR(G_1) & I_{n_1} \otimes b_n^T & O \\
O & (\lambda - 1) I_{m_1} & O & I_{m_1} \otimes c_n^T \\
I_{n_1} \otimes b_n^T & O & I_{n_1} \otimes (\lambda I_{n_2} - \mathcal{L}(G_2) \bullet B(G_2)) & O \\
I_{m_1} \otimes c_n^T & O & O & I_{m_1} \otimes (\lambda I_{n_3} - \mathcal{L}(G_3) \bullet B(G_3))
\end{pmatrix}.
\]

Denote by $X$ the elementary block matrices below,
\[
X = \begin{pmatrix}
I_{n_1} & O & -I_{n_1} \otimes (b_n^T(I_{n_2} - \mathcal{L}(G_2) \bullet B(G_2))^{-1}) & O \\
O & I_{m_1} & O & -I_{m_1} \otimes (c_n^T(I_{n_3} - \mathcal{L}(G_3) \bullet B(G_3))^{-1}) \\
O & O & I_{n_1} \otimes I_{n_2} & I_{m_1} \otimes I_{n_3}
\end{pmatrix}.
\]

Let $B = XB_0$. Then
\[
B = \begin{pmatrix}
(\lambda - 1 - \chi_2) I_{n_1} & aR(G_1) & O & O \\
anR(G_1)^T & (\lambda - 1 - \chi_3) I_{m_1} & O & O \\
I_{n_1} \otimes b_n^T & I_{m_1} \otimes c_n^T & I_{n_1} \otimes (\lambda I_{n_2} - \mathcal{L}(G_2) \bullet B(G_2)) & O \\
O & O & O & I_{m_1} \otimes (\lambda I_{n_3} - \mathcal{L}(G_3) \bullet B(G_3))
\end{pmatrix},
\]

where $\chi_2$ and $\chi_3$ refer to $\chi_{G_2}(B(G_2), b_{n_2}, \lambda)$ and $\chi_{G_3}(B(G_3), c_{n_3}, \lambda)$, respectively.

Set
\[
S_1 = \begin{pmatrix}
(\lambda - 1 - \chi_2) I_{n_1} & aR(G_1) \\
anR(G_1)^T & (\lambda - 1 - \chi_3) I_{m_1}
\end{pmatrix}.
\]

By applying Lemma 2.2, the result follows from
\[
\det(S_1) = \begin{vmatrix}
(\lambda - 1 - \chi_2) I_{n_1} & aR(G_1) \\
anR(G_1)^T & (\lambda - 1 - \chi_3) I_{m_1}
\end{vmatrix} = \det((\lambda - 1 - \chi_2) I_{n_1} - \frac{a^2}{\lambda - 1 - \chi_3} R(G_1) R(G_1)^T)
\]
\[
= (\lambda - 1 - \chi_3)^{m_1} \det((\lambda - 1 - \chi_2) I_{n_1} - \frac{a^2}{\lambda - 1 - \chi_3} R(G_1) R(G_1)^T)
\]
\[
= (\lambda - 1 - \chi_3)^{m_1} \det((\lambda - 1 - \chi_2) I_{n_1} - \frac{a^2}{\lambda - 1 - \chi_3} R(G_1) R(G_1)^T)
\]
\[
= (\lambda - 1 - \chi_3)(\lambda - 1 - \chi_2) I_{n_1} - a^2 r_1(2 I_{n_1} - \mathcal{L}(G_1))
\]
\[
= (\lambda - 1 - \chi_3)(\lambda - 1 - \chi_2) I_{n_1} - \frac{r_1(2 - \eta_1)}{(r_1 + n_2)(n_3 + 2)}.
\]

From Lemma 2.1, we can obtain that $R(G_1) R(G_1)^T = A(G_1) + r_1 I_{n_1}$. Combining the equation $A(G_1) = r_1 (I_{n_1} - \mathcal{L}(G_1))$, we get
\[
R(G_1) R(G_1)^T = r_1(2 I_{n_1} - \mathcal{L}(G_1)).
\]
As \( \mathcal{L}(G_2) \cdot B(G_2) = I_{n_2} - \frac{1}{r_2 + 1} A(G_2) \), we get
\[
\mathcal{L}(G_2) \cdot B(G_2) = \frac{1}{r_2 + 1} (I_{n_2} + r_2 \mathcal{L}(G_2)).
\]

Obviously, we have \( \mathcal{L}(G_3) \cdot B(G_3) = \frac{1}{r_3 + 1} (I_{n_3} + r_3 \mathcal{L}(G_3)) \).

Since \( (\mathcal{L}(G_2) \cdot B(G_2)) b_{n_2} = (I_{n_2} - \frac{1}{r_2 + 1} A(G_2)) b_{n_2} = (1 - \frac{r_2}{r_2 + 1}) b_{n_2} = \frac{1}{r_2 + 1} b_{n_2} \), we have
\[
(\lambda I_{n_2} - (\mathcal{L}(G_2) \cdot B(G_2))) b_{n_2} = (\lambda - \frac{1}{r_2 + 1}) b_{n_2}. \]

Also, \( b^T_n b_{n_2} = \frac{n_2}{(r_1 + n_2)(r_2 + 1)} \).

Moreover, the sum of all entries on every row of matrix \( \mathcal{L}(G_2) \cdot B(G_2) \) is \( \frac{1}{r_2 + 1} \), so
\[
\chi_2 = \beta^T n_2 (\lambda I_{n_2} - \mathcal{L}(G_2) \cdot B(G_2))^{-1} b_{n_2} = \frac{\beta^T n_2 b_{n_2}}{\lambda - \frac{1}{r_2 + 1}} = \frac{n_2}{(r_1 + n_2)(r_2 + 1)(\lambda - \frac{1}{r_2 + 1})}.
\]

The value of \( \chi_3 \) is similar to that of \( \chi_2 \), so
\[
\chi_3 = \alpha^T n_3 (\lambda I_{n_3} - \mathcal{L}(G_3) \cdot B(G_3))^{-1} c_{n_3} = \frac{\alpha^T n_3 c_{n_3}}{\lambda - \frac{1}{r_3 + 1}} = \frac{n_3}{(n_3 + 2)(r_3 + 1)(\lambda - \frac{1}{r_3 + 1})}.
\]

Note that \( \det(X) = 1 \). Then
\[
\Phi_{\mathcal{L}(G)}(\lambda) = \det(B_0) = \det(X^{-1}) \det(B) = \det(B),
\]
where
\[
\det(B) = \det(I_{n_1} \oplus (\lambda I_{n_2} - \mathcal{L}(G_2) \cdot B(G_2))) \cdot \det(I_{n_1} \oplus (\lambda I_{n_3} - \mathcal{L}(G_3) \cdot B(G_3))) \cdot \det(S_i).
\]

In summary, the normalized Laplacian characteristic polynomial of \( G_2^S \circ (G_2^V \cup G_3^E) \) is
\[
\Phi_{\mathcal{L}(G)}(\lambda) = \prod_{j=1}^{r_2} (\lambda - \frac{1}{r_2 + 1})^{n_1} \cdot \prod_{k=1}^{n_3} (\lambda - \frac{1}{r_3 + 1})^{m_1} \cdot \det(S_i)
\]
\[
= (\lambda - 1 - \frac{n_3}{(n_3 + 2)(r_3 + 1)(\lambda - \frac{1}{r_3 + 1})} )^{m_1 - n_1} \cdot \prod_{j=1}^{r_2} (\lambda - \frac{1}{r_2 + 1})^{n_1},
\]
\[
\prod_{k=1}^{n_3} (\lambda - \frac{1}{r_3 + 1})^{m_1} \cdot \prod_{i=1}^{n_3} ((\lambda - 1 - \frac{n_3}{(n_3 + 2)(r_3 + 1)(\lambda - \frac{1}{r_3 + 1})} )^{n_3} - \frac{r_1(2 - \eta_i)}{(r_1 + n_2)(r_2 + 1)}).
\]

(1) From the above we see that \( a \) and \( b \) are obtained, for \( \frac{1}{r_2 + 1} \) and \( \frac{1}{r_3 + 1} \) are extreme point of \( \chi_2 \) and \( \chi_3 \), respectively.

(2) Besides, the 2 eigenvalues are obtained from the equation
\[
\lambda - 1 - \frac{n_3}{(n_3 + 2)(r_3 + 1)(\lambda - \frac{1}{r_3 + 1})} = 0,
\]

and the eigenvalues repeat \( m_1 - n_1 \) in (c).

(3) The remaining 4\( n_1 \) eigenvalues of \( G_2^S \circ (G_2^V \cup G_3^E) \) are obtained by solving
\[
((\lambda - 1)(\lambda - \frac{1}{r_3 + 1}) - \frac{n_3}{(n_3 + 2)(r_3 + 1)})(\lambda - 1)(\lambda - \frac{1}{r_2 + 1}) - \frac{n_2}{(r_1 + n_2)(r_2 + 1)} - \frac{r_1(2 - \eta_i)}{(r_1 + n_2)(n_3 + 2)} = 0.
\]
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for each \( i = 1, 2, \ldots, n_1 \), and this yields the eigenvalues in (d). □

**Remark 3.3** By Theorem 3.2, we observe that the normalized Laplacian spectrum of \( G_1^S \circ (G^V_2 \cup G^E_3) \) depends on the degrees of regularities, number of vertices, number of edges and normalized Laplacian eigenvalues of regular graph \( G_i \) (\( i = 1, 2, 3 \)).

**Example 3.4** One can easily see that the normalized Laplacian eigenvalues of \( K_4 \) are 0 and \( \frac{4}{3} \) (multiplicity 3). The normalized Laplacian eigenvalues of \( K_3 \) are 0 and \( \frac{2}{3} \) (multiplicity 2). The normalized Laplacian eigenvalues of \( K_2 \) are 0 and 2. Let \( G_1 = K_4, G_2 = K_3 \) and \( G_3 = K_2 \). Then we consider the normalized Laplacian spectrum of \( G_1^S \circ (G^V_2 \cup G^E_3) \) (see Figure 2).

From Theorem 3.2, the normalized Laplacian spectrum of \( K_4^S \circ (K^V_3 \cup K^E_2) \) consists of: \( \frac{4}{3} \) (multiplicity 8), \( \frac{2}{3} \) (multiplicity 6), each root of the equation \( 4\lambda^2 - 6\lambda + 1 = 0 \) with multiplicity 2 (that is \( \frac{3 \pm \sqrt{5}}{4} \) (multiplicity 2)), each root of the equation \( 144\lambda^4 - 408\lambda^3 + 336\lambda^2 - 74\lambda + 4 = 0 \) with multiplicity 3, four roots of the equation \( 144\lambda^4 - 408\lambda^3 + 312\lambda^2 - 54\lambda = 0 \) (including 0 eigenvalue).

**Theorem 3.5** If \( G_i \) and \( H_i \) (not necessarily distinct) (\( i = 1, 2, 3 \)) are cospectral regular graphs, then \( G_1^S \circ (G^V_2 \cup G^E_3) \) and \( H_1^S \circ (H^V_2 \cup H^E_3) \) are \( \mathcal{L} \)-cospectral graphs.

**Proof** For an \( r \)-regular graph \( G \), we have \( \mathcal{L}(G) = I_n - \frac{1}{r}A(G) \). In other words, the normalized Laplacian spectrum of regular graph is determined by their adjacency spectrum. Since \( G_i \) and \( H_i \) (\( i = 1, 2, 3 \)) are cospectral regular graphs, \( G_i \) and \( H_i \) are \( \mathcal{L} \)-cospectral graphs. From Remark 3.3, the subdivision vertex-edge corona graphs in the theorem statement must then be \( \mathcal{L} \)-cospectral. □

**Example 3.6** Using MATLAB 7.0 software we obtain the two cospectral graphs \( G_1 \) and \( H_1 \).
The normalized Laplacian spectrum of subdivision vertex-edge corona for graphs

(see Figure 3) on 14 vertices. The $L$-characteristic polynomial of $G_1$ and $H_1$ is

$$\Phi_{G_1}(\lambda) = \Phi_{H_1}(\lambda) = \lambda^{14} - 14\lambda^{13} + \frac{266}{3}\lambda^{12} - \frac{9068}{27}\lambda^{11} + \frac{26270}{31}\lambda^{10} - \frac{49541}{33}\lambda^9 + \frac{155276}{81}\lambda^8 - \frac{30299}{17}\lambda^7 + \frac{16644}{13}\lambda^6 - \frac{24988}{43}\lambda^5 + \frac{11474}{59}\lambda^4 - \frac{7534}{177}\lambda^3 + \frac{1643}{302}\lambda^2 - \frac{233}{763}\lambda.$$

From Theorem 3.5, no regular graphs $G_1^S \circ (K_3^Y \cup K_2^F)$ and $H_1^S \circ (K_3^Y \cup K_2^F)$ (show in Figure 4) are $L$-cospectral graphs.

**Figure 4** $G_1^S \circ (K_3^Y \cup K_2^F)$ and $H_1^S \circ (K_3^Y \cup K_2^F)$

**Theorem 3.7** Let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges, $i = 1, 2, 3$. The multiplicative degree-Kirchhoff index of $G = G_1^S \circ (G_2^Y \cup G_3^F)$ is related as follows:

$$Kf^*(G) = 2(2m_1 + n_1n_2 + m_1n_3 + n_1m_2 + m_1m_3)
\left(\sum_{j=2}^{n_2} \frac{n_1(r_2 + 1)}{1 + r_2\mu_j} + \sum_{k=2}^{n_3} \frac{m_1(r_3 + 1)}{1 + r_3\eta_k} + \frac{\sum_{i=2}^{n_1} 2(r_1 + n_2)(r_2 + 2) + r_1(n_3 + 2)(r_3 + 2) - n_1(n_2 + 1)(r_2 + r_3 + 2)}{r_1\eta_i} + \frac{(n_3r_3 + 2n_3 + 2r_3 + 4)(m_1 - n_1)}{2} + \frac{2(r_2 + 1)(n_2 - r_1r_3) + (n_3 + 2)(r_1r_3 + 1) + (r_1 + n_2)(r_2 + 2)(r_3 + 2))}{2(r_1 + n_2)(r_2 + 2) + r_1(n_3 + 2)(r_3 + 2) - 2r_1(r_2 + r_3 + 2)}\right).$$

**Proof** By Definition 2.6, $Kf^*(G) = 2m \sum_{k=2}^{n} \frac{1}{\lambda_k}$, then the multiplicative degree-Kirchhoff index $Kf^*(G)$ can be computed in the following way:

From Theorem 3.2(c), let $\alpha_1$ and $\alpha_2$ be the eigenvalues of equation

$$(n_3r_3 + n_3 + 2r_3 + 2)\lambda^2 - (n_3r_3 + 2n_3 + 2r_3 + 4)\lambda + 2 = 0. \quad (3.1)$$

By Vieta Theorem, we have

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2} = \frac{n_3r_3 + 2n_3 + 2r_3 + 4}{2}.$$  

In light of Theorem 3.2(d), for $i = 2, 3, \ldots, n_1$, let $\beta_1, \beta_2, \beta_3$ and $\beta_4$ be the eigenvalues of equation

$$(r_1 + n_2)(n_3 + 2)(r_3 + 1)\lambda^2 - (n_3 + 2)(r_3 + 2)\lambda + 2)((r_1 + n_2)(r_2 + 1)\lambda^2 - (r_1 + n_2)(r_2 + 2)\lambda + r_1 - r_1 - n_1(n_3 + 2)(r_3 + 1) - (n_3 + 2)).$$
Let $G$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges, $i = 1, 2, 3$. The Kemeny's constant of $G = G_1^S \circ (G_2^V \cup G_3^F)$ is related as follows:

$$K(G) = \sum_{i=2}^{n_1} \frac{2(r_1 + n_2)(r_2 + 2) + r_1(n_3 + 2)(r_3 + 2) - r_1(2 - \eta_i)(r_2 + r_3 + 2)}{r_1 \eta_i} + \frac{(n_3r_3 + 2n_3 + 2r_3 + 4)(m_1 - n_1)}{2} + \sum_{j=2}^{n_2} \frac{n_1(r_2 + 1)}{1 + r_2 \mu_j} + \sum_{k=2}^{n_3} \frac{m_1(r_3 + 1)}{1 + r_3 \theta_k} + \frac{2(r_2 + 1)(n_2 - r_1r_3) + (n_3 + 2)(r_1(r_3 + 1) + (r_1 + n_2)(r_2 + 2)(r_3 + 2))}{2(r_1 + n_2)(r_2 + 2) + r_1(n_3 + 2)(r_3 + 2) - 2r_1(r_2 + r_3 + 2)}.$$ 

Note that by Definition 2.6, we have $Kf^*(G) = 2E \cdot K(G)$, so the result given in Corollary below is immediate.

**Corollary 3.8** Let $G_i$ be an $r_i$-regular graph with $n_i$ vertices and $m_i$ edges, $i = 1, 2, 3$. The Kemeny's constant of $G = G_1^S \circ (G_2^V \cup G_3^F)$ is related as follows:

$$K(G) = \sum_{i=2}^{n_1} \frac{2(r_1 + n_2)(r_2 + 2) + r_1(n_3 + 2)(r_3 + 2) - r_1(2 - \eta_i)(r_2 + r_3 + 2)}{r_1 \eta_i} + \frac{(n_3r_3 + 2n_3 + 2r_3 + 4)(m_1 - n_1)}{2} + \sum_{j=2}^{n_2} \frac{n_1(r_2 + 1)}{1 + r_2 \mu_j} + \sum_{k=2}^{n_3} \frac{m_1(r_3 + 1)}{1 + r_3 \theta_k} + \frac{2(r_2 + 1)(n_2 - r_1r_3) + (n_3 + 2)(r_1(r_3 + 1) + (r_1 + n_2)(r_2 + 2)(r_3 + 2))}{2(r_1 + n_2)(r_2 + 2) + r_1(n_3 + 2)(r_3 + 2) - 2r_1(r_2 + r_3 + 2)}.$$ 

A known result from Chung [1] allows the calculation of spanning trees from the normalized
Laplacian spectrum and the degrees of all the vertices, that is
\[ t(G) = \frac{\prod_{i=1}^{n} d_i \prod_{j=2}^{n} \lambda_i}{\sum_{i=1}^{n} d_i}. \]
Thus, we give closed formulas for the spanning trees below:

**Corollary 3.9** Let \( G = G^S \circ (G^Y_2 \cup G^E_3) \). If \( G_i \) is an \( r_i \)-regular graph with \( n_i \) vertices and \( m_i \) edges \((i = 1, 2, 3)\), then
\[
t(G) = \frac{\prod_{i=2}^{n} (r_1 \eta_i) \cdot \prod_{j=2}^{n} (1 + r_2 \mu_j) \cdot \prod_{k=2}^{n} (1 + r_3 \theta_k)^m_1 \cdot 2^{m_1 - n_1 - 1} \cdot (r_1 r_3 n_3 + 2 r_1 n_3 + 2 r_2 n_2 + 4 n_2 + 4 r_1)}{2 m_1 + n_1 n_2 + m_1 n_3 + n_1 m_2 + m_1 m_3}.
\]

**Proof** In order to get the result, we consider the normalized Laplacian eigenvalues of \( G \) in the following way:

From Theorem 3.7(3.1), we have
\[ \alpha_1 \alpha_2 = 2 \frac{n_3 r_3 + n_3 + 2 r_3 + 2}{n_3 r_3 + n_3 + 2 r_3 + 2}. \]
By Theorem 3.7(3.2), we have
\[ \beta_1 \beta_2 \beta_3 \beta_4 = \frac{r_3 \eta_1}{(r_1 + n_2)(r_2 + 1)(n_3 + 2)(r_3 + 1)}. \]
By means of Theorem 3.7(3.3), we obtain that
\[ \gamma_1 \gamma_2 \gamma_3 = \frac{r_1 r_3 n_3 + 2 r_1 n_3 + 2 r_2 n_2 + 4 n_2 + 4 r_1}{(r_1 + n_2)(r_2 + 1)(n_3 + 2)(r_3 + 1)}. \]
From the above we see that
\[
t(G) = \frac{\prod_{i=1}^{n} d_i \prod_{j=2}^{n} \lambda_i}{\sum_{i=1}^{n} d_i}
= \frac{(r_1 + n_2)^{n_1} (n_3 + 2)^{m_1} (r_2 + 1)^{n_2} (r_3 + 1)^{m_1 n_3}}{2 (2 m_1 + n_1 n_2 + m_1 n_3 + n_1 m_2 + m_1 m_3)} \left( \prod_{k=2}^{n} \frac{1 + r_3 \theta_k}{r_3 + 1} \right) \left( \prod_{i=2}^{n} \frac{r_3 \eta_i}{r_1 + n_2 + 2 r_1 n_3 (r_2 + 1)(n_3 + 2)(r_3 + 1)} \right) \left( \prod_{j=2}^{n} (1 + r_2 \mu_j)^{n_1} \cdot \prod_{k=2}^{n} (1 + r_3 \theta_k)^{m_1} \cdot 2^{m_1 - n_1 - 1} \cdot (r_1 r_3 n_3 + 2 r_1 n_3 + 2 r_2 n_2 + 4 n_2 + 4 r_1) \right)
= \frac{2 m_1 + n_1 n_2 + m_1 n_3 + n_1 m_2 + m_1 m_3}{2 m_1 + n_1 n_2 + m_1 n_3 + n_1 m_2 + m_1 m_3}.
\]

**Example 3.10** From Example 3.4, for a graph \( K^S_3 \circ (K^Y_3 \cup K^E_2) \) (see Figure 2) we know that
\[
\prod_{i=2}^{n} (r_1 \eta_i) = 64, \quad \prod_{j=2}^{n} (1 + r_2 \mu_j)^{n_1} = 4^8, \quad \prod_{k=2}^{n} (1 + r_3 \theta_k)^{m_1} = 3^6, \quad 2^{m_1 - n_1 - 1} = 2.
\]
Also, \( r_1 r_3 n_3 + 2 r_1 n_3 + 2 r_2 n_2 + 4 n_2 + 4 r_1 = 54 \). The number of edges of graph \( K^S_3 \circ (K^Y_3 \cup K^E_2) \) is \( 2 m_1 + n_1 n_2 + m_1 n_3 + n_1 m_2 + m_1 m_3 = 54 \). Hence,
\[
t(G) = \frac{64 \times 4^8 \times 3^6 \times 2 \times 54}{54} = 1458 \times 4^{11}.
\]
According to Song [11], we know that

$$t(G) = t(K_4) \cdot 2^{n_1 - n_1 + 1} \cdot \prod_{i=2}^{n_2} (1 + \nu_i(K_3))^n_1 \cdot \prod_{i=2}^{n_3} (1 + \nu_i(K_2))^m_1,$$

where $$\nu_i(G)$$ is non-zero Laplacian eigenvalue of $$G$$. So

$$t(G) = 4^2 \times 2^3 \times (1 + 3)^8 \times (1 + 2)^6 = 1458 \times 4^{11}.$$ 

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