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The Normalized Laplacian Spectrum of Subdivision Vertex-Edge Corona for Graphs

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Abstract A subdivision vertex-edge corona $G_1^S \circ (G_2^V \cup G_3^E)$ is a graph that consists of $S(G_1)$, $|V(G_1)|$ copies of G_2 and $|I(G_1)|$ copies of G_3 by joining the i-th vertex in $V(G_1)$ to each vertex in the i-th copy of G_2 and i-th vertex of $I(G_1)$ to each vertex in the i-th copy of G_3. In this paper, we determine the normalized Laplacian spectrum of $G_1^S \circ (G_2^V \cup G_3^E)$ in terms of the corresponding normalized Laplacian spectra of three connected regular graphs G_1, G_2 and G_3. As applications, we construct some non-regular normalized Laplacian cospectral graphs. In addition, we also give the multiplicative degree-Kirchhoff index, the Kemeny's constant and the number of the spanning trees of $G_1^S \circ (G_2^V \cup G_3^E)$ on three regular graphs.

Keywords normalized Laplacian spectrum; cospectral graphs; spanning trees; subdivision vertex-edge corona

MR(2010) Subject Classification 05C50

1. Introduction

Throughout this paper, all graphs considered are simple undirected and connected. Let $G = (V(G), E(G))$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, where $|V(G)| = n$ and $|E(G)| = m$. Let $d_G(v_i)$ be the degree of the vertex v_i in G. The incidence matrix of G, denoted by $R(G)$, is the $n \times m$ matrix whose (i, j)-entry is 1 if v_i and e_j are adjacent in G and 0 otherwise. As usual, we denote by $A(G)$ and $D(G)$ the adjacency matrix and the degree diagonal matrix of G, respectively. The Laplacian matrix of G is $L(G) = D(G) - A(G)$ and the signless Laplacian matrix of G is $Q(G) = D(G) + A(G)$. Chung [1] introduced the normalized Laplacian matrix of G, denoted by $\mathcal{L}(G) = D^{-1/2}(G)(D(G) - A(G))D^{-1/2}(G) = I - D^{-1/2}(G)A(G)D^{-1/2}(G)$, which is a square matrix with rows and columns being indexed by vertices of G. The $\mathcal{L}$-characteristic polynomial of G is defined as $\Phi_{\mathcal{L}(G)}(\lambda) = \det(\lambda I - \mathcal{L}(G))$. Since $\mathcal{L}(G)$ is real symmetric, their eigenvalues are real number. The multiset of eigenvalues of $\mathcal{L}(G)$ is called the $\mathcal{L}$-spectrum of G and the $\mathcal{L}$-eigenvalues are arranged as $0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \le 2$. Graphs G and H are said to be A-cospectral (resp., $\mathcal{L}$-cospectral) if they share the same A-spectrum (resp., $\mathcal{L}$-spectrum). Furthermore, K_n and P_n denote, respectively, the complete graph and the path on n vertices.

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Graph operations are becoming increasingly useful mathematical models for a broad range of applications, such as complex systems theory, computer security [2], and so on. Recently, many graph operations such as the disjoint union, the corona, the edge corona and the neighborhood corona have been introduced, and their adjacency, Laplacian and signless Laplacian spectra are computed in [3–8], respectively. Banerjee [9] investigated how the normalized Laplacian spectrum is affected by operations like joining. For the aspect of the $\mathcal{L}$-cospectral spectrum, Butler [10] produced large families of non-bipartite, non-regular graphs which are mutually $\mathcal{L}$-cospectral. In 2016, Song [11] obtained the A-spectrum and L-spectrum by graph operation of the subdivision vertex-edge corona $G_1^S \circ (G_2^V \cup G_3^E)$, which is the graph described below.

For a graph G_1, let $S(G_1)$ be the subdividing graph of G_1 whose vertex set has two parts: one the original vertices $V(G_1)$, another, denoted by $I(G_1)$, the inserting vertices corresponding to the edges of G_1. Let G_2 and G_3 be other two disjoint graphs.

Definition 1.1 ([11]) *The subdivision vertex-edge corona (briefly SVE-corona) of G_1 with G_2 and G_3, denoted by $G_1^S \circ (G_2^V \cup G_3^E)$, is the graph consisting of $S(G_1)$, $|V(G_1)|$ copies of G_2 and $|I(G_1)|$ copies of G_3 by joining the i-th vertex in $V(G_1)$ to each vertex in the i-th copy of G_2 and i-th vertex of $I(G_1)$ to each vertex in the i-th copy of G_3. (for example, see $P_4^S \circ (P_3^V \cup P_2^E)$ in Figure 1)*

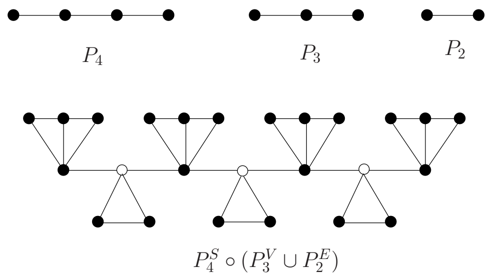

Figure 1 $P_4^S \circ (P_3^V \cup P_2^E)$

One can easily check that $G_1^S \circ (G_2^V \cup G_3^E)$ has $n = n_1 + m_1 + n_1n_2 + m_1n_3$ vertices and $m = 2m_1 + n_1n_2 + m_1n_3 + n_1m_2 + m_1m_3$ edges, where n_i and m_i are the number of vertices and edges of G_i for $i = 1, 2, 3$. We see that $G_1^S \circ (G_2^V \cup G_3^E)$ will be a subdivision-vertex corona if G_3 is null, and will be a subdivision-edge corona if G_2 is null. Thus subdivision vertex-edge corona can be viewed as the generalizations of both subdivision-vertex corona (denoted by $G_1 \odot G_2$) (see [12]) and subdivision-edge corona (denoted by $G_1 \ominus G_2$).

Calculating the spectra of graphs as well as formulating the characteristic polynomials of graphs is a fundamental and very meaningful work in spectral graph theory. In this paper, we determine the normalized Laplacian spectrum of $G_1^S \circ (G_2^V \cup G_3^E)$ in terms of the corresponding normalized Laplacian spectra of three connected regular graphs G_1, G_2 and G_3. As applications, we construct some non-regular normalized Laplacian cospectral graphs. In addition, we also give the multiplicative degree-Kirchhoff index, the Kemeny's constant and the number of the spanning trees of $G_1^S \circ (G_2^V \cup G_3^E)$ for three regular graphs.

2. Preliminaries

In this section we give some useful established results which are required in the proof of the main result.

Lemma 2.1 ([3]) *For a graph G, let $R(G)$ be the incidence matrix of G. Then*

$$R(G)R(G)^T = D(G) + A(G) = Q(G).$$

Lemma 2.2 ([13]) *Let M_1, M_2, M_3, M_4 be respectively $p \times p$, $p \times q$, $q \times p$, $q \times q$ matrices with M_1 and M_4 invertible. Then*

$$\det \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \det(M_4) \cdot \det(M_1 - M_2 M_4^{-1} M_3)$$

$$= \det(M_1) \cdot \det(M_4 - M_3 M_1^{-1} M_2),$$

where $M_1 - M_2 M_4^{-1} M_3$ and $M_4 - M_3 M_1^{-1} M_2$ are called the Schur complements of M_4 and M_1.

Lemma 2.3 ([14]) *The Kronecker product $A \otimes B$ of two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{p\times q}$ is the $mp \times nq$ matrix obtained from A by replacing each element a_{ij} by $a_{ij}B$. It is known that:*

(a) $(M \otimes P)(N \otimes Q) = MN \otimes PQ$, *for matrices M, N, P, Q of suitable sizes;*

(b) $(M \otimes N)^{-1} = M^{-1} \otimes N^{-1}$, *for non-singular matrices M and N;*

(c) $\det(M \otimes N) = (\det M)^s (\det N)^k$, *where M is a matrix of order k and N is a matrix of order s;*

(d) $(M \otimes N)^T = M^T \otimes N^T$, *for any two matrices M and N.*

The reader is referred to [14] for other properties of the Kronecker product not mentioned here.

Definition 2.4 ([14]) *For two matrices $A = (a_{ij})_{m\times n}$ and $B = (b_{ij})_{m\times n}$, the Hadamard product $A \bullet B$ is a matrix of size $m \times n$ with entries given by*

$$(A \bullet B)_{ij} = a_{ij} \cdot b_{ij}.$$

Definition 2.5 ([15]) *Let matrix $B = cJ_n - (c-1)I_n$ where c is a constant and J_n denotes the matrix of size n whose entry equal to one, and C denotes the column vector of dimension n, respectively. For the regular graph G with n vertices and parameter λ, we have*

$$\chi_G(B, C, \lambda) = C^T (\lambda I_n - (\mathcal{L}(G) \bullet B(G)))^{-1} C,$$

where the notion $\chi_G(B, C, \lambda)$ is similar to the notion 'coronal' in [13].

Definition 2.6 ([16]) *The multiplicative degree-Kirchhoff index of G is defined as:*

$$Kf^*(G) = \sum_{i<j} d_i d_j r_{ij},$$

where r_{ij} denotes the resistance distance between v_i and v_j. It has been proved [16] that $Kf^*(G)$ can be expressed by the edge number m and the normalized Laplacian spectrum $\mathrm{Spec}_{\mathcal{L}}(G) =$

$\{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ *below:*

$$Kf^*(G) = 2m \sum_{k=2}^{n} \frac{1}{\lambda_k}, \quad \text{where } \; 0 = \lambda_1 < \lambda_2 \le \cdots \le \lambda_n \le 2.$$

For a graph G, Kemeny's constant $K(G)$, is the expected number of steps required for the transition from a starting vertex i to a destination vertex, which is independent of the selection of starting vertex i (see [17]). Moreover, Kemeny's [18] constant can be computed from the normalized Laplacian spectrum as follows:

$$K(G) = \sum_{k=2}^{n} \frac{1}{\lambda_k}.$$

3. Main results

In this section, we present the normalized Laplacian matrix, $\mathcal{L}$-spectra and some applications of subdivision vertex-edge corona for three regular graphs. For convenience, let η_i, μ_j and θ_k be an eigenvalue of $\mathcal{L}(G_1)$, $\mathcal{L}(G_2)$ and $\mathcal{L}(G_3)$, respectively.

For $i = 1, 2, 3$, let G_i be an r_i-regular graph with n_i vertices and m_i edges. First we label the vertices of $G = G_1^S \circ (G_2^V \cup G_3^E)$: $V(G_1) = \{v_1, v_2, \ldots, v_{n_1}\}$, $I(G_1) = \{e_1, e_2, \ldots, e_{m_1}\}$, $V(G_2) = \{u_1, u_2, \ldots, u_{n_2}\}$ and $V(G_3) = \{w_1, w_2, \ldots, w_{n_3}\}$; for $i = 1, 2, \ldots, n_1$, let $U_i = \{u_1^i, u_2^i, \ldots, u_{n_2}^i\}$ denote the vertices of the i-th copy of G_2 in G, and $W_j = \{w_1^j, w_2^j, \ldots, w_{n_3}^j\}$ $(j = 1, 2, \ldots, m_1)$ the j-th copy of G_3 in G. Then the vertices of G are partitioned by

$$V(G_1) \cup I(G_1) \cup (U_1 \cup U_2 \cup \cdots \cup U_{n_1}) \cup (W_1 \cup W_2 \cup \cdots \cup W_{m_1}).$$

Clearly, the degrees of the vertices of $G = G_1^S \circ (G_2^V \cup G_3^E)$ are:

$$d_G(v_i) = d_{G_1}(v_i) + n_2, \quad i = 1, 2, \ldots, n_1;$$

$$d_G(e_i) = n_3 + 2, \quad i = 1, 2, \ldots, m_1;$$

$$d_G(u_j^i) = d_{G_2}(u_j) + 1, \quad j = 1, 2, \ldots, n_2, \; i = 1, 2, \ldots, n_1;$$

$$d_G(w_j^i) = d_{G_3}(w_j) + 1, \quad j = 1, 2, \ldots, n_3, \; i = 1, 2, \ldots, m_1.$$

Theorem 3.1 *Let $G = G_1^S \circ (G_2^V \cup G_3^E)$. If G_i is an r_i-regular graph with n_i vertices and m_i edges $(i = 1, 2, 3)$, then*

$$\mathcal{L}(G) = \begin{pmatrix} I_{n_1} & -aR(G_1) & -I_{n_1} \otimes b_{n_2}^T & O_{n_1 \times m_1 n_3} \\ -aR(G_1)^T & I_{m_1} & O_{m_1 \times n_1 n_2} & -I_{m_1} \otimes c_{n_3}^T \\ -I_{n_1} \otimes b_{n_2} & O_{n_1 n_2 \times m_1} & I_{n_1} \otimes (\mathcal{L}(G_2) \bullet B(G_2)) & O_{n_1 n_2 \times m_1 n_3} \\ O_{m_1 n_3 \times n_1} & -I_{m_1} \otimes c_{n_3} & O_{m_1 n_3 \times n_1 n_2} & I_{m_1} \otimes (\mathcal{L}(G_3) \bullet B(G_3)) \end{pmatrix},$$

where b_{n_2} and c_{n_3} are the column vector of size n_2 and n_3 with all entries equal to $\frac{1}{\sqrt{(r_1+n_2)(r_2+1)}}$ and $\frac{1}{\sqrt{(n_3+2)(r_3+1)}}$, respectively. $B(G_2)$ is the $n_2 \times n_2$ matrix whose all diagonal entries are 1 and *off-diagonal entries are* $\frac{r_2}{r_2+1}$*,* $B(G_3)$ *is the* $n_3 \times n_3$ *matrix whose all diagonal entries are 1 and off-diagonal entries are* $\frac{r_3}{r_3+1}$*,* O *is zero matrix and* a *is the constant whose value is* $\frac{1}{\sqrt{(r_1+n_2)(n_3+2)}}$.

Proof The adjacency matrix and the degree diagonal matrix of $G_1^S \circ (G_2^V \cup G_3^E)$ can be represented in the form of block-matrix according to the ordering of $V(G_1), I(G_1), U_1, \ldots, U_{n_1}, W_1, \ldots, W_{m_1}$ as follows:

$$A(G) = \begin{pmatrix} O_{n_1\times n_1} & R(G_1) & I_{n_1}\otimes \mathbf{1}_{n_2}^T & O_{n_1\times m_1n_3} \\ R(G_1)^T & O_{m_1\times m_1} & O_{m_1\times n_1n_2} & I_{m_1}\otimes \mathbf{1}_{n_3}^T \\ I_{n_1}\otimes \mathbf{1}_{n_2} & O_{n_1n_2\times m_1} & I_{n_1}\otimes A(G_2) & O_{n_1n_2\times m_1n_3} \\ O_{m_1n_3\times n_1} & I_{m_1}\otimes \mathbf{1}_{n_3} & O_{m_1n_3\times n_1n_2} & I_{m_1}\otimes A(G_3) \end{pmatrix},$$

where $\mathbf{1}_{n_2}$ is the column vector of size n_2 with all entries equal to 1.

$$D(G) = \begin{pmatrix} (r_1+n_2)I_{n_1} & & & \\ & (n_3+2)I_{m_1} & & \\ & & (r_2+1)I_{n_1n_2} & \\ & & & (r_3+1)I_{m_1n_3} \end{pmatrix}.$$

Since G_2 is an r_2-regular graph, we have $\mathcal{L}(G_2) = I_{n_2} - \frac{1}{r_2}A(G_2)$. So

$$\mathcal{L}(G_2) \bullet B(G_2) = (I_{n_2} - \frac{1}{r_2}A(G_2)) \bullet B(G_2) = I_{n_2} - \frac{1}{r_2+1}A(G_2).$$

Thus

$$I_{n_1n_2} - \frac{1}{r_2+1}I_{n_1}\otimes A(G_2) = I_{n_1}\otimes(\mathcal{L}(G_2)\bullet B(G_2)).$$

Furthermore, we can obtain that

$$I_{m_1n_3} - \frac{1}{r_3+1}I_{m_1}\otimes A(G_3) = I_{m_1}\otimes(\mathcal{L}(G_3)\bullet B(G_3)).$$

By $\mathcal{L}(G) = I - D(G)^{-1/2}A(G)D(G)^{-1/2}$, the required normalized Laplacian matrix is given below:

$$\mathcal{L}(G) = \begin{pmatrix} I_{n_1} & -aR(G_1) & -I_{n_1}\otimes b_{n_2}^T & O_{n_1\times m_1n_3} \\ -aR(G_1)^T & I_{m_1} & O_{m_1\times n_1n_2} & -I_{m_1}\otimes c_{n_3}^T \\ -I_{n_1}\otimes b_{n_2} & O_{n_1n_2\times m_1} & I_{n_1}\otimes(\mathcal{L}(G_2)\bullet B(G_2)) & O_{n_1n_2\times m_1n_3} \\ O_{m_1n_3\times n_1} & -I_{m_1}\otimes c_{n_3} & O_{m_1n_3\times n_1n_2} & I_{m_1}\otimes(\mathcal{L}(G_3)\bullet B(G_3)) \end{pmatrix}. \quad \square$$

Theorem 3.2 *Let* $G = G_1^S \circ (G_2^V \cup G_3^E)$*. If* G_i *is an* r_i*-regular graph with* n_i *vertices and* m_i *edges* $(i = 1, 2, 3)$*, then the normalized Laplacian spectrum of* G *consists of:*

(a) $\frac{1+r_2\mu_j}{r_2+1}$ *repeated* n_1 *times for each eigenvalue* μ_j *of* $\mathcal{L}(G_2)$*,* $j = 2, 3, \ldots, n_2$*;*

(b) $\frac{1+r_3\theta_k}{r_3+1}$ *repeated* m_1 *times for each eigenvalue* θ_k *of* $\mathcal{L}(G_3)$*,* $k = 2, 3, \ldots, n_3$*;*

(c) two roots of the equation $(n_3r_3 + n_3 + 2r_3 + 2)\lambda^2 - (n_3r_3 + 2n_3 + 2r_3 + 4)\lambda + 2 = 0$*, where each root repeats* $m_1 - n_1$ *times;*

(d) four roots of the equation

$$(r_1+n_2)(n_3+2)((n_3+2)(r_3+1)\lambda^2 - (n_3+2)(r_3+2)\lambda + 2)((r_1+n_2)(r_2+1)\lambda^2 -$$

$$(r_1+n_2)(r_2+2)\lambda+r_1)-r_1(2-\eta_i)((n_3+2)(r_3+1)\lambda-$$
$$(n_3+2))((r_1+n_2)(r_2+1)\lambda-(r_1+n_2))=0,$$

where each eigenvalue η_i *of* $\mathcal{L}(G_1)$, $i=1,2,\ldots,n_1$.

Proof According to Theorem 3.1, the normalized Laplacian characteristic polynomial of $G_1^S\circ(G_2^V\cup G_3^E)$ is

$$\Phi_{\mathcal{L}(G)}(\lambda)=\det(\lambda I_n-\mathcal{L}(G))=\det(B_0),$$

where

$$B_0=\begin{pmatrix} (\lambda-1)I_{n_1} & aR(G_1) & I_{n_1}\otimes b_{n_2}^T & O\\ aR(G_1)^T & (\lambda-1)I_{m_1} & O & I_{m_1}\otimes c_{n_3}^T\\ I_{n_1}\otimes b_{n_2} & O & I_{n_1}\otimes(\lambda I_{n_2}-\mathcal{L}(G_2)\bullet B(G_2)) & O\\ O & I_{m_1}\otimes c_{n_3} & O & I_{m_1}\otimes(\lambda I_{n_3}-\mathcal{L}(G_3)\bullet B(G_3))\end{pmatrix}.$$

Denote by X the elementary block matrices below,

$$X=\begin{pmatrix} I_{n_1} & O & -I_{n_1}\otimes(b_{n_2}^T(\lambda I_{n_2}-\mathcal{L}(G_2)\bullet B(G_2))^{-1}) & O\\ O & I_{m_1} & O & -I_{m_1}\otimes(c_{n_3}^T(\lambda I_{n_3}-\mathcal{L}(G_3)\bullet B(G_3))^{-1})\\ O & O & I_{n_1}\otimes I_{n_2} & O\\ O & O & O & I_{m_1}\otimes I_{n_3}\end{pmatrix}.$$

Let $B=XB_0$. Then

$$B=\begin{pmatrix} (\lambda-1-\chi_2)I_{n_1} & aR(G_1) & O & O\\ aR(G_1)^T & (\lambda-1-\chi_3)I_{m_1} & O & O\\ I_{n_1}\otimes b_{n_2} & O & I_{n_1}\otimes(\lambda I_{n_2}-\mathcal{L}(G_2)\bullet B(G_2)) & O\\ O & I_{m_1}\otimes c_{n_3} & O & I_{m_1}\otimes(\lambda I_{n_3}-\mathcal{L}(G_3)\bullet B(G_3))\end{pmatrix},$$

where χ_2 and χ_3 refer to $\chi_{G_2}(B(G_2),b_{n_2},\lambda)$ and $\chi_{G_3}(B(G_3),c_{n_3},\lambda)$, respectively.
Set

$$S_1=\begin{pmatrix}(\lambda-1-\chi_2)I_{n_1} & aR(G_1)\\ aR(G_1)^T & (\lambda-1-\chi_3)I_{m_1}\end{pmatrix}.$$

By applying Lemma 2.2, the result follows from

$$\begin{aligned}\det(S_1)&=\begin{vmatrix}(\lambda-1-\chi_2)I_{n_1} & aR(G_1)\\ aR(G_1)^T & (\lambda-1-\chi_3)I_{m_1}\end{vmatrix}\\
&=\det((\lambda-1-\chi_3)I_{m_1})\cdot\det((\lambda-1-\chi_2)I_{n_1}-\frac{a^2}{\lambda-1-\chi_3}R(G_1)R(G_1)^T)\\
&=(\lambda-1-\chi_3)^{m_1}\cdot\det((\lambda-1-\chi_2)I_{n_1}-\frac{a^2}{\lambda-1-\chi_3}R(G_1)R(G_1)^T)\\
&=(\lambda-1-\chi_3)^{m_1-n_1}\cdot\det((\lambda-1-\chi_3)(\lambda-1-\chi_2)I_{n_1}-a^2r_1(2I_{n_1}-\mathcal{L}(G_1)))\\
&=(\lambda-1-\chi_3)^{m_1-n_1}\cdot\prod_{i=1}^{n_1}((\lambda-1-\chi_3)(\lambda-1-\chi_2)-\frac{r_1(2-\eta_i)}{(r_1+n_2)(n_3+2)}).\end{aligned}$$

From Lemma 2.1, we can obtain that $R(G_1)R(G_1)^T=A(G_1)+r_1I_{n_1}$. Combining the equation $A(G_1)=r_1(I_{n_1}-\mathcal{L}(G_1))$, we get

$$R(G_1)R(G_1)^T=r_1(2I_{n_1}-\mathcal{L}(G_1)).$$

As $\mathcal{L}(G_2)\bullet B(G_2)=I_{n_2}-\frac{1}{r_2+1}A(G_2)$, we get

$$\mathcal{L}(G_2)\bullet B(G_2)=\frac{1}{r_2+1}(I_{n_2}+r_2\mathcal{L}(G_2)).$$

Obviously, we have $\mathcal{L}(G_3)\bullet B(G_3)=\frac{1}{r_3+1}(I_{n_3}+r_3\mathcal{L}(G_3))$.

Since $(\mathcal{L}(G_2)\bullet B(G_2))b_{n_2}=(I_{n_2}-\frac{1}{r_2+1}A(G_2))b_{n_2}=(1-\frac{r_2}{r_2+1})b_{n_2}=\frac{1}{r_2+1}b_{n_2}$, we have $(\lambda I_{n_2}-(\mathcal{L}(G_2)\bullet B(G_2)))b_{n_2}=(\lambda-\frac{1}{r_2+1})b_{n_2}$. Also, $b_{n_2}^Tb_{n_2}=\frac{n_2}{(r_1+n_2)(r_2+1)}$. Moreover, the sum of all entries on every row of matrix $\mathcal{L}(G_2)\bullet B(G_2)$ is $\frac{1}{r_2+1}$, so

$$\chi_2=b_{n_2}^T(\lambda I_{n_2}-\mathcal{L}(G_2)\bullet B(G_2))^{-1}b_{n_2}=\frac{b_{n_2}^Tb_{n_2}}{\lambda-\frac{1}{r_2+1}}=\frac{n_2}{(r_1+n_2)(r_2+1)(\lambda-\frac{1}{r_2+1})}.$$

The value of χ_3 is similar to that of χ_2, so

$$\chi_3=c_{n_3}^T(\lambda I_{n_3}-\mathcal{L}(G_3)\bullet B(G_3))^{-1}c_{n_3}=\frac{c_{n_3}^Tc_{n_3}}{\lambda-\frac{1}{r_3+1}}=\frac{n_3}{(n_3+2)(r_3+1)(\lambda-\frac{1}{r_3+1})}.$$

Note that $\det(X)=1$. Then

$$\Phi_{\mathcal{L}(G)}(\lambda)=\det(B_0)=\det(X^{-1})\det(B)=\det(B),$$

where

$$\det(B)=\det(I_{n_1}\otimes(\lambda I_{n_2}-\mathcal{L}(G_2)\bullet B(G_2)))\cdot\det(I_{m_1}\otimes(\lambda I_{n_3}-\mathcal{L}(G_3)\bullet B(G_3)))\cdot\det(S_1).$$

In summary, the normalized Laplacian characteristic polynomial of $G_1^S\circ(G_2^V\cup G_3^E)$ is

$$\begin{aligned}\Phi_{\mathcal{L}(G)}(\lambda)=&\prod_{j=1}^{n_2}(\lambda-\frac{1+r_2\mu_j}{r_2+1})^{n_1}\cdot\prod_{k=1}^{n_3}(\lambda-\frac{1+r_3\theta_k}{r_3+1})^{m_1}\cdot\det(S_1)\\ =&(\lambda-1-\frac{n_3}{(n_3+2)(r_3+1)(\lambda-\frac{1}{r_3+1})})^{m_1-n_1}\cdot\prod_{j=1}^{n_2}(\lambda-\frac{1+r_2\mu_j}{r_2+1})^{n_1}\cdot\\ &\prod_{k=1}^{n_3}(\lambda-\frac{1+r_3\theta_k}{r_3+1})^{m_1}\cdot\prod_{i=1}^{n_1}((\lambda-1-\frac{n_3}{(n_3+2)(r_3+1)(\lambda-\frac{1}{r_3+1})})\cdot\\ &(\lambda-1-\frac{n_2}{(r_1+n_2)(r_2+1)(\lambda-\frac{1}{r_2+1})})-\frac{r_1(2-\eta_i)}{(r_1+n_2)(n_3+2)}).\end{aligned}$$

(1) From the above we see that (a) and (b) are obtained, for $\frac{1}{r_2+1}$ and $\frac{1}{r_3+1}$ are extreme point of χ_2 and χ_3, respectively.

(2) Besides, the 2 eigenvalues are obtained from the equation

$$\lambda-1-\frac{n_3}{(n_3+2)(r_3+1)(\lambda-\frac{1}{r_3+1})}=0,$$

and the eigenvalues repeat m_1-n_1 in (c).

(3) The remaining $4n_1$ eigenvalues of $G_1^S\circ(G_2^V\cup G_3^E)$ are obtained by solving

$$((\lambda-1)(\lambda-\frac{1}{r_3+1})-\frac{n_3}{(n_3+2)(r_3+1)})((\lambda-1)(\lambda-\frac{1}{r_2+1})-\frac{n_2}{(r_1+n_2)(r_2+1)})-\frac{(\lambda-\frac{1}{r_3+1})(\lambda-\frac{1}{r_2+1})r_1(2-\eta_i)}{(r_1+n_2)(n_3+2)}=0$$

for each $i = 1, 2, \ldots, n_1$, and this yields the eigenvalues in (d). $\square$

Remark 3.3 By Theorem 3.2, we observe that the normalized Laplacian spectrum of $G_1^S \circ (G_2^V \cup G_3^E)$ depends on the degrees of regularities, number of vertices, number of edges and normalized Laplacian eigenvalues of regular graph G_i $(i = 1, 2, 3)$.

Example 3.4 One can easily see that the normalized Laplacian eigenvalues of K_4 are 0 and $\frac{4}{3}$ (multiplicity 3). The normalized Laplacian eigenvalues of K_3 are 0 and $\frac{3}{2}$ (multiplicity 2). The normalized Laplacian eigenvalues of K_2 are 0 and 2. Let $G_1 = K_4$, $G_2 = K_3$ and $G_3 = K_2$. Then we consider the normalized Laplacian spectrum of $K_4^S \circ (K_3^V \cup K_2^E)$ (see Figure 2).

From Theorem 3.2, the normalized Laplacian spectrum of $K_4^S \circ (K_3^V \cup K_2^E)$ consists of: $\frac{4}{3}$ (multiplicity 8), $\frac{3}{2}$ (multiplicity 6), each root of the equation $4\lambda^2 - 6\lambda + 1 = 0$ with multiplicity 2 (that is $\frac{3+\sqrt{5}}{4}$ (multiplicity 2), $\frac{3-\sqrt{5}}{4}$ (multiplicity 2)), each root of the equation $144\lambda^4 - 408\lambda^3 + 336\lambda^2 - 74\lambda + 4 = 0$ with multiplicity 3, four roots of the equation $144\lambda^4 - 408\lambda^3 + 312\lambda^2 - 54\lambda = 0$ (including 0 eigenvalue).

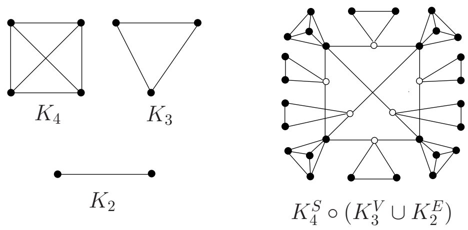

Figure 2 $K_4^S \circ (K_3^V \cup K_2^E)$

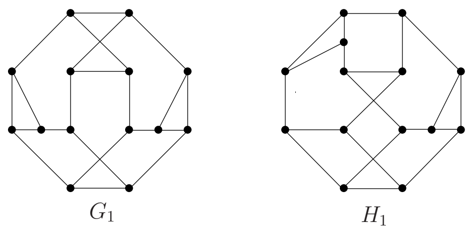

Figure 3 G_1 and H_1

Theorem 3.5 *If G_i and H_i (not necessarily distinct) $(i = 1, 2, 3)$ are cospectral regular graphs, then $G_1^S \circ (G_2^V \cup G_3^E)$ and $H_1^S \circ (H_2^V \cup H_3^E)$ are $\mathcal{L}$-cospectral graphs.*

Proof For an r-regular graph G, we have $\mathcal{L}(G) = I_n - \frac{1}{r}A(G)$. In other words, the normalized Laplacian spectrum of regular graph is determined by their adjacency spectrum. Since G_i and H_i $(i = 1, 2, 3)$ are cospectral regular graphs, G_i and H_i are $\mathcal{L}$-cospectral graphs. From Remark 3.3, the subdivision vertex-edge corona graphs in the theorem statement must then be $\mathcal{L}$-cospectral. $\square$

Example 3.6 Using MATLAB 7.0 software we obtain the two cospectral graphs G_1 and H_1

(see Figure 3) on 14 vertices. The $\mathcal{L}$-characteristic polynomial of G_1 and H_1 is

$$\begin{aligned}\Phi_{G_1}(\lambda) =\Phi_{H_1}(\lambda) = \lambda^{14} - 14\lambda^{13} + \frac{266}{3}\lambda^{12} - \frac{9068}{27}\lambda^{11} + \frac{26270}{31}\lambda^{10} - \frac{49541}{33}\lambda^{9} + \\ \frac{155276}{81}\lambda^{8} - \frac{30299}{17}\lambda^{7} + \frac{15644}{13}\lambda^{6} - \frac{24988}{43}\lambda^{5} + \\ \frac{11474}{59}\lambda^{4} - \frac{7534}{177}\lambda^{3} + \frac{1643}{302}\lambda^{2} - \frac{233}{763}\lambda.\end{aligned}$$

From Theorem 3.5, no regular graphs $G_1^S \circ (K_3^V \cup K_2^E)$ and $H_1^S \circ (K_3^V \cup K_2^E)$ (show in Figure 4) are $\mathcal{L}$-cospectral graphs.

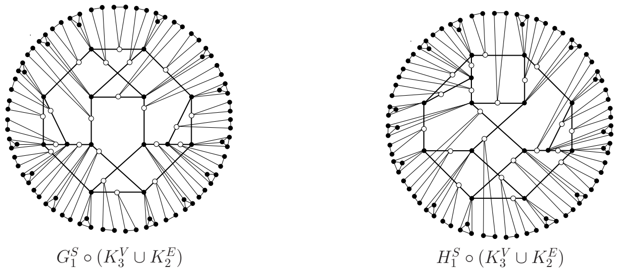

Figure 4 $G_1^S \circ (K_3^V \cup K_2^E)$ and $H_1^S \circ (K_3^V \cup K_2^E)$

Theorem 3.7 *Let G_i be an r_i-regular graph with n_i vertices and m_i edges, $i = 1, 2, 3$. The multiplicative degree-Kirchhoff index of $G = G_1^S \circ (G_2^V \cup G_3^E)$ is related as follows:*

$$\begin{aligned}Kf^*(G) = 2(2m_1 + n_1n_2 + m_1n_3 + n_1m_2 + m_1m_3)\Big(\sum_{j=2}^{n_2} \frac{n_1(r_2+1)}{1+r_2\mu_j} + \sum_{k=2}^{n_3} \frac{m_1(r_3+1)}{1+r_3\theta_k} + \\ \sum_{i=2}^{n_1} \frac{2(r_1+n_2)(r_2+2) + r_1(n_3+2)(r_3+2) - r_1(2-\eta_i)(r_2+r_3+2)}{r_1\eta_i} + \\ \frac{(n_3r_3 + 2n_3 + 2r_3 + 4)(m_1 - n_1)}{2} + \\ \frac{2(r_2+1)(n_2 - r_1r_3) + (n_3+2)(r_1(r_3+1) + (r_1+n_2)(r_2+2)(r_3+2))}{2(r_1+n_2)(r_2+2) + r_1(n_3+2)(r_3+2) - 2r_1(r_2+r_3+2)}\Big).\end{aligned}$$

Proof By Definition 2.6, $Kf^*(G) = 2m\sum_{k=2}^{n}\frac{1}{\lambda_k}$, then the multiplicative degree-Kirchhoff index $Kf^*(G)$ can be computed in the following way:

From Theorem 3.2(c), let α_1 and α_2 be the eigenvalues of equation

$$(n_3r_3 + n_3 + 2r_3 + 2)\lambda^2 - (n_3r_3 + 2n_3 + 2r_3 + 4)\lambda + 2 = 0. \tag{3.1}$$

By Vieta Theorem, we have

$$\frac{1}{\alpha_1} + \frac{1}{\alpha_2} = \frac{\alpha_1 + \alpha_2}{\alpha_1\alpha_2} = \frac{n_3r_3 + 2n_3 + 2r_3 + 4}{2}.$$

In light of Theorem 3.2(d), for $i = 2, 3, \ldots, n_1$, let β_1, β_2, β_3 and β_4 be the eigenvalues of equation

$$\begin{aligned}(r_1+n_2)(n_3+2)((n_3+2)(r_3+1)\lambda^2 - (n_3+2)(r_3+2)\lambda + 2)((r_1+n_2)(r_2+1)\lambda^2 - \\ (r_1+n_2)(r_2+2)\lambda + r_1) - r_1(2-\eta_i)((n_3+2)(r_3+1)\lambda - (n_3+2))\cdot\end{aligned}$$

$$((r_1+n_2)(r_2+1)\lambda-(r_1+n_2))=0. \tag{3.2}$$

By Vieta Theorem, we have

$$\begin{aligned}\frac{1}{\beta_1}+\frac{1}{\beta_2}+\frac{1}{\beta_3}+\frac{1}{\beta_4}&=\frac{\beta_2\beta_3\beta_4+\beta_1\beta_3\beta_4+\beta_1\beta_2\beta_4+\beta_1\beta_2\beta_3}{\beta_1\beta_2\beta_3\beta_4}\\&=\frac{2(r_1+n_2)(r_2+2)+r_1(n_3+2)(r_3+2)-r_1(2-\eta_i)(r_2+r_3+2)}{r_1\eta_i}.\end{aligned}$$

Note that $\eta_1=0$. Let γ_1, γ_2, γ_3 and 0 be the eigenvalues of equation

$$\begin{aligned}&(r_1+n_2)(n_3+2)((n_3+2)(r_3+1)\lambda^2-(n_3+2)(r_3+2)\lambda+2)((r_1+n_2)(r_2+1)\lambda^2-\\&\quad(r_1+n_2)(r_2+2)\lambda+r_1)-2r_1((n_3+2)(r_3+1)\lambda-(n_3+2))\cdot\\&\quad((r_1+n_2)(r_2+1)\lambda-(r_1+n_2))=0.\end{aligned} \tag{3.3}$$

By Vieta Theorem, we have

$$\begin{aligned}\frac{1}{\gamma_1}+\frac{1}{\gamma_2}+\frac{1}{\gamma_3}&=\frac{\gamma_2\gamma_3+\gamma_1\gamma_3+\gamma_1\gamma_2}{\gamma_1\gamma_2\gamma_3}\\&=\frac{2(r_2+1)(n_2-r_1r_3)+(n_3+2)\big(r_1(r_3+1)+(r_1+n_2)(r_2+2)(r_3+2)\big)}{2(r_1+n_2)(r_2+2)+r_1(n_3+2)(r_3+2)-2r_1(r_2+r_3+2)}.\end{aligned}$$

In summary, the multiplicative degree-Kirchhoff index of $G=G_1^S\circ(G_2^V\cup G_3^E)$ is related as

$$\begin{aligned}Kf^*(G)=&2(2m_1+n_1n_2+m_1n_3+n_1m_2+m_1m_3)\Big(\sum_{j=2}^{n_2}\frac{n_1(r_2+1)}{1+r_2\mu_j}+\sum_{k=2}^{n_3}\frac{m_1(r_3+1)}{1+r_3\theta_k}+\\&\sum_{i=2}^{n_1}\frac{2(r_1+n_2)(r_2+2)+r_1(n_3+2)(r_3+2)-r_1(2-\eta_i)(r_2+r_3+2)}{r_1\eta_i}+\\&\frac{(n_3r_3+2n_3+2r_3+4)(m_1-n_1)}{2}+\\&\frac{2(r_2+1)(n_2-r_1r_3)+(n_3+2)(r_1(r_3+1)+(r_1+n_2)(r_2+2)(r_3+2))}{2(r_1+n_2)(r_2+2)+r_1(n_3+2)(r_3+2)-2r_1(r_2+r_3+2)}\Big).\ \Box\end{aligned}$$

Note that by Definition 2.6, we have $Kf^*(G)=2E\cdot K(G)$, so the result given in Corollary below is immediate.

Corollary 3.8 *Let G_i be an r_i-regular graph with n_i vertices and m_i edges, $i=1,2,3$. The Kemeny's constant of $G=G_1^S\circ(G_2^V\cup G_3^E)$ is related as follows:*

$$\begin{aligned}K(G)=&\sum_{i=2}^{n_1}\frac{2(r_1+n_2)(r_2+2)+r_1(n_3+2)(r_3+2)-r_1(2-\eta_i)(r_2+r_3+2)}{r_1\eta_i}+\\&\frac{(n_3r_3+2n_3+2r_3+4)(m_1-n_1)}{2}+\sum_{j=2}^{n_2}\frac{n_1(r_2+1)}{1+r_2\mu_j}+\sum_{k=2}^{n_3}\frac{m_1(r_3+1)}{1+r_3\theta_k}+\\&\frac{2(r_2+1)(n_2-r_1r_3)+(n_3+2)\big(r_1(r_3+1)+(r_1+n_2)(r_2+2)(r_3+2)\big)}{2(r_1+n_2)(r_2+2)+r_1(n_3+2)(r_3+2)-2r_1(r_2+r_3+2)}.\end{aligned}$$

A known result from Chung [1] allows the calculation of spanning trees from the normalized

Laplacian spectrum and the degrees of all the vertices, that is

$$t(G)=\frac{\prod_{i=1}^{n}d_i\prod_{i=2}^{n}\lambda_i}{\sum_{i=1}^{n}d_i}.$$

Thus, we give closed formulas for the spanning trees below:

Corollary 3.9 *Let $G=G_1^S\circ(G_2^V\cup G_3^E)$. If G_i is an r_i-regular graph with n_i vertices and m_i edges $(i=1,2,3)$, then*

$$t(G)=\frac{\prod_{i=2}^{n_1}(r_1\eta_i)\cdot\prod_{j=2}^{n_2}(1+r_2\mu_j)^{n_1}\cdot\prod_{k=2}^{n_3}(1+r_3\theta_k)^{m_1}\cdot 2^{m_1-n_1-1}\cdot(r_1r_3n_3+2r_1n_3+2r_2n_2+4n_2+4r_1)}{2m_1+n_1n_2+m_1n_3+n_1m_2+m_1m_3}.$$

Proof In order to get the result, we consider the normalized Laplacian eigenvalues of G in the following way:

From Theorem 3.7(3.1), we have

$$\alpha_1\alpha_2=\frac{2}{n_3r_3+n_3+2r_3+2}.$$

By Theorem 3.7(3.2), we have

$$\beta_1\beta_2\beta_3\beta_4=\frac{r_1\eta_i}{(r_1+n_2)(r_2+1)(n_3+2)(r_3+1)}.$$

By means of Theorem 3.7(3.3), we obtain that

$$\gamma_1\gamma_2\gamma_3=\frac{r_1r_3n_3+2r_1n_3+2r_2n_2+4n_2+4r_1}{(r_1+n_2)(r_2+1)(n_3+2)(r_3+1)}.$$

From the above we see that

$$\begin{aligned}t(G)=&\frac{\prod_{i=1}^{n}d_i\prod_{i=2}^{n}\lambda_i}{\sum_{i=1}^{n}d_i}\\
=&\frac{(r_1+n_2)^{n_1}(n_3+2)^{m_1}(r_2+1)^{n_1n_2}(r_3+1)^{m_1n_3}}{2(2m_1+n_1n_2+m_1n_3+n_1m_2+m_1m_3)}\Big(\Big(\prod_{k=2}^{n_3}\frac{1+r_3\theta_k}{r_3+1}\Big)^{m_1}\cdot\Big(\frac{2}{n_3r_3+n_3+2r_3+2}\Big)^{m_1-n_1}\times\\
&\Big(\prod_{j=2}^{n_2}\frac{1+r_2\mu_j}{r_2+1}\Big)^{n_1}\cdot\Big(\prod_{i=2}^{n_1}\frac{r_1\eta_i}{(r_1+n_2)(r_2+1)(n_3+2)(r_3+1)}\Big)\cdot\frac{r_1r_3n_3+2r_1n_3+2r_2n_2+4n_2+4r_1}{(r_1+n_2)(r_2+1)(n_3+2)(r_3+1)}\Big)\\
=&\frac{\prod_{i=2}^{n_1}(r_1\eta_i)\cdot\prod_{j=2}^{n_2}(1+r_2\mu_j)^{n_1}\cdot\prod_{k=2}^{n_3}(1+r_3\theta_k)^{m_1}\cdot 2^{m_1-n_1-1}\cdot(r_1r_3n_3+2r_1n_3+2r_2n_2+4n_2+4r_1)}{2m_1+n_1n_2+m_1n_3+n_1m_2+m_1m_3}.\ \square\end{aligned}$$

Example 3.10 From Example 3.4, for a graph $K_4^S\circ(K_3^V\cup K_2^E)$ (see Figure 2) we know that

$$\prod_{i=2}^{n_1}(r_1\eta_i)=64,\ \prod_{j=2}^{n_2}(1+r_2\mu_j)^{n_1}=4^8,\ \prod_{k=2}^{n_3}(1+r_3\theta_k)^{m_1}=3^6,\ 2^{m_1-n_1-1}=2.$$

Also, $r_1r_3n_3+2r_1n_3+2r_2n_2+4n_2+4r_1=54$. The number of edges of graph $K_4^S\circ(K_3^V\cup K_2^E)$ is $2m_1+n_1n_2+m_1n_3+n_1m_2+m_1m_3=54$. Hence,

$$t(G)=\frac{64\times4^8\times3^6\times2\times54}{54}=1458\times4^{11}.$$

According to Song [11], we know that

$$t(G) = t(K_4) \cdot 2^{m_1-n_1+1} \cdot \prod_{i=2}^{n_2}(1+\nu_i(K_3))^{n_1} \cdot \prod_{i=2}^{n_3}(1+\nu_i(K_2))^{m_1},$$

where $\nu_i(G)$ is non-zero Laplacian eigenvalue of G. So

$$t(G) = 4^2 \times 2^3 \times (1+3)^8 \times (1+2)^6 = 1458 \times 4^{11}.$$

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