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On the Distance Spectra of Several Double Neighbourhood Corona Graphs

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Abstract Let G be a connected graph of order n and $D(G)$ be its distance matrix. The distance eigenvalues of G are the eigenvalues of its distance matrix. Its distance eigenvalues and their multiplicities constitute the distance spectrum of G. In this article, we give a complete description of the eigenvalues and the corresponding eigenvectors of a block matrix D_{NC}. Further, we give a complete description of the eigenvalues and the corresponding eigenvectors of distance matrix of double neighbourhood corona graphs $G^{(S)}\bullet\{G_1,G_2\}$, $G^{(Q)}\bullet\{G_1,G_2\}$, $G^{(R)}\bullet\{G_1,G_2\}$, $G^{(T)}\bullet\{G_1,G_2\}$, where G is a complete graph and G_1, G_2 are regular graphs.

Keywords corona; distance spectrum; double neighbourhood corona graph; block matrix

MR(2010) Subject Classification 15A18; 05C50

1. Introduction

Throughout this article we consider only simple graphs. Let $G=(V,E)$ be a graph with vertex set $V=\{1,2,\ldots,n\}$ and edge set $E=\{e_1,e_2,\ldots,e_m\}$. Let $M(G)$ be the vertex-edge incidence matrix of G and $A(G)$ be the adjacency matrix of G. The distance matrix $D(G)=[d_{ij}]$ of a graph G is the matrix indexed by the vertices $\{v_1,v_2,\ldots,v_n\}$ of G, where $d_{ij}=d(v_i,v_j)$ is the distance between the vertices v_i and v_j, i.e., the length of a shortest path between v_i and v_j. Since $D(G)$ is a real symmetric matrix, its eigenvalues, called distance eigenvalues of G, are all real. The spectrum of $D(G)$ is its set of eigenvalues together with their multiplicities and is called the distance spectrum of the graph G. The spectrum of $A(G)$ is denoted by $\mathrm{spec}_A(G)=\{\lambda_1,\lambda_2,\ldots,\lambda_n\}$ and is called the adjacency spectrum of the graph G.

We shall use the following notation throughout this paper. The Kronecker product of matrices $A=[a_{ij}]$ and B is defined to be the partitioned matrix $[a_{ij}B]$ and is denoted by $A\otimes B$. The $m\times 1$ vector with i-th entry equal to one and all other entries zero is denoted by ϵ_i. The $n\times 1$ vector with each entry 1 is denoted by 1_n. By J_n, we denote the matrix of all ones of order n. By I_n, we denote the identity matrix of order n. K_n denotes the complete graph of order n.

Let G be a connected graph on n vertices and m edges and H be any graph. Then it is well known that the corona $G\circ H$ of G and H is the graph obtained by taking one copy of G and n copies of H and then by joining the i-th vertex of G to every vertex in the i-th copy of H.

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In [1], Barik, Pati and Sarma have characterized both adjacency and Laplacian eigenvalues and eigenvectors of corona graph of two graphs. The neighbourhood corona graph of G and H has been defined by Gopalapillai in [2], which is based on the idea that the i-th neighbouring vertices of G are connected to every vertex in the i-th copy of H. The edge corona graph of two graphs is defined similarly, see [3] for definition and its spectral characterization. In [4,5], the graphs such as R-vertex corona graph, R-edge corona graph, R-vertex neighbourhood corona graph, R-edge neighbourhood corona graph, subdivision-vertex neighbourhood corona graph and subdivision-edge neighbourhood corona graph are considered and the coronal technique is used to find the spectrum of these graphs. Recently, in [6], Indulal and Stevanović describe the distance spectrum of corona $G \circ H$ and cluster $G\{H\}$ of two graphs, where G is connected distance regular and H is regular.

This work is motivated from [7] in which Sasmita Barik and Gopinath Sahoo describe the distance spectra of coronas $G \circ H$, where G is connected transmission regular and H is regular. In [8], the authors describe the Laplacian spectra of some variants of corona graphs. Motivated by all these we describe the distance eigenvalues and eigenvectors of several double neighbourhood corona graphs.

Definition 1.1 ([8]) *Let G be a connected graph on n vertices and m edges. The subdivision graph $S(G)$ of G is the graph obtained by inserting a new vertex into every edge of G. The $Q(G)$-graph of G is the graph obtained from G by inserting a new vertex into every edge of G and by joining by edges those pairs of these new vertices which lie on adjacent edges of G. The $R(G)$-graph of G is defined as the graph obtained from G by adding a new vertex corresponding to each edge of G and by joining each new vertex to the end points of the edge corresponding to it. The total graph of G, denoted by $T(G)$, is the graph whose set of vertices is the union of the set of vertices and set of edges of G, with two vertices of $T(G)$ being adjacent if and only if the corresponding elements of G are adjacent or incident.*

Example 1.2 ([8]) Consider the graphs $G = C_4$, where C_n denotes the cycle of order n. Figure 1 describes the four graphs $S(C_4)$, $Q(C_4)$, $R(C_4)$ and $T(C_4)$.

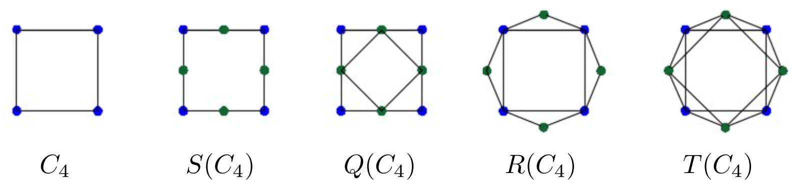

Figure 1 C_4, $S(C_4)$, $Q(C_4)$, $R(C_4)$ and $T(C_4)$

Definition 1.3 ([8]) *Let G be a connected graph on n vertices and m edges. Let G_1 and G_2 be graphs on n_1 and n_2 vertices, respectively. The subdivision double neighbourhood corona graph of G, G_1 and G_2, denoted by $G^{(S)} \bullet \{G_1, G_2\}$, is the graph obtained by taking one copy of $S(G)$, n copies of G_1 and m copies of G_2 and then by joining the neighbourhood vertices of the i-th old-vertex of $S(G)$ to every vertex of the i-th copy of G_1 and the neighbourhood* *vertices of the j-th new-vertex of $S(G)$ to every vertex of the j-th copy of G_2. In place of $S(G)$, if we take $Q(G)(R(G),T(G))$, then the resulting graph is called Q-graph (R-graph, total) double neighbourhood corona graph and denoted by $G^{(Q)} \bullet \{G_1, G_2\}(G^{(R)} \bullet \{G_1, G_2\}, G^{(T)} \bullet \{G_1, G_2\})$.*

Note that all the above four graphs contain $n(n_1+1)+m(n_2+1)$ number of vertices.

Example 1.4 ([8]) Consider the graphs $G = C_4, G_1 = P_3$ and $G_2 = P_2$, where C_n and P_n denote the cycle and the path of order n. Figure 2 describes the four graphs $C_4^{(S)} \bullet \{P_3, P_2\}$, $C_4^{(Q)} \bullet \{P_3, P_2\}$, $C_4^{(R)} \bullet \{P_3, P_2\}$ and $C_4^{(T)} \bullet \{P_3, P_2\}$.

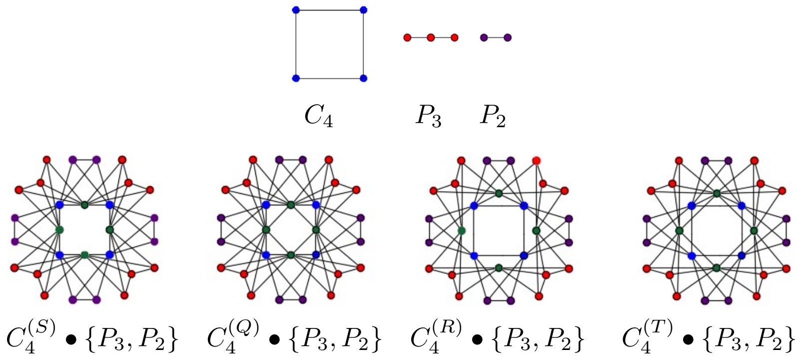

Figure 2 Subdivision (Q-graph, R-graph, total) double neighbourhood corona graph

In Section 2, looking at the similarities in the proofs of the results describing the distance spectra of several double neighbourhood corona graphs, we define a block matrix and determine its spectra. Using the spectra of the matrice we obtain the distance eigenvalues and eigenvectors of several double neighbourhood corona graphs in successive sections.

2. Block matrix D_{NC}

Let $n, m \in \mathbf{N}$, $n \le m$, $n_1, n_2 \in \mathbf{N} \cup \{0\}$, where $\mathbf{N}$ is the set of positive integers. Let $\mathcal{A}, \mathcal{G}, \mathcal{K}$ be real square matrices of order n, n_1, n_2, respectively, and $\mathcal{B}, \mathcal{C}$ be an $n \times m$ real matrix. Let $\mathcal{D}, \mathcal{F}, \mathcal{H}$ be real square matrices of order m and $\mathcal{E}$ be an $m \times n$ real matrix. Consider the following one real square matrix of order $n(n_1+1)+m(n_2+1)$:

$$D_{NC} = \left[\begin{array}{cc|cc} \mathcal{A} & \mathcal{B} & 1_{n_1}^T \otimes 2J_n & 1_{n_2}^T \otimes \mathcal{C} \\ \mathcal{B}^{\mathcal{T}} & \mathcal{D} & 1_{n_1}^T \otimes \mathcal{E} & 1_{n_2}^T \otimes \mathcal{F} \\ \hline 1_{n_1} \otimes 2J_n & 1_{n_1} \otimes \mathcal{E}^{\mathcal{T}} & J_{n_1} \otimes 2(J_n - I) + \mathcal{G} \otimes I_n & J_{n_1 \times n_2} \otimes 3J_{n \times m} \\ 1_{n_2} \otimes \mathcal{C}^{\mathcal{T}} & 1_{n_2} \otimes \mathcal{F} & J_{n_2 \times n_1} \otimes 3J_{m \times n} & J_{n_2} \otimes \mathcal{H} + \mathcal{K} \otimes I_m \end{array}\right].$$

We call block matrix D_{NC} if it satisfies the following conditions:

(i) If X_i and Y_i are the singular vector pairs corresponding to singular values b_i of $\mathcal{B}$ for $i = 2, 3, \ldots, n$ and $\mathcal{C}Y_i = c_i X_i, \mathcal{C}^{\mathcal{T}} X_i = c_i Y_i, \mathcal{E} X_i = e_i Y_i, \mathcal{E}^{\mathcal{T}} Y_i = e_i X_i$, then X_i and Y_i are orthogonal eigenvectors of $\mathcal{A}, \mathcal{D}$ and $\mathcal{F}$, respectively. That is, if $\mathcal{B} Y_i = b_i X_i$ and $\mathcal{B}^{\mathcal{T}} X_i = b_i Y_i$ for

$i = 2, \ldots, n$ then $\mathcal{A}X_i = a_iX_i, \mathcal{D}Y_i = d_iY_i$ and $\mathcal{F}Y_i = f_iY_i$ where a_i, d_i and f_i are the eigenvalues of $\mathcal{A}, \mathcal{D}$ and $\mathcal{F}$, respectively.

(ii) If $\mathcal{B}\hat{Y}_j = 0_n, \mathcal{C}\hat{Y}_j = 0_n, \mathcal{E}^{\mathcal{T}}\hat{Y}_j = 0_n$ for $j = 1, 2, \ldots, m-n$ (This is true as $n \le m$), then $\hat{Y}_j$ are orthogonal eigenvectors of $\mathcal{D}, \mathcal{F}, \mathcal{H}$, that is $\mathcal{D}\hat{Y}_j = \hat{d}_j\hat{Y}_j, \mathcal{F}\hat{Y}_j = \hat{f}_j\hat{Y}_j, \mathcal{H}\hat{Y}_j = \hat{h}_j\hat{Y}_j$ for $j = 1, \ldots, m-n$, where $\hat{d}_j, \hat{f}_j, \hat{h}_j$ are eigenvalues of $\mathcal{D}, \mathcal{F}, \mathcal{H}$.

(iii) 1_{n_1} is an eigenvector of $\mathcal{G}$.

(iv) 1_{n_2} is an eigenvector of $\mathcal{K}$.

Let β_1 ($= g$ say), $\beta_2, \ldots, \beta_{n_1}$ be the eigenvalues of $\mathcal{G}$ with the corresponding eigenvectors as $1_{n_1} = Z_1, Z_2, \ldots, Z_{n_1}$, respectively. Similarly, let η_1 ($= k$ say), $\eta_2, \ldots, \eta_{n_2}$ be the eigenvalues of $\mathcal{K}$ with the corresponding eigenvectors as $1_{n_2} = W_1, W_2, \ldots, W_{n_2}$, respectively.

The following result gives all the eigenvalues and the corresponding eigenvectors of the block matrix D_{NC}.

Theorem 2.1 *Let D_{NC} be a block matrix of order $n(n_1+1) + m(n_2+1)$ as defined above. Then the spectrum of D_{NC} consists of*

(i) all the roots of the following equation

$$\begin{aligned}&\lambda^4 - (a_i + d_i + k + g - 2n_1 + h_in_2)\lambda^3 + [(h_in_2 + k)(a_i + d_i + g - 2n_1) + (d_i + g - 2n_1)a_i - \\ &\quad (c_i^2 + f_i^2)n_2 - e_i^2n_1 + d_i(g - 2n_1) - b_i^2]\lambda^2 + [(h_in_2 + k)(b_i^2 + e_i^2n_1 - a_id_i - \\ &\quad (a_i + d_i)(g - 2n_1)) + (e_i^2n_1 + f_i^2n_2 - d_i(g - 2n_1))a_i + (b_i^2 + f_i^2n_2)(g - 2n_1) + \\ &\quad (d_i + g - 2n_1)c_i^2n_2 - 2c_in_2f_ib_i]\lambda + (h_in_2 + k)((a_id_i - b_i^2)(g - 2n_1) - a_ie_i^2n_1) + \\ &\quad (e_i^2n_1 - d_i(g - 2n_1))c_i^2n_2 - a_if_i^2(g - 2n_1)n_2 + 2c_in_2f_ib_i(g - 2n_1) \\ &= 0 \text{ for } i = 2, 3, \ldots, n;\end{aligned}$$

(ii) $\frac{(\hat{h}_jn_2 + k + \hat{d}_j) \pm \sqrt{(\hat{h}_jn_2 + k + \hat{d}_j)^2 - 4[(\hat{h}_jn_2 + k)\hat{d}_j) - \hat{f}_j^2n_2]}}{2}$*, for* $j = 1, \ldots, m-n$;

(iii) β_j repeated n times, for $j = 2, 3, \ldots, n_1$;

(iv) η_j repeated m times, for $j = 2, 3, \ldots, n_2$;

(v) all the roots of the following equation $\det(\lambda I - \mathbf{D}) = 0$*, where*

$$\mathbf{D} = \begin{bmatrix} a_1 & b_1 & 2nn_1 & c_1n_2 \\ b_1' & d_1 & e_1n_1 & f_1n_2 \\ 2n & e_1' & 2(n-1)n_1 + g & 3mn_2 \\ c_1' & f_1 & 3nn_1 & h_1n_2 + k \end{bmatrix}$$

$\mathcal{A}1_n = a_11_n$, $\mathcal{B}1_m = b_11_n$, $\mathcal{B}^{\mathcal{T}}1_n = b_1'1_m$, $\mathcal{C}1_m = c_11_m$, $\mathcal{C}^{\mathcal{T}}1_n = c_1'1_m$, $\mathcal{D}1_m = d_11_m$, $\mathcal{E}1_n = e_11_m$, $\mathcal{E}^{\mathcal{T}}1_m = e_1'1_n$, $\mathcal{F}1_m = f_11_m$, $\mathcal{H}1_m = h_11_m$.

Proof (a) To prove (i), we suppose that the vector $\phi = \begin{bmatrix} k_1X_i \\ k_2Y_i \\ k_31_{n_1} \otimes X_i \\ k_41_{n_2} \otimes Y_i \end{bmatrix}$ is an eigenvector of D_{NC} corresponding to the eigenvalue λ.

Consider the matrix equation

$$D_{NC}\begin{bmatrix} k_1X_i \\ k_2Y_i \\ k_3\mathbf{1}_{n_1}\otimes X_i \\ k_4\mathbf{1}_{n_2}\otimes Y_i \end{bmatrix} = \lambda \begin{bmatrix} k_1X_i \\ k_2Y_i \\ k_3\mathbf{1}_{n_1}\otimes X_i \\ k_4\mathbf{1}_{n_2}\otimes Y_i \end{bmatrix}$$

where k_1, k_2, k_3, k_4 and λ are the unknown constants to be determined. Comparing both sides, we obtain

$$\begin{cases} a_ik_1 + b_ik_2 + c_in_2k_4 = \lambda k_1, \\ b_ik_1 + d_ik_2 + e_in_1k_3 + f_in_2k_4 = \lambda k_2, \\ e_ik_2 + (g-2n_1)k_3 = \lambda k_3, \\ c_ik_1 + f_ik_2 + (h_in_2 + k)k_4 = \lambda k_4. \end{cases}$$

Let $k_4 = 1$. Eliminating k_1, k_2 and k_3 from these equations, we get

$$\begin{aligned} &\lambda^4 - (a_i + d_i + k + g - 2n_1 + h_in_2)\lambda^3 + [(h_in_2 + k)(a_i + d_i + g - 2n_1) + (d_i + g - 2n_1)a_i - \\ &\quad (c_i^2 + f_i^2)n_2 - e_i^2n_1 + d_i(g-2n_1) - b_i^2]\lambda^2 + [(h_in_2 + k)(b_i^2 + e_i^2n_1 - a_id_i - \\ &\quad (a_i + d_i)(g - 2n_1)) + (e_i^2n_1 + f_i^2n_2 - d_i(g-2n_1))a_i + (b_i^2 + f_i^2n_2)(g - 2n_1) + \\ &\quad (d_i + g - 2n_1)c_i^2n_2 - 2c_in_2f_ib_i]\lambda + (h_in_2 + k)((a_id_i - b_i^2)(g - 2n_1) - a_ie_i^2n_1) + \\ &\quad (e_i^2n_1 - d_i(g-2n_1))c_i^2n_2 - a_if_i^2(g - 2n_1)n_2 + 2c_in_2f_ib_i(g-2n_1) \\ &\quad = 0 \text{ for } i = 2, 3, \ldots, n. \end{aligned}$$

Hence the proof of (i) follows.

(b) As $n \le m$, there exists $m - n$ orthogonal vectors $\hat{Y}_j$ for $j = 1, 2, \ldots, m-n$ such that $\mathcal{B}\hat{Y}_j = 0_n$, $\mathcal{C}\hat{Y}_j = 0_n$, $\mathcal{E}^T\hat{Y}_j = 0_n$ and we have $\mathcal{D}\hat{Y}_j = \hat{d}_j\hat{Y}_j$, $\mathcal{F}\hat{Y}_j = \hat{f}_j\hat{Y}_j$, $\mathcal{H}\hat{Y}_j = \hat{h}_j\hat{Y}_j$ for $j = 1, 2, \ldots, m-n$.

To prove (ii), we suppose that the vector $\phi = \begin{pmatrix} 0_n \\ k_1\hat{Y}_j \\ 0_{n_1n} \\ k_2\mathbf{1}_{n_2}\otimes\hat{Y}_j \end{pmatrix}$ is an eigenvector of D_{NC} corresponding to the eigenvalue λ.

Consider the matrix equation

$$D_{NC}\begin{pmatrix} 0_n \\ k_1\hat{Y}_j \\ 0_{n_1n} \\ k_2\mathbf{1}_{n_2}\otimes\hat{Y}_j \end{pmatrix} = \lambda\begin{pmatrix} 0_n \\ k_1\hat{Y}_j \\ 0_{n_1n} \\ k_2\mathbf{1}_{n_2}\otimes\hat{Y}_j \end{pmatrix}$$

where k_1, k_2 and λ are the unknown constants to be determined. Comparing both sides, we get

$$\begin{cases} \hat{d}_jk_1 + \hat{f}_jn_2k_2 = \lambda k_1, \\ \hat{f}_jk_1 + (\hat{h}_jn_2 + k)k_2 = \lambda k_2. \end{cases}$$

Eliminating k_1 and k_2 from these equations, we get

$$\lambda^2 - (\hat{h}_jn_2 + k + \hat{d}_j)\lambda + [(\hat{h}_jn_2 + k)\hat{d}_j - \hat{f}_j^2n_2] = 0.$$

Hence the proof of (ii) follows.

(c) To prove (iii), observe that

$$D_{NC}\begin{pmatrix}0_n\\0_m\\Z_j\otimes\epsilon_i\\0_{n_2m}\end{pmatrix}=\lambda\begin{pmatrix}0_n\\0_m\\Z_j\otimes\epsilon_i\\0_{n_2m}\end{pmatrix}$$

for $j=2,\dots,n_1$ and $i=1,2,\dots,n$, where ϵ_i is a vector of length n whose all components are zero except the i-th component which is 1. We obtain $\lambda=\beta_j$.

(d) Similarly it can be observed that

$$D_{NC}\begin{pmatrix}0_n\\0_m\\0_{n_1n}\\W_j\otimes\epsilon_i\end{pmatrix}=\lambda\begin{pmatrix}0_n\\0_m\\0_{n_1n}\\W_j\otimes\epsilon_i\end{pmatrix}$$

for $j=2,\dots,n_2$ and $i=1,2,\dots,n$, where ϵ_i is a vector of length n whose all components are zero except the i-th component which is 1. We obtain $\lambda=\eta_j$.

(e) To prove (iv), we suppose that the vector $\phi=\begin{pmatrix}k_11_n\\k_21_m\\k_31_{n_1}\otimes 1_n\\k_41_{n_2}\otimes 1_m\end{pmatrix}$ is an eigenvector of D_{NC} corresponding to the eigenvalue λ.

Suppose that

$$\mathcal{A}1_n=a_11_n,\ \mathcal{B}1_m=b_11_n,\ \mathcal{B}^{\mathcal{T}}1_n=b_1'1_m,\ \mathcal{C}1_m=c_11_m,\ \mathcal{C}^{\mathcal{T}}1_n=c_1'1_m,\ \mathcal{D}1_m=d_11_m,$$
$$\mathcal{E}1_n=e_11_m,\ \mathcal{E}^{\mathcal{T}}1_m=e_1'1_n,\ \mathcal{F}1_m=f_11_m,\ \mathcal{H}1_m=h_11_m.$$

Consider the matrix equation

$$D_{NC}\begin{pmatrix}k_11_n\\k_21_m\\k_31_{n_1}\otimes 1_n\\k_41_{n_2}\otimes 1_m\end{pmatrix}=\lambda\begin{pmatrix}k_11_n\\k_21_m\\k_31_{n_1}\otimes 1_n\\k_41_{n_2}\otimes 1_m\end{pmatrix}$$

where k_1,k_2,k_3,k_4 and λ are the unknown constants to be determined. Comparing both sides, we obtain

$$\begin{cases}a_1k_1+b_1k_2+2nn_1k_3+c_1n_2k_4=\lambda k_1,\\ b_1'k_1+d_1k_2+e_1n_1k_3+f_1n_2k_4=\lambda k_2,\\ 2nk_1+e_1'k_2+[g+2(n-1)n_1]k_3+3mn_2k_4=\lambda k_3,\\ c_1'k_1+f_1k_2+3nn_1k_3+(h_1n_2+k)k_4=\lambda k_4.\end{cases}$$

as k_1,k_2,k_3,k_4 are not all 0, by Cramers rule, the determinant of coefficient of the homogeneous linear equations satisfy $det(\lambda I - \mathbf{D}) = 0$, where

$$\mathbf{D} = \begin{bmatrix} a_1 & b_1 & 2nn_1 & c_1 n_2 \\ b_1' & d_1 & e_1 n_1 & f_1 n_2 \\ 2n & e_1' & 2(n-1)n_1 + g & 3mn_2 \\ c_1' & f_1 & 3nn_1 & h_1 n_2 + k \end{bmatrix}.$$

This finishes (v).

Thus we have listed all the $n(n_1 + 1) + m(n_2 + 1)$ eigenvalues of D_{NC} which completes the proof. □

3. Distance spectra of several double neighbourhood corona graphs

In this section, we describe the distance spectra of the four double neighbourhood corona graphs defined earlier by using Theorem 2.1. The following result describes all the distance eigenvalues of subdivision double neighbourhood corona graph.

Proposition 3.1 *Let G be a complete graph on n vertices and m edges. Let G_1 be a r_1-regular graph on n_1 vertices with an adjacency matrix $A(G_1)$ and $\mathrm{spec}_A(G_1) = \{r_1 = \lambda_1(G_1), \lambda_2(G_1), \ldots, \lambda_{n_1}(G_1)\}$. Let G_2 be a r_2-regular graph on n_2 vertices with an adjacency matrix $A(G_2)$ and $\mathrm{spec}_A(G_2) = \{r_2 = \lambda_1(G_2), \lambda_2(G_2), \ldots, \lambda_{n_2}(G_2)\}$. Then the distance spectrum of $G^{(S)} \bullet \{G_1, G_2\}$ consists of*

(i) all the roots of the equation

$$\begin{aligned}&\lambda^4 + (2n + r_1 + r_2 + (2n-6)n_2)\lambda^3 + [2r_1 + (2n-4)(r_1+2) - n_2(4n + (2n-6)^2 - 8) + \\ &\quad (2n + r_1)(r_2 - 2n_2 + n_2(2n-4) + 2) - n_1(4n-8) + 4]\lambda^2 - [2n_2(2n-6)^2 - \\ &\quad 2(2n-4)(r_1+2) - ((2n-2)(r_1+2) - n_1(4n-8))(r_2 - 2n_2 + n_2(2n-4) + 2) + \\ &\quad (r_1+2)(4n + n_2(2n-6)^2 - 8) + 2n_1(4n-8) + n_2(4n-8)(2n + r_1 - 2) - \\ &\quad 2n_2(2n-6)((4n-8)^2)^{1/2}]\lambda + 2n_2(2n-6)((4n-8)^2)^{1/2}(r_1+2) - \\ &\quad n_2(4n-8)((2n-4)(r_1+2) - n_1(4n-8)) - n_1(8n-16)(r_2 - 2n_2 + n_2(2n-4) + 2) - \\ &\quad n_2(2n-6)(4n-12)(r_1+2) \\ &= 0, \ \text{for } i = 2, 3, \ldots, n;\end{aligned}$$

(ii) $\frac{2(n_2-1) - r_2 \pm \sqrt{(2(n_2-1)-r_2)^2 + 16n_2}}{2}$, *for $j = 1, \ldots, m-n$;*

(iii) $-\lambda_i(G_1) - 2$ repeated n times, for $i = 2, 3, \ldots, n_1$;

(iv) $-\lambda_i(G_2) - 2$ repeated m times, for $i = 2, 3, \ldots, n_2$;

(v) all the roots of the following equation $\det(\lambda I - \mathbf{D}) = 0$, where

$$\mathbf{D} = \begin{bmatrix} 2(n-1) & 3m - 2(n-1) & 2nn_1 & (3m - 2(n-1))n_2 \\ 3n-4 & 4m - 4(n-1) & (3n-4)n_1 & (4m - 4n + 6)n_2 \\ 2n & 3m - 2(n-1) & 2nn_1 - r_1 - 2 & 3mn_2 \\ 3n-4 & 4m - 4n + 6 & 3nn_1 & (4m - 4n + 6)n_2 - r_2 - 2 \end{bmatrix}.$$

Proof By a harmonious labelings of vertex set, the distance matrix of $G^{(S)} \bullet \{G_1, G_2\}$ can be expressed in the form

$$D_{NC}(G^{(S)} \bullet \{G_1, G_2\}) =$$

$$\left[\begin{array}{cc|cc} 2(J_n - I_n) & 3J - 2M & 1_{n_1}^T \otimes 2J_n & 1_{n_2}^T \otimes (3J - 2M) \\ 3J_{m\times n} - 2M^T & 4J - 2M^T M & 1_{n_1}^T \otimes (3J_{m\times n} - 2M^T) & \begin{array}{c} 1_{n_2}^T \otimes (4J \\ -2M^T M + 2I) \end{array} \\ \hline 1_{n_1} \otimes 2J_n & 1_{n_1} \otimes (3J - 2M)) & A^* & J_{n_1\times n_2} \otimes 3J \\ 1_{n_2} \otimes (3J - 2M^T) & \begin{array}{c} 1_{n_2} \otimes (4J \\ -2M^T M + 2I) \end{array} & J_{n_2\times n_1} \otimes 3J & B^* \end{array}\right]$$

where M is incidence matrix of G and

$$A^* = J_{n_1} \otimes 2(J - I) + (2(J - I) - A(G_1)) \otimes I_n,$$
$$B^* = J_{n_2} \otimes [4J - 2M^T M] + (2(J - I) - A(G_2)) \otimes I_m.$$

Comparing with the super neighbourhood corona distance matrix D_{NC}, we have

$\mathcal{A} = 2(J_n - I_n), \mathcal{B} = 3J - 2M, \mathcal{C} = 3J - 2M, \mathcal{D} = 4J - 2M^T M, \mathcal{E} = 3J - 2M^T,$
$\mathcal{F} = 4J - 2M^T M + 2I, \mathcal{G} = 2(J - I) - A(G_1), \mathcal{H} = 4J - 2M^T M, \mathcal{K} = 2(J - I) - A(G_2).$

Since $MM^T = 2(n-1)I_n - (nI_n - J_n)$, we have

$a_i = -2, b_i^2 = 4n - 8, c_i^2 = 4n - 8, d_i = 4 - 2n,$ for $i = 2, \ldots, n;$
$e_i^2 = 4n - 8, f_i = 6 - 2n, h_i = 4 - 2n,$ for $i = 2, \ldots, n;$
$\hat{d}_j = \hat{h}_j = 0, \hat{f}_j = 2,$ for $j = 1, 2, \ldots, m - n;$
$\beta_i = -\lambda_i(G_1) - 2,$ for $i = 2, 3, \ldots, n_1; g = 2(n_1 - 1) - r_1;$
$\eta_i = -\lambda_i(G_2) - 2,$ for $i = 2, 3, \ldots, n_2; k = 2(n_2 - 1) - r_2;$
$a_1 = 2(n-1), b_1 = 3m - 2(n-1), b_1' = 3n - 4, c_1 = 3m - 2(n-1), c_1' = 3n - 4,$
$d_1 = 4m - 4(n-1), e_1 = 3n - 4, e_1' = 3m - 2(n-1),$
$f_1 = 4m - 4n + 6, h_1 = 4m - 4(n-1).$

Hence, the result follows from substituting these values into Theorem 2.1. □

Example 3.1.1 Consider the subdivision double neighbourhood corona graph $K_4^{(S)} \bullet \{K_3, P_2\}$. Then the distance matrix of $K_4^{(S)} \bullet \{K_3, P_2\}$ is

$$D_C(K_4^{(S)} \bullet \{K_3, P_2\}) =$$

$$\left[\begin{array}{cc|cc} 2(J_4 - I_4) & 3J_{4\times 6} - 2M(K_4) & 1_3^T \otimes 2J_4 & 1_2^T \otimes (3J - 2M) \\ 3J_{6\times 4} - 2M^T & 4J_{6\times 6} - 2M^T M & 1_3^T \otimes (3J_{6\times 4} - 2M^T) & 1_2^T \otimes (4J - 2M^T M + 2I) \\ \hline 1_3 \otimes 2J_4 & 1_3 \otimes (3J - 2M) & A^* & J_{3\times 2} \otimes 3J \\ 1_2 \otimes (3J - 2M^T) & 1_2 \otimes (4J - 2M^T M + 2I) & J_{2\times 3} \otimes (3J_{6\times 4}) & B^* \end{array}\right]$$

$$A^* = J_3 \otimes [2(J_4 - I_4)] + [2(J - I) - A(K_3)] \otimes I_4,$$

$$B^* = J_2 \otimes (4J - 2M^T M) + [2(J - I) - A(P_2)] \otimes I_6.$$

Solution 1. Through a matlab program, the distance spectrum of $K_4^{(S)} \bullet \{K_3, P_2\}$ is

$\{-13.7140, -12.1286^{(3)}, -6.7740, -2.3723^{(2)}, -1.1454, -1^{(14)}, -0.0974^{(3)}, 0.8733, 2^{(3)}, 3.3723^{(2)}, 76.9861\}$.

Solution 2. Applying Theorem 2.1, the distance spectrum of $K_4^{(S)} \bullet \{K_3, P_2\}$ contains:

(i) All the roots of equation $\lambda^4 + 17\lambda^3 + 46\lambda^2 - 160\lambda - 16 = 0$, for $i = 2, 3, 4$. With matlab, the roots of the equation are 2, -0.09737, -6.7740, -12.1286 ($i = 2, 3, 4$);

(ii) 3.3723 and -2.3723 ($j = 1, 2$);

(iii) -1 repeated 4 times ($i = 2, 3$);

(iv) -1 repeated 6 times ($i = 2$);

(v) The roots of the following equation $\det(\lambda I - \mathbf{D}) = 0$ are 76.9861, 0.8733, -1.1454, -13.7140, where

$$\mathbf{D} = \begin{bmatrix} 6 & 12 & 24 & 24 \\ 8 & 12 & 24 & 28 \\ 8 & 12 & 20 & 36 \\ 8 & 14 & 36 & 25 \end{bmatrix}$$

We solved the distance spectrum of $K_4^{(S)} \bullet \{K_3, P_2\}$ by two different methods and the result is the same. Then Theorem 2.1 and Proposition 3.1 are accurate.

The distance eigenvalues of Q-graph double neighbourhood coronae graph are shown in the following result.

Proposition 3.2 *Let G be a complete graph on n vertices and m edges. Let G_1 be a r_1-regular graph on n_1 vertices with an adjacency matrix $A(G_1)$ and $\mathrm{spec}_A(G_1) = \{r_1 = \lambda_1(G_1), \lambda_2(G_1), \ldots, \lambda_{n_1}(G_1)\}$. Let G_2 be a r_2-regular graph on n_2 vertices with an adjacency matrix $A(G_1)$ and $\mathrm{spec}_A(G_2) = \{r_2 = \lambda_1(G_2), \lambda_2(G_2), \ldots, \lambda_{n_2}(G_2)\}$. Then the distance spectrum of $G^{(Q)} \bullet \{G_1, G_2\}$ consists of*

(i) All the roots of the equation

$$\begin{aligned}
&\lambda^4 + (2n + r_1 + r_2 + (2n-6)n_2 + 4)\lambda^3 + [n + 2r_1 - n_2(4n + (n-3)^2 - 8) + \\
&(n + r_1 + 2)(r_2 - 2n_2 + n_2(2n-4) + 2) - n_1(n-2) + (n-2)(r_1+2) + 2]\lambda^2 - \\
&[2n_1(n-2) - (n - n_1(n-2) + n(r_1+2) - 2)(r_2 - 2n_2 + n_2(2n-4) + 2) + \\
&(r_1+2)(n + n_2(n-3)^2 - 2) + 2n_2(n-3)^2 - 2(n-2)(r_1+2) + n_2(4n-8)(n+r_1) - \\
&2n_2((4n-8)(n-2))^{1/2}(n-3)]\lambda + n_2(4n-8)(n_1(n-2) - (n-2)(r_1+2)) - \\
&(n_1(2n-4) - (n-2)(r_1+2))(r_2 - 2n_2 + n_2(2n-4) + 2) + \\
&2n_2((4n-8)(n-2))^{1/2}(n-3)(r_1+2) - n_2(2n-6)(n-3)(r_1+2) \\
&= 0, \text{ for } i = 2, 3, \ldots, n;
\end{aligned}$$

(ii) $\frac{2(n_2-1)-r_2\pm\sqrt{(2(n_2-1)-r_2)^2+4n_2}}{2}$, *for* $j=1,\dots,m-n$;

(iii) $-\lambda_i(G_1)-2$ *repeated* n *times, for* $i=2,3,\dots,n_1$;

(iv) $-\lambda_i(G_2)-2$ *repeated* m *times, for* $i=2,3,\dots,n_2$;

(v) *All the roots of the following equation* $\det(\lambda I-\mathbf{D})=0$, *where*

$$\mathbf{D}=\begin{bmatrix} 2(n-1) & 2m-(n-1) & 2nn_1 & (3m-2(n-1))n_2 \\ 2n-2 & 2m-2(n-1) & (2n-2)n_1 & (3m-2n+3)n_2 \\ 2n & 2m-(n-1) & 2nn_1-r_1-2 & 3mn_2 \\ 3n-4 & 3m-2n+6 & 3nn_1 & (4m-4n+6)n_2-r_2-2 \end{bmatrix}.$$

Proof By a harmonious labelings of vertex set, the distance matrix of $G^{(Q)}\bullet\{G_1,G_2\}$ can be expressed in the form

$$D_{NC}(G^{(Q)}\bullet\{G_1,G_2\})=$$

$$\left[\begin{array}{cc|cc} 2(J_n-I_n) & 2J-M & 1_{n_1}^T\otimes 2J_n & 1_{n_2}^T\otimes(3J-2M) \\ 2J_{m\times n}-M^T & 2J-M^TM & 1_{n_1}^T\otimes(2J_{m\times n}-M^T) & 1_{n_2}^T\otimes(3J-M^TM+I) \\ \hline 1_{n_1}\otimes 2J_n & 1_{n_1}\otimes(2J-M)) & A^* & J_{n_1\times n_2}\otimes 3J \\ 1_{n_2}\otimes(3J-2M^T) & 1_{n_2}\otimes(3J-M^TM+I) & J_{n_1\times n_2}\otimes 3J & B^* \end{array}\right]$$

where M is incidence matrix of G and

$$A^*=J_{n_1}\otimes 2(J-I)+(2(J-I)-A(G_1))\otimes I_n,$$
$$B^*=J_{n_2}\otimes(4J-2M^TM)+(2(J-I)-A(G_2))\otimes I_m.$$

Comparing with the super neighbourhood corona distance matrix D_{NC}, we have

$$\mathcal{A}=2(J_n-I_n),\mathcal{B}=2J-M,\mathcal{C}=3J-2M,\mathcal{D}=2J-M^TM,\mathcal{E}=2J-2M^T,$$
$$\mathcal{F}=3J-M^TM+I,\mathcal{G}=2(J-I)-A(G_1),\mathcal{H}=4J-2M^TM,\mathcal{K}=2(J-I)-A(G_2).$$

Since $MM^T=2(n-1)I_n-(nI_n-J_n)$, we have

$$a_i=-2,b_i^2=n-2,c_i^2=4n-8,d_i=2-n,\ \text{for } i=2,\dots,n;$$
$$e_i^2=n-2,f_i=3-n,h_i=4-2n,\ \text{for } i=2,\dots,n;$$
$$\hat{d}_j=\hat{h}_j=0,\hat{f}_j=1,\ \text{for } j=1,2,\dots,m-n;$$
$$\beta_i=-\lambda_i(G_1)-2,\ \text{for } i=2,3,\dots,n_1; g=2(n_1-1)-r_1;$$
$$\eta_i=-\lambda_i(G_2)-2,\ \text{for } i=2,3,\dots,n_2; k=2(n_2-1)-r_2;$$
$$a_1=2(n-1),b_1=2m-(n-1),b_1'=2n-2,c_1=3m-2(n-1),c_1'=3n-4,$$
$$d_1=2m-2(n-1),e_1=2n-2,e_1'=2m-(n-1),$$
$$f_1=3m-2n+3,h_1=4m-4(n-1).$$

Hence, the result follows from substituting these values into Theorem 2.1. □

Example 3.2.1 Consider the subdivision double neighbourhood corona graph $K_4^{(Q)} \bullet \{K_3, P_2\}$. Then the distance matrix of $K_4^{(Q)} \bullet \{K_3, P_2\}$ is

$D_C(K_4^{(Q)} \bullet \{K_3, P_2\}) =$

$$\left[\begin{array}{cc|cc} 2(J_4 - I_4) & 2J_{4\times 6} - M(K_4) & 1_3^T \otimes 2J_4 & 1_2^T \otimes (3J - 2M) \\ 2J_{6\times 4} - M^T & 2J_{6\times 6} - M^T M & 1_3^T \otimes (2J_{6\times 4} - M^T) & 1_2^T \otimes (3J - M^T M + I) \\ \hline 1_3 \otimes 2J_4 & 1_3 \otimes (2J - M) & A^* & J_{3\times 2} \otimes 3J \\ 1_2 \otimes (3J - 2M^T) & 1_2 \otimes (3J - M^T M + I) & J_{2\times 3} \otimes 3J_{6\times 4} & B^* \end{array}\right]$$

$$A^* = J_3 \otimes [2(J_4 - I_4)] + [2(J - I) - A(K_3)] \otimes I_4,$$

$$B^* = J_2 \otimes (4J - 2M^T M) + [2(J - I) - A(P_2)] \otimes I_6.$$

Solution 1. Through a matlab program, the distance spectrum of $K_4^{(Q)} \bullet \{K_3, P_2\}$ is

$$\{-14.4182, -9.7688^{(3)}, -5.3934^{(3)}, -1.3222, -1^{(16)}, -0.4387, -0.4030^{(3)}, 0.5652^{(3)}, 2^{(2)}, 73.1791\}.$$

Solution 2. Applying Theorem 2.1, the distance spectrum of $K_4^{(Q)} \bullet \{K_3, P_2\}$ contains:

(i) All the roots of equation $\lambda^4 + 15\lambda^3 + 50\lambda^2 - 12\lambda - 12 = 0$, for $i = 2, 3, 4$. With matlab, the roots of the equation are 0.5652, -0.4030, -5.3934, -9.7688 (i=2, 3, 4);

(ii) 2 and -1 ($j = 1, 2$);

(iii) -1 repeated 4 times ($i = 2, 3$);

(iv) -1 repeated 6 times ($i = 2$);

(v) The roots of the following equation $\det(\lambda I - \mathbf{D}) = 0$ are 73.1791, -14.4182, -0.4387, -1.3222, where

$$\mathbf{D} = \begin{bmatrix} 6 & 9 & 24 & 24 \\ 6 & 6 & 18 & 26 \\ 8 & 9 & 20 & 36 \\ 8 & 13 & 36 & 25 \end{bmatrix}$$

We solved the distance spectrum of $K_4^{(Q)} \bullet \{K_3, P_2\}$ by two different methods and the result is the same. Then Theorem 2.1 and Proposition 3.2 are accurate.

The distance eigenvalues of R-graph double neighbourhood coronae graph are shown in the following result.

Proposition 3.3 *Let G be a complete graph on n vertices and m edges. Let G_1 be a r_1-regular graph on n_1 vertices with an adjacency matrix $A(G_1)$ and $\text{spec}_A(G_1) = \{r_1 = \lambda_1(G_1), \lambda_2(G_1), \ldots, \lambda_{n_1}(G_1)\}$. Let G_2 be a r_2-regular graph on n_2 vertices with an adjacency matrix $A(G_1)$ and $\text{spec}_A(G_2) = \{r_2 = \lambda_1(G_2), \lambda_2(G_2), \ldots, \lambda_{n_2}(G_2)\}$. Then the distance spectrum of $G^{(R)} \bullet \{G_1, G_2\}$ consists of*

(i) All the roots of the equation

$$\lambda^4 + (n + r_1 + r_2 + (n-3)n_2 + 4)\lambda^3 + [r_1 - n_2(n + (n-3)^2 - 2) +$$

$(n+r_1+2)(r_2-2n_2+n_2(n-1)+2)-n_1(4n-8)+(n-1)(r_1+2)+3]\lambda^2-$
$[(r_1+2)(n+n_2(n-3)^2-2)-(n(r_1+2)-n_1(4n-8)+1)(r_2-2n_2+n_2(n-1)+2)+$
$n_1(4n-8)+n_2(n-3)^2-(n-1)(r_1+2)-2n_2((n-2)^2)^{1/2}(n-3)+$
$n_2(n-2)(n+r_1+1)]\lambda+(r_1-n_1(4n-8)+2)(r_2-2n_2+n_2(n-1)+2)-$
$n_2(n-3)^2(r_1+2)+n_2(n_1(4n-8)-(n-1)(r_1+2))(n-2)+$
$2n_2((n-2)^2)^{1/2}(n-3)(r_1+2)=0,$ *for* $i=2,3,\ldots,n$;

(ii) $\frac{n_2-r_2-3\pm\sqrt{(n_2-r_2-3)^2+4(2n_2-r_2-2)}}{2}$, *for* $j=1,\ldots,m-n$;

(iii) $-\lambda_i(G_1)-2$ *repeated* n *times, for* $i=2,3,\ldots,n_1$;

(iv) $-\lambda_i(G_2)-2$ *repeated* m *times, for* $i=2,3,\ldots,n_2$;

(v) All the roots of the following equation $\det(\lambda I-\mathbf{D})=0$, *where*

$$\mathbf{D}=\begin{bmatrix} n-1 & 2m-(n-1) & 2nn_1 & (2m-(n-1))n_2 \\ 2n-2 & 2m-2n+1 & (3n-4)n_1 & (3m-2n+3)n_2 \\ 2n & 3m-2(n-1)-1 & 2nn_1-r_1-2 & 3mn_2 \\ 2n-2 & 3m-2n+3 & 3nn_1 & (3m-2n+3)n_2-r_2-2 \end{bmatrix}.$$

Proof By a harmonious labelings of vertex set, the distance matrix of $G^{(R)}\bullet\{G_1,G_2\}$ can be expressed in the form

$$D_{NC}(G^{(R)}\bullet\{G_1,G_2\})=$$
$$\left[\begin{array}{cc|cc} J_n-I_n & 2J-M & 1_{n_1}^T\otimes 2J_n & 1_{n_2}^T\otimes(2J-M) \\ 2J_{m\times n}-M^T & 3J-M^TM-I & 1_{n_1}^T\otimes(3J_{m\times n}-2M^T) & 1_{n_2}^T\otimes(3J-M^TM+I) \\ \hline 1_{n_1}\otimes 2J_n & 1_{n_1}\otimes(3J-2M)) & A^* & J_{n_1\times n_2}\otimes 3J \\ 1_{n_2}\otimes(2J-M^T) & 1_{n_2}\otimes(3J-M^TM+I) & J_{n_1\times n_2}\otimes 3J & B^* \end{array}\right]$$

where M is incidence matrix of G and

$$A^*=J_{n_1}\otimes 2(J-I)+(2(J-I)-A(G_1))\otimes I_n,$$
$$B^*=J_{n_2}\otimes[3J-M^TM-I]+(2(J-I)-A(G_2))\otimes I_m.$$

Comparing with the super neighbourhood corona distance matrix D_{NC}, we have

$$\mathcal{A}=J_n-I_n,\mathcal{B}=2J-M,\mathcal{C}=2J-M,\mathcal{D}=3J-M^TM+I,\mathcal{E}=3J-2M^T,$$
$$\mathcal{F}=3J-M^TM+I,\mathcal{G}=2(J-I)-A(G_1),\mathcal{H}=3J-M^TM,\mathcal{K}=2(J-I)-A(G_2).$$

Since $MM^T=2(n-1)I_n-(nI_n-J_n)$, we have

$$a_i=-1,b_i^2=n-2,c_i^2=n-2,d_i=1-n,e_i^2=4n-8,$$
$$f_i=3-n,h_i=1-n,\text{ for } i=2,\ldots,n;$$
$$\hat{d}_j=\hat{h}_j=-1,\hat{f}_j=1,\text{ for } j=1,2,\ldots,m-n;$$

$$\beta_i = -\lambda_i(G_1) - 2, \text{ for } i = 2, 3, \ldots, n_1; g = 2(n_1 - 1) - r_1;$$
$$\eta_i = -\lambda_i(G_2) - 2, \text{ for } i = 2, 3, \ldots, n_2; k = 2(n_2 - 1) - r_2;$$
$$a_1 = n - 1, b_1 = 2m - (n - 1), b_1' = 2n - 2, c_1 = 2m - (n - 1), c_1' = 2n - 2,$$
$$d_1 = 3m - 2(n - 1) - 1, e_1 = 3n - 4, e_1' = 3m - 2(n - 1),$$
$$f_1 = 3m - 2n + 3, h_1 = 3m - 2n + 1.$$

Hence, the result follows from substituting these values into Theorem 2.1. □

Example 3.3.1 Consider the subdivision double neighbourhood corona graph $K_4^{(R)} \bullet \{K_3, P_2\}$. Then the distance matrix of $K_4^{(R)} \circ \{K_3, P_2\}$ is

$$D_C(K_4^{(R)} \circ \{K_3, P_2\}) =$$
$$\left[\begin{array}{cc|cc} J_4 - I_4 & 2J_{4\times 6} - M(K_4) & 1_3^T \otimes 2J_4 & 1_2^T \otimes (2J - M) \\ 2J_{6\times 4} - M^T & 3J_{6\times 6} - M^TM - I & 1_3^T \otimes (3J_{6\times 4} - 2M^T) & 1_2^T \otimes (3J - M^TM + I) \\ \hline 1_3 \otimes 2J_4 & 1_3 \otimes (3J - 2M) & A^* & J_{3\times 2} \otimes 3J \\ 1_2 \otimes (2J - M^T) & 1_2 \otimes (3J - M^TM + I) & J_{2\times 3} \otimes (3J_{6\times 4}) & B^* \end{array}\right]$$

$$A^* = J_3 \otimes [2(J_4 - I_4)] + [2(J - I) - A(K_3)] \otimes I_4,$$
$$B^* = J_2 \otimes (3J - M^TM - I) + [2(J - I) - A(P_2)] \otimes I_6.$$

Solution 1. Through a matlab program, the distance spectrum of $K_4^{(R)} \bullet \{K_3, P_2\}$ is

$$\{-15.1237, -8.9753^{(3)}, -5.5636^{(3)}, -2.4142^{(2)}, -1.3625, -1^{(14)}, -0.3042^{(3)},$$
$$0.1350, 0.4142^{(2)}, 1.8432^{(3)}, 73.3512\}.$$

Solution 2. Applying Theorem 2.1, the distance spectrum of $K_4^{(S)} \bullet \{K_3, P_2\}$ contains:

(i) All the roots of equation $\lambda^4 + 13\lambda^3 + 27\lambda^2 - 85\lambda - 28 = 0$, for $i = 2, 3, 4$. With matlab, the roots of the equation are 1.8432, -0.3042, -5.5636, -8.9753 (i=2, 3, 4);

(ii) 0.4142 and -2.4142 ($j = 1, 2$);

(iii) -1 repeated 4 times ($i = 2, 3$);

(iv) -1 repeated 6 times ($i = 2$);

(v) The roots of the following equation $\det(\lambda I - \mathbf{D}) = 0$ are 73.3512, -15.1237, -1.3625, 0.1350, where

$$\mathbf{D} = \begin{bmatrix} 3 & 9 & 24 & 18 \\ 6 & 11 & 24 & 26 \\ 8 & 12 & 20 & 36 \\ 6 & 13 & 36 & 23 \end{bmatrix}$$

We solved the distance spectrum of $K_4^{(R)} \bullet \{K_3, P_2\}$ by two different methods and the result is the same. Then Theorem 2.1 and Proposition 3.3 are accurate.

The distance eigenvalues of T-graph double neighbourhood coronae graph are shown in the following result.

Proposition 3.4 *Let G be a complete graph on n vertices and m edges. Let G_1 be a r_1-regular graph on n_1 vertices with an adjacency matrix $A(G_1)$ and $\mathrm{spec}_A(G_1) = \{r_1 = \lambda_1(G_1), \lambda_2(G_1), \ldots, \lambda_{n_1}(G_1)\}$. Let G_2 be a r_2-regular graph on n_2 vertices with an adjacency matrix $A(G_1)$ and $\mathrm{spec}_A(G_2) = \{r_2 = \lambda_1(G_2), \lambda_2(G_2), \ldots, \lambda_{n_2}(G_2)\}$. Then the distance spectrum of $G^{(T)} \bullet \{G_1, G_2\}$ consists of*

(i) All the roots of the equation

$$\begin{aligned}&\lambda^4 + (n + r_1 + r_2 + (n-3)n_2 + 3)\lambda^3 + [r_1 - n_1(n-2) - n_2(n + (n-3)^2 - 2) + \\ &\quad (n + r_1 + 1)(r_2 - 2n_2 + n_2(n-1) + 2) + (n-2)(r_1+2) + 2]\lambda^2 - \\ &\quad [(n_1(n-2) - (n-1)(r_1+2))(r_2 - 2 * n_2 + n_2(n-1) + 2) + n_1(n-2) + \\ &\quad (r_1+2)(n + n_2(n-3)^2 - 2) + n_2(n-3)^2 - (n-2)(r_1+2) - \\ &\quad 2n_2((n-2)^2)^{1/2}(n-3) + n_2(n+r_1)(n-2)]\lambda + \\ &\quad n_2(n-2)(n_1(n-2) - (n-2)(r_1+2)) - n_2(n-3)^2(r_1+2) - \\ &\quad n_1(n-2)(r_2 - 2n_2 + n_2(n-1) + 2) + 2n_2((n-2)^2)^(1/2)(n-3)(r_1+2) \\ &= 0, \text{ for } i = 2, 3, \ldots, n;\end{aligned}$$

(ii) $\frac{n_2 - r_2 - 2 \pm \sqrt{(n_2 - r_2 - 2)^2 + 4n_2}}{2}$, *for $j = 1, \ldots, m-n$;*

(iii) $-\lambda_i(G_1) - 2$ *repeated n times, for $i = 2, 3, \ldots, n_1$;*

(iv) $-\lambda_i(G_2) - 2$ *repeated m times, for $i = 2, 3, \ldots, n_2$;*

(v) All the roots of the following equation $det(\lambda I - D\mathbf{D}) = 0$, where

$$\mathbf{D} = \begin{bmatrix} n-1 & 2m-(n-1) & 2nn_1 & (2m-(n-1))n_2 \\ 2n-2 & 2m-2n+2 & (2n-2)n_1 & (3m-2n+3)n_2 \\ 2n & 2m-(n-1) & 2nn_1 - r_1 - 2 & 3mn_2 \\ 2n-2 & 3m-2n+3 & 3nn_1 & (3m-2n+3)n_2 - r_2 - 2 \end{bmatrix}.$$

Proof By a harmonious labelings of vertex set, the distance matrix of $G^{(T)} \bullet \{G_1, G_2\}$ can be expressed in the form

$$D_{NC}(G^{(T)} \bullet \{G_1, G_2\}) = $$
$$\left[\begin{array}{cc|cc} J_n - I_n & 2J - M & 1_{n_1}^T \otimes 2J_n & 1_{n_2}^T \otimes (2J - M) \\ 2J_{m\times n} - M^T & 2J - M^TM & 1_{n_1}^T \otimes (2J_{m\times n} - M^T) & 1_{n_2}^T \otimes (3J - M^TM + I) \\ \hline 1_{n_1} \otimes 2J_n & 1_{n_1} \otimes (2J - M)) & A^* & J_{n_1\times n_2} \otimes 3J \\ 1_{n_2} \otimes (2J - M^T) & 1_{n_2} \otimes (3J - M^TM + I) & J_{n_1\times n_2} \otimes 3J & B^* \end{array}\right]$$

where, M is incidence matrix of G, and

$$A^* = J_{n_1} \otimes 2(J - I) + (2(J - I) - A(G_1)) \otimes I_n,$$
$$B^* = J_{n_2} \otimes [3J - M^TM - I] + (2(J - I) - A(G_2)) \otimes I_m.$$

Comparing with the super neighbourhood corona distance matrix D_{NC}, we have

$$\mathcal{A} = J_n - I_n, \mathcal{B} = 2J - M, \mathcal{C} = 2J - M, \mathcal{D} = 2J - M^T M, \mathcal{E} = 2J - M^T,$$
$$\mathcal{F} = 3J - M^T M + I, \mathcal{G} = 2(J - I) - A(G_1), \mathcal{H} = 3J - M^T M - I, \mathcal{K} = 2(J - I) - A(G_2).$$

Since $MM^T = 2(n-1)I_n - (nI_n - J_n)$, we have

$$a_i = -1, b_i^2 = n-2, c_i^2 = n-2, d_i = 2-n, e_i^2 = n-2,$$
$$f_i = 3-n, h_i = 1-n, \text{ for } i = 2, \dots, n;$$
$$\hat{d}_j = 0, \hat{h}_j = -1, \hat{f}_j = 1, \text{ for } j = 1, 2, \dots, m-n;$$
$$\beta_i = -\lambda_i(G_1) - 2, \text{ for } i = 2, 3, \dots, n_1; g = 2(n_1 - 1) - r_1;$$
$$\eta_i = -\lambda_i(G_2) - 2, \text{ for } i = 2, 3, \dots, n_2; k = 2(n_2 - 1) - r_2;$$
$$a_1 = n-1, b_1 = 2m - (n-1), b_1' = 2n-2, c_1 = 2m - (n-1), c_1' = 2n-2,$$
$$d_1 = 2m - 2(n-1), e_1 = 2n-2, e_1' = 2m - (n-1),$$
$$f_1 = 3m - 2n + 3, h_1 = 3m - 2n + 1.$$

Hence, the result follows from substituting these values into Theorem 2.1. □

Example 3.4.1 Consider the subdivision double neighbourhood corona graph $K_4^{(T)} \bullet \{K_3, P_2\}$. Then the distance matrix of $K_4^{(T)} \bullet \{K_3, P_2\}$ is

$$D_C(K_4^{(T)} \bullet \{K_3, P_2\}) =$$
$$\left[\begin{array}{cc|cc} J_4 - I_4 & 2J_{4\times6} - M(K_4) & 1_3^T \otimes 2J_4 & 1_2^T \otimes (2J - M) \\ 2J_{6\times4} - M^T & 2J_{6\times6} - M^T M & 1_3^T \otimes (2J_{6\times4} - M^T) & 1_2^T \otimes (J - M^T M + I) \\ \hline 1_3 \otimes 2J_4 & 1_3 \otimes (2J - M) & A^* & J_{3\times2} \otimes 3J \\ 1_2 \otimes (2J - M^T) & 1_2 \otimes (3J - M^T M + I) & J_{2\times3} \otimes 3J_{6\times4} & B^* \end{array}\right]$$

$$A^* = J_3 \otimes [2(J_4 - I_4)] + [2(J - I) - A(K_3)] \otimes I_4,$$

$$B^* = J_2 \otimes (3J - M^T M - I) + [2(J - I) - A(P_2)] \otimes I_6.$$

Solution 1. Through a matlab program, the distance spectrum of $K_4^{(T)} \bullet \{K_3, P_2\}$ is

$$\{-16.0848, -7^{(3)}, -4.9173^{(3)}, -2^{(2)}, -1.5143, -1.1943, -1^{(14)}, -0.6804^{(3)},$$
$$0.5977^{(3)}, 1^{(2)}, 70.7924\}.$$

Solution 2. Applying Theorem 2.1, the distance spectrum of $K_4^{(T)} \bullet \{K_3, P_2\}$ contains:

(i) All the roots of equation $\lambda^4 + 12\lambda^3 + 35\lambda^2 - 2\lambda - 14 = 0$, for $i = 2, 3, 4$. With matlab, the roots of the equation are -7, 0.5977, -0.6804, -4.9173 (i=2, 3, 4);

(ii) 1 and -2 ($j = 1, 2$);

(iii) -1 repeated 4 times ($i = 2, 3$);

(iv) -1 repeated 6 times ($i = 2$);

(v) The roots of the following equation $\det(\lambda I - \mathbf{D}) = 0$ are 70.7924, -16.0848, -1.5143, -1.1943, where

$$\mathbf{D} = \begin{bmatrix} 3 & 9 & 24 & 18 \\ 6 & 6 & 18 & 26 \\ 8 & 9 & 20 & 36 \\ 6 & 13 & 36 & 23 \end{bmatrix}.$$

We solved the distance spectrum of $K_4^{(T)} \bullet \{K_3, P_2\}$ by two different methods and the result is the same. Then Theorem 2.1 and Proposition 3.4 are accurate.

References

[1] S. BARIK, S. PATI, B. K. SARMA. *The spectrum of the corona of two graphs*. SIAM J. Discrete Math., 2007, **21**(1): 47–56.

[2] I. GOPALAPILLAI. *The spectrum of neighbourhood corona of graphs*. Kragujevac J. Math., 2011, **35**(3): 493–500.

[3] Yaoping HOU, W. C. SHIU. *The spectrum of the edge corona of two graphs*. Electron. J. Linear Algebra, 2010, **20**(1): 586–594.

[4] Jie LAN, Bo ZHOU. *Spectra of graph operations based on R-graph*. Linear Multilinear Algebra, 2015, **63**(7): 1401–1422.

[5] Xiaogang LIU, Pengli LU. *Spectra of subdivision-vertex and subdivision-edge neighbourhood coronae*. Linear Algebra Appl., 2013, **438**(8): 3547–3559.

[6] G. INDULAL, D. STEVANOVIĆ. *The distance spectrum of corona and cluster of two graphs*. AKCE Int. J. Graphs Comb., 2015, **12**(2-3): 186–192.

[7] S. BARIK, G. SAHOO. *On the distance spectra of coronas*. Linear Multilinear Algebra, 2016, **65**(8): 1617–1628.

[8] S. BARIK, G. SAHOO. *On the Laplacian spectra of some variants of corona*. Linear Algebra Appl., 2017, **521**: 32–47.