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Bipartite Version of the Erdős-Sós Conjecture

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Abstract The Erdős-Sós Conjecture states that every graph on n vertices and more than $\frac{n(k-2)}{2}$ edges contains every tree of order k as a subgraph. In this note, we study a weak (bipartite) version of Erdős-Sós Conjecture. Based on a basic lemma, we show that every bipartite graph on n vertices and more than $\frac{n(k-2)}{2}$ edges contains the following families of trees of order k: (1) trees of diameter at most five; (2) trees with maximum degree at least $\lfloor\frac{k-1}{2}\rfloor$; (3) almost balanced trees, these results are better than the corresponding known results for the general version of the Erdős-Sós Conjecture.

Keywords Erdős-Sós conjecture; bipartite graphs; trees

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1. Introduction

In this paper, all graphs considered are simple and finite. Let $G=(V,E)$ be a graph. For $S\subset V(G)$, $G-S$ denotes the graph obtained from G by deleting all the vertices of S and all the edges with at least one end in S. If $xy\in E(G)$, we say that x is adjacent to y and y is a neighbor of x. We call $|V(G)|$ the order of G. Let $N_G^i(u)=\{v\in V(G)\mid d_G(v,u)=i\}$, where $d_G(v,u)$ is the distance between vertices u and v. For a tree T, we call a vertex of degree one a leaf of T and use $L(T)$ to denote the set of leaves of T. We denote a bipartite graph G with bipartition (X,Y) by $G[X,Y]$. A tree T is called almost balanced if the bipartition (V_1,V_2) of $V(T)$ satisfies that $||V_1|-|V_2||\le 2$.

Let H and G be two graphs. A map $\varphi: V(H)\to V(G)$ is called an embedding of H in G if φ is an injection and for every edge $uv\in E(H)$, $\varphi(u)\varphi(v)\in E(G)$. Clearly, H is a subgraph of G if and only if there exists an embedding φ of H into G.

The following conjecture was proposed first by Erdős and Sós [1].

Conjecture 1.1 (Erdős-Sós Conjecture or ESC) *If G is a graph on n vertices with $e(G)>\frac{n(k-2)}{2}$, then G contains every tree of order k.*

There are many partial results in the study of the conjecture. Roughly speaking, there are

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two ways to attack the conjecture. One way is to prove the conjecture holds for special family of trees on k vertices, such as

(a) Path (Erdős and Gallai [2]);

(b) Spiders (Fan [3], Fan and Sun [4], Fan, Hong and Liu [5]);

(c) Trees of diameter at most four (McLennan [6]);

(d) Trees with a vertex joined to at least $\lfloor \frac{k}{2} \rfloor - 1$ vertices of degree one (Sidorenko [7]).

The other is to show that the conjecture is true for special family of host graphs, such as

- Graphs with girth at least 5 (Brandt and Dobson [8]);
- Graphs with its complement having girth four (Wang, Li, and Liu [9]);
- Graphs without cycle of length four (Saclé and Woźniak [10]).

These results motivate us to consider the following weak version of the Erdős-Sós Conjecture.

Conjecture 1.2 (Weak version of ESC) *If G is a bipartite graph on n vertices with $e(G) > \frac{n(k-2)}{2}$, then G contains every tree of order k.*

Clearly, Conjecture 1.1 implies Conjecture 1.2. So all partial results of ESC are also true for Conjecture 1.2, in this note we will show that we can do better than ESC.

In fact, we prove that Conjecture 1.2 holds for: trees of diameter no more than 5, which is better than (c); trees with maximum degree at least $\lfloor \frac{k-1}{2} \rfloor$, which is better than (d); almost balanced trees (including path), which is new for Conjecture 1.2, as we have known, there is no similar results for the Erdős-Sós Conjecture.

The rest of the paper is arranged as follows. In Section 2, we will give some lemmas used in the proofs, we prove our main results in Section 3, some remarks and discussions are given in the last section.

2. Lemmas

A k-degenerate graph is a graph in which every subgraph has a vertex of degree at most k: that is, some vertex in the subgraph touches k or fewer of the subgraph's edges.

Lemma 2.1 ([11]) *Every k-degenerate graph on n vertices has at most $nk - \frac{k(k+1)}{2}$ edges.*

The following lemma can be seen as a consequence of the above lemma.

Lemma 2.2 *If G is a graph on n vertices with $e(G) > \frac{n(k-2)}{2}$, then there is a subgraph $H \subset G$ with $e(H) > \frac{|V(H)|(k-2)}{2}$ and $\delta(H) \geq \lfloor \frac{k}{2} \rfloor$.*

Proof We claim that G cannot be $(\lfloor \frac{k}{2} \rfloor - 1)$-degenerate. Otherwise, by Lemma 2.1, $e(G) \leq n(\lfloor \frac{k}{2} \rfloor - 1) - \frac{\lfloor \frac{k}{2} \rfloor (\lfloor \frac{k}{2} \rfloor - 1)}{2} < \frac{n(k-2)}{2}$, a contradiction. So G must contain a subgraph H with minimum degree at least $\lfloor \frac{k}{2} \rfloor$. $\square$

Lemma 2.3 *Let $G[X, Y]$ be a bipartite graph and let T be a tree with bipartition (V_1, V_2). If $\max\{|V_1|, |V_2|\} \leq \delta(G)$, then $T \subset G$. Moreover, for any $v_0 \in V(T)$ and $x_0 \in V(G)$, there is an embedding φ of T into G such that $\varphi(v_0) = x_0$.*

Proof The proof is by induction on the order of T. For $|V(T)| = 1$, the lemma is true clearly. Suppose $k \geq 2$ and the result is true for all trees of order less than k. Let T be a tree of order k. We may assume $v_0 \in V_1$ and $x_0 \in X$. As $k \geq 2$, T has at least two leaves. Choose a leaf $u_0 \in V(T)$ such that $u_0 \neq v_0$. Let w_0 be the unique neighbor of u_0. Without loss of generality, assume $u_0 \in V_1$. Let $T' = T - \{u_0\}$. Then T' is a tree of order $k-1$ with bipartition $(V_1 \setminus \{u_0\}, V_2)$. Clearly, $\max\{|V_1 \setminus \{u_0\}|, |V_2|\} \leq \delta(G)$. By the induction hypothesis, there exists an embedding $\varphi' : V(T') \to V(G)$ with $\varphi'(v_0) = x_0$. Then $\varphi'(V_1 \setminus \{u_0\}) \subset X$ and $\varphi'(V_2) \subset Y$. Let $\varphi'(w_0) = y_0 \in Y$. As $|\varphi'(V_1 \setminus \{u_0\})| = |V_1 \setminus \{u_0\}| = |V_1| - 1 \leq \delta(G) - 1$, there must exist a vertex $z_0 \in N_G(y_0) \setminus \varphi'(V_1 \setminus \{u_0\})$. So embed u_0 to z_0, we get a desired embedding of T. $\square$

Lemma 2.4 *Let $G[X, Y]$ be a bipartite graph and let T be a tree with bipartition (V_1, V_2) and $|V_1| \leq \delta(G)$. Suppose $A \subset L(T) \cap V_1$ and $T' = T - A$. If T' can be embedded in G, then so do T.*

Proof Let $U_1 = V_1 \setminus A$ and $U_2 = V_2$. Then T' is a tree with bipartition (U_1, U_2). Since T' can be embedded in G, there exists an embedding φ' that embeds T' in G. Without loss of generality, we may assume $\varphi'(U_1) \subset X$ and $\varphi'(U_2) \subset Y$. Denote $A = \{v_1, v_2, \ldots, v_m\}$ and for any $v_i \in A$, let u_i be the support vertex of v_i in T. Clearly $u_i \in U_2$, $i = 1, 2, \ldots, m$. As $\varphi'(U_2) \subset Y, \varphi'(u_i) \in Y$. Denote $\varphi'(u_i) = w_i \in Y$. Then $N_G(w_i) \subset X$. To embed T into G, it suffices to show that every w_i has enough (at least m) neighbors not used by φ'. In fact, $|N_G(w_i) \setminus \varphi'(V(T'))| = |N_G(w_i) \setminus \varphi'(U_1)| \geq |N_G(w_i)| - |\varphi'(U_1)| \geq \delta(G) - |\varphi'(U_1)| \geq |V_1| - |\varphi'(U_1)| \geq m$. This completes the proof. $\square$

3. Main results

In this section, we give our main results.

Theorem 3.1 *Let $k \geq 2, m \geq 1$ be integers and let G be a bipartite graph of order n with $e(G) > \frac{n(k-2)}{2}$. If any tree of order k with diameter $2m$ is a subgraph of G, then so do any tree of order k with diameter $2m+1$.*

Proof Let T be a tree of order k with diameter $2m+1$ and let P be a path of length $2m+1$ in T. Denote P by $x_m x_{m-1} \cdots x_1 x_0 y_0 y_1 \cdots y_m$. Then for any $x \in V(T)$, either $d_T(x, x_0) \leq m$ or $d_T(x, y_0) \leq m$. By Lemma 2.2, we may assume $\delta(G) \geq \lfloor \frac{k}{2} \rfloor$. Let (V_1, V_2) be the bipartition of $V(T)$ with $x_0 \in V_1$ and $y_0 \in V_2$. Without loss of generality, we may assume $|V_1| \leq \lfloor \frac{k}{2} \rfloor \leq \delta(G)$.

If m is even, then $N_T^{m+1}(y_0) \subset V_1$ and for every $x \in N_T^{m+1}(y_0)$, $d_T(x) = 1$. If m is odd, then $N_T^{m+1}(x_0) \subset V_1$ and for every $x \in N_T^{m+1}(x_0)$, $d_T(x) = 1$. Let $A = N_T^{m+1}(y_0)$ (m is even) or $A = N_T^{m+1}(x_0)$ (m is odd). Then $A \subset L(T) \cap V_1$. Let $T' = T - A$. Then T' is a tree of diameter $2m$. So $T' \subset G$. By Lemma 2.4, $T \subset G$. $\square$

We have mentioned that the Erdős-Sós conjecture holds for trees of diameter four ((c), McLennan [6]). Combining this result with Theorem 3.1, we have

Corollary 3.2 *If G is a bipartite graph on n vertices with $e(G) > \frac{n(k-2)}{2}$, then G contains* *every tree of order k with diameter at most five.*

Theorem 3.3 *If G is a bipartite graph on n vertices with $e(G) > \frac{n(k-2)}{2}$, then G contains every tree T of order k and $\Delta(T) \geq \lfloor \frac{k-1}{2} \rfloor$.*

Proof By Lemma 2.2, we may assume $\delta(G) \geq \lfloor \frac{k}{2} \rfloor$. Since $e(G) > \frac{n(k-2)}{2}$, there must exist a vertex $x_0 \in V(G)$ with $d_G(x_0) \geq k-1$.

Let $u_0 \in V(T)$ with $d_T(u_0) \geq \lfloor \frac{k-1}{2} \rfloor$. Let $A = \{u \in N_T(u_0) \mid d_T(u) = 1\}$ and $B = N_T(u_0) \backslash A$. Denote $\alpha = |A|$ and $\beta = |B|$, then $\alpha + \beta \geq \lfloor \frac{k-1}{2} \rfloor$. Let $T' = T - A$ and (V_1, V_2) be the bipartition of T' with $u_0 \in V_1$. Then $B \subset V_2$ and for every $x \in B, d_{T'}(x) \geq 2$. This implies that $\left|N^2_{T'}(u_0)\right| \geq |B| = \beta$.

If $N^3_{T'}(u_0) = \emptyset$, then the diameter of T is at most four. By Corollary 3.2, $T \subset G$.

Now assume $N^3_{T'}(u_0) \neq \emptyset$. As $\{u_0\} \cup N^2_{T'}(u_0) \subset V_1$ and $N^1_{T'}(u_0) \cup N^3_{T'}(u_0) \subset V_2$, we have $\left|\{u_0\} \cup N^2_{T'}(u_0)\right| \geq \beta + 1$ and $\left|N^1_{T'}(u_0) \cup N^3_{T'}(u_0)\right| \geq \beta + 1$. Therefore,

$$|V_i| \leq |V(T')| - \beta - 1 = k - \alpha - \beta - 1 \leq \lfloor \frac{k}{2} \rfloor \leq \delta(G) \text{ for } i = 1, 2.$$

By Lemma 2.3, there exists an embedding φ' that embeds T' in G such that $\varphi'(u_0) = x_0$. So, to embed T in G, it suffices to show that x_0 has enough unused neighbors to embed vertices of A in G. This is clearly true since $d_G(x_0) - |V(T')| \geq k - 1 - (k - 1 - \alpha) = \alpha$. $\Box$

Theorem 3.4 *Every bipartite graph G on n vertices with $e(G) > \frac{n(k-2)}{2}$ contains every almost balanced tree of order k.*

Proof Let (V_1, V_2) be the bipartition of T with $|V_1| \leq |V_2|$. Then

$$\lfloor \frac{k}{2} \rfloor - 1 \leq |V_1| \leq |V_2| \leq \lfloor \frac{k}{2} \rfloor + 1.$$

It is a basic fact that V_2 contains at least one leaf. By Lemma 2.2, we may assume $\delta(G) \geq \lfloor \frac{k}{2} \rfloor$. By $e(G) > \frac{n(k-2)}{2}$, there exists a vertex $x_0 \in V(G)$ with $d_G(x_0) \geq k-1$. Let $v_0 \in V_2$ be a leaf and u_0 be its unique neighbor. Then $u_0 \in V_1$. Let $T' = T - \{v_0\}$. Then the bipartition $(V_1, V_2 \setminus \{v_0\})$ of T' satisfies

$$\max\{|V_1|, |V_2 \setminus \{v_0\}|\} = \lfloor \frac{k}{2} \rfloor \leq \delta(G).$$

By Lemma 2.3, there is an embedding φ' that embeds T' into G so that $\varphi'(u_0) = x_0$. As $d_G(x_0) \geq k - 1 = |V(T')|$, there exists a vertex $y_0 \in N_G(x_0) \setminus \varphi'(V(T'))$ such that we can map v_0 to y_0. Therefore, we get an embedding of T in G. $\Box$

4. Remarks and discussions

It should be mentioned that Ajtai, Komlós, Simonovits, and Szemerédi announced (unpublished) a proof of Erdős-Sós Conjecture for sufficiently large k. It is still interesting to prove ESC for unrestricted n and k. In this note, we initially start the study of ESC when the host graph is bipartite and obtain some stronger results than the original version. Unfortunately, Conjecture

1.2 is still open, we leave this as an open problem.

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