Journal of Mathematical Research with Applications
May, 2019, Vol. 39, No. 3, pp. 254–258
DOI:10.3770/j.issn:2095-2651.2019.03.004
Http://jmre.dlut.edu.cn

A Criterion on the Finite p-Nilpotent Groups

Xiangyang XU[1], **Yangming LI**[2,*]
1. *Department of Mathematics, Nanchang Normal University, Jiangxi* 330032, *P. R. China;*
2. *Department of Mathematics, Guangdong University of Education, Guangdong* 510310, *P. R. China*

Abstract Let G be a finite group. Suppose that H is a subgroup of G. We say that H is s-semipermutable in G if $HG_p = G_pH$ for any Sylow p-subgroup G_p of G with $(p, |H|) = 1$, where p is a prime dividing the order of G. We give a p-nilpotent criterion of G under the hypotheses that some subgroups of G are s-semipermutable in G. Our result is a generalization of the famous Burnside's p-nilpotent criterion.

Keywords p-nilpotent group; k-th center of a group; s-semipermutable subgroup

MR(2010) Subject Classification 20D10; 20D15

1. Introduction and statements of results

All groups considered in this paper are finite. We use conventional notions and notation, as in Huppert [1]. G always denotes a finite group and p is a prime, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$, $O^p(G)$ is the subgroup of G generated by all p'-elements of G.

We also need the following notions. For an integer k, $Z_k(G)$ is the k-th center of G, $k \geq 1$. In fact, $Z_1(G) = Z(G)$, the center of G, and

$$Z_k(G)/Z_{k-1}(G) = Z(G/Z_{k-1}(G))$$

for $k > 1$.

A subgroup H of G is said to be quasinormal [2] or permutable [3] in G if $HK = KH$ for all subgroups K of G; H is said to be s-permutable [4] (or s-quasinormal, π-quasinormal) in G if $HP = PH$ for any Sylow subgroup P of G. As a generalization of s-permutability, a subgroup H of G is said to be s-semipermutable [5] in G if H permutes with every Sylow p-subgroup G_p of G with $(|H|, p) = 1$. Many authors are interesting in this concept. According to the Math Review, there are more than 20 papers applying this concepts, for example, [5–9], etc.

A group G is called p-nilpotent if p does not divide the order of $O^p(G)$, or equivalently, $P \cap O^p(G) = 1$ for any Sylow p-subgroup P of G. For the p-nilpotent criterion, the most famous

Received June 1, 2018; Accepted November 8, 2018
Supported by the National Natural Science Foundation of China (Grant No. 11271085), the Major Projects in Basic Research and Applied Research (Natural Science) of Guangdong Province (Grant No. 2017KZDXM058), Funds of Guangzhou Science and Technology (Grant No. 201804010088) and the Science and Technology Research Foundation of Education Department of Jiangxi Province (Grant No. GJJ171109).
* Corresponding author
E-mail address: 1043991895@qq.com (Xiangyang XU); liyangming@gdei.edu.cn (Yangming LI)

one is Burnside's result.

Theorem 1.1 ([10, Theorem 7.2.1]) *Let p be a prime and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if P is abelian.*

P. Hall extended Burnside's result as follows.

Theorem 1.2 ([11]) *Let p be a prime and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if the nilpotency class of P is less than p, i.e., $P \leq Z_{p-1}(P)$.*

In the literature, there are many extensions of Hall's results. For the recent results in this line please refer to [12–15], etc.

In this paper, we obtain a criterion on the finite p-nilpotent groups which is a paralleled result of [14, Theorem 1.7] by replacing partial CAP-property by s-semipermutability. It is easy to see that partial CAP-property and s-semipermutability are different [16].

Theorem 1.3 *Let p be a fixed prime in $\pi(G)$ and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if there exists a normal subgroup H of P contained in $\Phi(P)$ such that H is s-semipermutable in G and the nilpotency class of P/H is less than p, i.e., $P/H \leq Z_{p-1}(P/H)$.*

Remark 1.4 If P is abelian, then $P' = 1$. Obviously, the trivial subgroup 1 is an s-semipermutable subgroup of G. Hence Theorem 1.3 is a generalization of Burnside's result.

Immediately, we have the following corollaries.

Corollary 1.5 *Let p be a prime in $\pi(G)$ and P a Sylow p-subgroup of G. Suppose that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if and only if P' is s-semipermutable in G.*

It is easy to prove that G is p-nilpotent if every maximal subgroup of any Sylow p-subgroup of G is s-permutable in G, where p is the smallest prime in $\pi(G)$ (see [8, Theorem 3.2]). Bearing this result in mind, we can see that [17, Theorem 3.1] follows immediately from Corollary 1.5.

Corollary 1.6 ([17, Theorem 3.1]) *Let p be the smallest prime in $\pi(G)$ and P a Sylow p-subgroup of G. Then G is p-nilpotent if every maximal subgroup of P is an s-permutable subgroup of $N_G(P)$ and P' is an s-permutable subgroup of G.*

2. Proofs

We first give some properties of s-semipermutable subgroups of finite groups.

Lemma 2.1 *Suppose that H is an s-semipermutable subgroup of G. Then*

(1) If $H \leq K \leq G$, then H is s-semipermutable in K;

(2) Let N be a normal subgroup of G. If H is a p-group for some prime $p \in \pi(G)$, then HN/N is s-semipermutable in G/N;

(3) If $H \leq O_p(G)$, then H is s-permutable in G;

(4) If H is a p-subgroup of G for some prime $p \in \pi(G)$ and N is normal in G, then $H \cap N$

is also an s-semipermutable subgroup of G;

(5) If H is a p-subgroup of G for some prime $p \in \pi(G)$, then the normal closure H^G of H in G is solvable.

Proof By [8, Lemma 2.2], we get (1)–(4). (5) is Theorem A in [7]. □

Lemma 2.2 *Assume that A and B are two subgroups of a group G and $AB^g = B^gA$ holds for any $g \in G$.*

(1) If X and Y are two subsets of G, then the commutator $[\langle A^X\rangle, \langle B^Y\rangle]$ is a subnormal subgroup of G.

(2) If $G \neq AB$, then either A or B is contained in a proper normal subgroup of G.

Proof (1) is Theorem 1.1.9 in [18], (2) is [1, VI, 4.10]. □

Lemma 2.3 ([14, Lemma 2.2]) *Let p be a prime and P a Sylow p-subgroup of G. Assume that $N_G(P)$ is p-nilpotent. Then G is p-nilpotent if $P \cap O^p(G) \leq Z_{p-1}(P)$.*

Proof of Theorem 1.3 Suppose that the result is false and let G be a counterexample with minimal order. We will get a contradiction in several steps. Suppose that H is the subgroup quantified in the theorem. Then H is s-semipermutable in G by hypotheses.

Step 1. $O_{p'}(G) = 1$.

Clearly, the hypotheses hold for $G/O_{p'}(G)$ by Lemma 2.1. Hence the minimal choice of G implies that $O_{p'}(G) = 1$.

Step 2. For any subgroup S of G with $P \leq S < G$, S is p-nilpotent.

By Lemma 2.1, the hypotheses hold for S and thus S is p-nilpotent by the minimal choice of G.

Step 3. H is not trivial and G is not simple.

If $H = 1$, then applying the Hall's result (Theorem 1.2), we have that G is p-nilpotent, a contradiction.

If G is simple, then HQ^g is a proper subgroup of G for any Sylow q-subgroup Q and any $g \in G$, where $q \neq p$. By Lemma 2.2 we know that there is a proper normal subgroup L of G such that L contains H or Q, a contradiction.

Step 4. For any minimal normal subgroup N of G, G/N is p-nilpotent. Hence N is the unique minimal normal subgroup of G and $\Phi(G) = 1$.

Consider the factor group $\overline{G} = G/N$. Then $N_{\overline{G}}(\overline{P}) = \overline{N_G(P)}$ is p-nilpotent. By Lemma 2.1 we know that $\overline{H}$ satisfies the hypotheses of the theorem. By the minimal choice of G we have that G/N is p-nilpotent.

If $\Phi(G) \neq 1$, then $N \leq \Phi(G)$. Hence G is p-nilpotent as the class of p-nilpotent groups is saturated. Hence Step 4 follows.

Step 5. $N \cap H < N$.

If $N \cap H = N$, then $N \leq H \leq \Phi(P)$. Then $N \leq \Phi(G)$, contrary to Step 4.

Step 6. $O_p(G) = 1$. Hence N is not solvable. Furthermore, N is not p-nilpotent.

Assume that $O_p(G) \neq 1$. Then $N \leq O_p(G)$.

(6.1) $N = O_p(G)$.

Since $N \nleq \Phi(G)$ by Step 4, there exists a maximal subgroup M of G such that $N \nleq M$. Then $G = NM$. It is easy to see that $N \cap M$ is normal in G, we have $N \cap M = 1$ by the minimality of N. Therefore, $O_p(G) = O_p(G) \cap G = N(O_p(G) \cap M)$. Since $\Phi(O_p(G)) \leq \Phi(G) = 1$, we have that $O_p(G)$ is abelian. Thus $O_p(G) \cap M$ is a normal subgroup of G. Considering that N is the unique minimal normal subgroup of G, we have that $O_p(G) \cap M = 1$. Therefore, $N = O_p(G)$.

(6.2) $N \cap H = 1$.

By Lemma 2.1, $N \cap H$ is s-permutable in G. Since $N \cap H$ is normal in P, $H \cap N$ is normal in G. By Step 5 and minimality of N, we have $N \cap H = 1$.

(6.3) $N \leq Z_{p-1}(P)$.

Denote $\overline{P} = P/H$ and $\overline{N} = NH/H$. From hypotheses, we have

$$[\overline{N}, \underbrace{\overline{P}, \overline{P}, \ldots, \overline{P}}_{p-1}] = 1.$$

Since N is normal in P, we then have

$$[N, \underbrace{P, P, \ldots, P}_{p-1}] \leq N \cap H = 1.$$

It follows that $N \leq Z_{p-1}(P)$.

(6.4) Completing Step 6.

Let L be a Hall p'-subgroup of G. Then $LN \trianglelefteq G$. Since $G/(LN)$ is a p-group, $O^p(G) \leq LN$. It follows that $O^p(G) \cap P \leq P \cap LN = N \leq Z_{p-1}(P)$. Applying Lemma 2.3, G is p-nilpotent, contrary to the choice of G.

Step 7. $H \cap N = 1$.

If $H \cap N \neq 1$, then $N \leq (H \cap N)^G$. By Lemma 2.1 we know that $H \cap N$ is s-semipermutable and $(H \cap N)^G$ is solvable, this is contrary to Step 6.

Step 8. The final contradiction.

By Steps 1 and 6, we know that $N_p \neq 1$ and $N_G(N_p) < G$. Since $P \leq N_G(N_p)$, $N_G(N_p)$ is p-nilpotent by Step 2. Then $N_N(N_p)$ is p-nilpotent. Applying Step 7, we have

$$N_p \cong N_p/N_p \cap H \cong N_pH/H \leq P/H.$$

Hence the nilpotent class of N_p is less than p. By Hall's result (Theorem 1.2), we have that N is p-nilpotent, contrary to Step 6.

These complete the proof of the theorem. □

Proof of Corollary 1.5 Choose $N = P'$. Then P/N is abelian. □

Proof of Corollary 1.6 By [8, Theorem 3.2], we know that $N_G(P)$ is p-nilpotent. Applying Theorem 1.3, the corollary follows. □

References

[1] B. HUPPERT. *Endliche Gruppen I*. Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[2] O. ORE. *Contributions in the theory of groups of finite order*. Duke Math. J., 1939, **5**: 431–460.

[3] S. E. STONEHEWER. *Permutable subgroups in inifinite groups*. Math. Z., 1972, **125**: 1–16.

[4] O. H. KEGEL. *Sylow-Gruppen und Subnormalteiler endlicher Gruppen*. Math. Z., 1962, **78**: 205–221.

[5] Zhiming CHEN. *On a theorem of Srinivasan*. J. Southwest Normal Univ. Nat. Sci., 1987, **12**: 1–4.

[6] Y. BERKOVICH, I. M. ISAACS. *p-supersolvability and actions on p-groups stabilizing certain subgroups*. J. Algebra, 2014, **414**: 82–94.

[7] I. M. ISAACS. *Semipermutable p-subgroups*. Arch. Math., 2014, **102**: 1–6.

[8] Yangming LI, Shouhong QIAO, Ning SU, et al. *On weakly s-semipermutable subgroups of finite groups*. J. Algebra, 2012, **371**: 250–261.

[9] Qinhai ZHANG, Lifang WANG. *The influence of s-semipermutable subgroups on the structure of finite groups*. Acta Math. Sinica (Chin. Ser.), 2005, **48**(1): 81–88. (in Chinese)

[10] H. KURZWEIL, B. STELLMACHER. *The Theory of Finite Groups: An Introduction*. Springer, New York, 2003.

[11] P. HALL. *On a theorem of Frobenius*. Proc. London Math. Soc., 1936, **40**: 468–501.

[12] A. BALLESTER-BOLINCHES, Xiuyun GUO, Yangming LI, et al. *On finite p-nilpotent groups*. Monatsh. Math., 2016, **181**(1): 63–70.

[13] Yangming LI, Ning SU, Yanming WANG. *A generalization of Burnside's p-nilpotency criterion*. J. Group Theory, 2017, **20**(1): 185–192.

[14] Ning SU, Yangming LI, Yanming WANG. *A criterion on the finite p-nilpotent groups*. Adv. Math. (China), 2018, **47**(1): 65–70.

[15] T. S. WEIGEL. *Finite p-groups which determine p-nilpotency locally*. Hokkaido Math. J., 2012, **41**(1): 11–29.

[16] Yun FAN, Xiuyun GUO, K. P. SHUM. *Remarks on two generalizations of normality of subgroups*. Chinese Ann. Math. Ser. A, 2006, **27**(2): 169–176.

[17] Xiuyun GUO, Xianhe ZHAO. *π-quasinormality of the maximal subgroups of a Sylow subgroup in a local subgroup*. Acta Math. Sci. Ser. A (Chin. Ed.), 2008, **28**(6): 1222–1226.

[18] A. BALLESTER-BOLINCHES, R. ESTEBAN-ROMERO, M. ASAAD. *Product of Finite Groups*. Walter de Gruyter, Berlin-New York, 2010.