Journal of Mathematical Research with Applications
May, 2019, Vol. 39, No. 3, pp. 269–276
DOI:10.3770/j.issn:2095-2651.2019.03.006
Http://jmre.dlut.edu.cn

Hexagonal Patterns in the 2-D Lengyel-Epstein System

Li-na ZHANG*, **Fei XU**
Department of Mathematics and statistics, Northwest Normal University, Gansu 730070*, P. R. China*

Abstract Some qualitative behaviors of stationary solutions for the 2-D Lengyel-Epstein system are considered from the local bifurcation viewpoint in this paper. First, local bifurcation branches of hexagonal stationary solutions are constructed in the special case when the habitat domain is a rectangle. Next, the type of the bifurcation diagram near the bifurcation points is discussed.

Keywords Lengyel-Epstein system; bifurcation; hexagonal patterns

MR(2010) Subject Classification 35B32; 35B36

1. Introduction

One of the most fundamental problems in theoretical biology is to explain the mechanism by which patterns and forms are created in the living world. Attempting to model this mechanism, A. Turing [1] proposed the striking idea of "diffusion-driven instability" in 1952. The first experimental evidence of Turing's idea was observed in 1990 by De Kepper and her associates [2] on the CIMA reaction in an open unstirred gel reactor, almost 40 years after Turing's prediction. Lengyel and Epstein characterized this famous experiment using a system of 2×2 reaction-diffusion equations [3, 4], i.e., so-called Lengyel-Epstein model, which takes the form

$$\begin{cases} u_t = \Delta u + a - u - \dfrac{4uv}{1+u^2}, & \text{in } \Omega \times (0,\infty), \\ v_t = \sigma[c\Delta v + b(u - \dfrac{uv}{1+u^2})], & \text{in } \Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad v(x,0) = v_0(x), & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega \times (0,\infty), \end{cases} \tag{1.1}$$

where Ω is a bounded domain in R^n, with a smooth boundary $\partial\Omega$, ν is the unit outer normal to $\partial\Omega$; u and v denote the chemical concentrations of the activator iodide and the inhibitor chlorite, respectively; a and b are parameters related to the feed concentrations; c is the radio of the diffusion coefficients; $\sigma > 1$ is a rescaling parameter, enlarging the effective diffusion ratio to σc. We shall assume that all constants a, b, c and σ are positive.

Over the past decades, a number of rigorous mathematical investigations focus on the system (1.1) when the spatial domain is one-dimensional. Jang, Ni and Tang [5] discussed the global bifurcation structure of the set of the non-constant steady states by taking the effective diffusion

Received May 30, 2018; Accepted December 8, 2018
Supported by the National Natural Science Foundation of China (Grant Nos. 11761063; 61872227).
* Corresponding author
E-mail address: linazhang@nwnu.edu.cn (Li-na ZHANG); 1210038248@qq.com (Fei XU)

rate $d(=c/b)$ as bifurcation parameter. Yi, Wei and Shi [6] performed a detailed Hopf bifurcation analysis for both the ODE and PDE models, and investigated the direction of the Hopf bifurcation and the stability of the bifurcating spatially homogeneous periodic solutions. Du and Wang [7] gave the existence of multiple spatially nonhomogeneous periodic solutions though all the parameters of the system were spatially homogeneous. Wei, Wu and Guo [8] studied the steady state structures, especially the double bifurcation one, and their stability and multiplicity by the use of Lyapunov-Schmidt reduction technique and singularity theory. In [9], taking the feeding rate a as the bifurcation parameter, the authors proved that the PDE system (1.1) undergoes a sequence of bifurcations generating spatially nonhomogeneous time-periodic solutions and steady state solutions, which strongly suggested the richness of spatiotemporal dynamics.

When the spatial domain is high dimensional, although various numerical studies on the system (1.1) have been conducted [10–12], the mathematical progress (on the analytic aspects) has been very limited. Ni and Tang [13] obtained the existence of nonconstant steady states using the degree theory. As they pointed out, the drawback in the degree-theoretical approach is that they are not able to say much about the shape of the solution obtained this way.

Motivated by the above discussions, in this paper, we will discuss a detailed bifurcation structure of stationary solutions for the two-dimension case when the habitat domain is a rectangle. That is, we treat the corresponding stationary problem

$$\begin{cases} \Delta u + a - u - \dfrac{4uv}{1+u^2} = 0 & \text{in } \Omega, \\ d\Delta v + u - \dfrac{uv}{1+u^2} = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\Omega = (0, \pi/l) \times (0, \pi/(\sqrt{3}l))\ (l > 0), d = c/b$. Obviously, (1.2) has a unique constant solution $(u^*, v^*) := (\alpha, 1 + \alpha^2),\ \alpha = a/5$. In this paper, we shall maintain the basic hypothesis

$$0 < 3\alpha^2 - 5 < \sigma\alpha b, \tag{H}$$

which makes the system (1.1) a diffusion-free stable activator-inhibitor system [13].

Our main tool in the analysis is the bifurcation theorem by Crandall and Rabinowitz [14]. In the application of the theorem, we regard d as a bifurcation parameter. The organization of this paper is as follows. In Section 2, we find infinitely many degenerate (bifurcation) points on the positive constant solution set (trivial branch) $\Gamma := \{(d; u^*, v^*) : d > 0\}$. Section 3 obtains the local branch of nonconstant solutions arising from the double bifurcation following the methed of Nishida et al. [15] for the bifurcation problem in hydrodynamics. In Section 4, we discuss the type of the bifurcation diagram near the bifurcation points.

2. Degeneracy points on the constant solution set

In this section, we show the existence of nonconstant solutions bifurcating from the positive constant solution $(u^*, v^*) = (\alpha, 1 + \alpha^2)$. For the framework of the bifurcation argument, we introduce two Hilbert spaces X and Y defined by

$$X = H_\nu^2(\Omega) \times H_\nu^2(\Omega), \qquad Y = L^2(\Omega) \times L^2(\Omega).$$

By regarding d as the bifurcation parameter, we can find the bifurcation points on the trivial branch $\Gamma = \{(d; u^*, v^*) : d > 0\} \subset \mathbb{R} \times X$.

For convenience, let

$$f(u,v) = a - u - \frac{4uv}{1+u^2}, \quad g(u,v) = u - \frac{uv}{1+u^2}.$$

Then the system (1.2) can be written as

$$\begin{cases} \Delta u + f(u,v) = 0 & \text{in } \Omega, \\ d\Delta v + g(u,v) = 0 & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Denote

$$f_0 = \frac{\partial f}{\partial u}(u^*, v^*) = \frac{3\alpha^2 - 5}{1+\alpha^2}, \quad f_1 = \frac{\partial f}{\partial v}(u^*, v^*) = -\frac{4\alpha}{1+\alpha^2},$$
$$g_0 = \frac{\partial g}{\partial u}(u^*, v^*) = \frac{2\alpha^2}{1+\alpha^2}, \quad g_1 = \frac{\partial g}{\partial v}(u^*, v^*) = -\frac{\alpha}{1+\alpha^2}.$$

For the application of the local bifurcation theory by Crandall and Rabinowitz [14], we define the operator $F : \mathbb{R} \times X \to Y$ associated with (2.1) by

$$F(d; u, v) = \begin{pmatrix} \Delta u + f(u,v) \\ d\Delta v + g(u,v) \end{pmatrix}.$$

By virtue of the implicit function theorem, the Frechet derivative of F with respect to (u, v) at (u^*, v^*) is given by

$$F_{(u,v)}(d) \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} \Delta h + f_0 h + f_1 k \\ d\Delta k + g_0 h + g_1 k \end{pmatrix} \tag{2.2}$$

and must be degenerate at any bifurcation point. So, we seek d such that $F_{(u,v)}(d)$ has a zero eigenvalue. To do so, we consider the linear elliptic boundary value problem

$$\begin{cases} \Delta h + f_0 h + f_1 k = 0 & \text{in } \Omega, \\ d\Delta k + g_0 h + g_1 k = 0 & \text{in } \Omega, \\ \dfrac{\partial h}{\partial \nu} = \dfrac{\partial k}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Let a solution (h, k) of (2.3) be represented by the Fourier expansion formula

$$h(x,y) = \sum_{m,n=0}^{\infty} h_{mn}\phi_m(x)\psi_n(y), \quad k(x,y) = \sum_{m,n=0}^{\infty} k_{mn}\phi_m(x)\psi_n(y), \tag{2.4}$$

where

$$\phi_m(x) := \cos(lmx), \quad \psi_n(y) := \cos(\sqrt{3}lny). \tag{2.5}$$

Substituting (2.4) into (2.3), we show that (h_{mn}, k_{mn}) satisfies

$$\begin{cases} \sum_{m,n=0}^{\infty}[(-\lambda_{mn}+f_0)h_{mn}+f_1k_{mn}]\phi_m(x)\psi_n(y)=0, \\ \sum_{m,n=0}^{\infty}[g_0h_{mn}+(g_1-d\lambda_{mn})k_{mn}]\phi_m(x)\psi_n(y)=0, \end{cases} \tag{2.6}$$

where $\lambda_{mn}=(m^2+3n^2)l^2$. Since $\{\phi_m(x)\psi_n(y)\}_{m,n=0}^{\infty}$ forms a complete orthogonal base of X, (2.6) can be reduced to the algebraic equations

$$\begin{pmatrix} -\lambda_{mn}+f_0 & f_1 \\ g_0 & -d\lambda_{mn}+g_1 \end{pmatrix}\begin{pmatrix} h_{mn} \\ k_{mn} \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{2.7}$$

for all $m,n\in N\cup\{0\}$. Since $\lambda_{00}=0$ and $f_0g_1-f_1g_0=5\alpha/(1+\alpha^2)>0$, $h_{00}=k_{00}=0$. Therefore, (2.3) has nontrivial solutions if and only if (2.7) has nontrivial solutions for some (m,n) satisfying $m^2+n^2>0$, that is,

$$\begin{vmatrix} -\lambda_{mn}+f_0 & f_1 \\ g_0 & -d\lambda_{mn}+g_1 \end{vmatrix}=0,\quad m^2+n^2>0. \tag{2.8}$$

Therefore, Ker $F_{(u,v)}(d)$ is nontrivial if and only if

$$d=\frac{-g_1\lambda_{mn}+f_0g_1-f_1g_0}{\lambda_{mn}(f_0-\lambda_{mn})}=\frac{\alpha}{1+\alpha^2}\frac{\lambda_{mn}+5}{\lambda_{mn}(f_0-\lambda_{mn})}=:d(m,n),\quad m^2+n^2>0. \tag{2.9}$$

Remark 2.1 It follows that $d(m,n)>0$ if and only if $f_0>\lambda_{mn}$.

Remark 2.2 If the parameters satisfy some appropriate conditions, then $\dim\mathrm{Ker}F_{(u,v)}(d(m,n))=1$. For example, $\dim\mathrm{Ker}F_{(u,v)}(d(1,0))=1$ if $l^2<f_0<3l^2$. Because there is only λ_{10} satisfying $f_0>\lambda_{mn}$ and $m^2+n^2>0$.

Remark 2.3 $(m,n)\longmapsto d(m,n)$ is not a one-to-one correspondence. For example, $d(1,1)=d(2,0)$ and $d(1,3)=d(5,1)=d(4,2)$. On the other hand, $d(m,n)$ is not monotonous function for λ_{mn}. It is easy to see that $d(m,n)=d(i,j)$ if λ_{ij} and λ_{mn} satisfy $\lambda_{mn}\lambda_{ij}+5(\lambda_{ij}+\lambda_{mn})-5f_0=0$. Therefore, in most cases $\dim\mathrm{Ker}F_{(u,v)}(d(m,n))>1$.

Remark 2.4 If $f_0<13l^2$, then there are following λ_{mn}:

$$\lambda_{10}=l^2,\lambda_{20}=4l^2,\lambda_{30}=9l^2,\lambda_{01}=3l^2,\lambda_{11}=4l^2,\lambda_{21}=7l^2,\lambda_{31}=12l^2,\lambda_{02}=12l^2$$

satisfying $f_0>\lambda_{mn}$ and $m^2+n^2>0$. It is easy to see $d(1,1)=d(2,0)$ and $d(3,1)=d(0,2)$. Furthermore, assume that $d(m,n)\neq d(i,j)$ if $\lambda_{mn}\neq\lambda_{ij}$. It is verified that

$$\dim\mathrm{Ker}F_{(u,v)}(d(2,0))=\dim\mathrm{Ker}F_{(u,v)}(d(1,1))=2, \tag{2.10}$$

$$\dim\mathrm{Ker}F_{(u,v)}(d(3,1))=\dim\mathrm{Ker}F_{(u,v)}(d(0,2))=2.$$

Similarly, we can also make $\dim\mathrm{Ker}F_{(u,v)}(d(m,n))=3$ under some appropriate assumptions on the parameters.

For the one-dimensional degenerate cases, that is, dim Ker$F(u,v)(d(m,n))=1$, we already discussed nonconstant solutions of (2.1) in [16]. In this paper, we discuss dim Ker$F_{(u,v)}(d(m,n))=2$ as a simple example of the dimensional degenerate cases in Section 3.

3. Local bifurcation branch of hexagonal patterns

In this section, we study the case (2.10) as a typical case when that dim $\mathrm{Ker}F_{(u,v)}(d(m,n)) = 2$. Among other things, we will show that a local branch of the hexagonal solutions bifurcates from the trivial branch Γ at $d = d(2,0) = d(1,1)$. It is easy to see that

$$\mathrm{Ker}F_{(u,v)}(d(2,0)) = \mathrm{Ker}F_{(u,v)}(d(1,1)) = \mathrm{span}\left\{ \begin{pmatrix} \phi_2(x) \\ k_{20}\phi_2(x) \end{pmatrix}, \begin{pmatrix} \phi_1(x)\psi_1(y) \\ k_{11}\phi_1(x)\psi_1(y) \end{pmatrix} \right\}, \quad (3.1)$$

where

$$k_{mn} = (\lambda_{mn} - f_0)/f_1. \quad (3.2)$$

In this case, we cannot apply the local bifurcation theory in X. Therefore, we introduce the closed subspace H^2_{hexa} of $H^2_\nu(\Omega)$, defined by

$$H^2_{\mathrm{hexa}} = \{w(x,y) = \sum_{m+n=\mathrm{even}}^{\infty} \beta_{mn}(\phi_m(x)\psi_n(y) + \cos\frac{l(m-3n)x}{2}\cos\frac{\sqrt{3}l(m+n)y}{2} + \cos\frac{l(m+3n)x}{2}\cos\frac{\sqrt{3}l(m-n)y}{2}) : \sum_{m+n=\mathrm{even}}^{\infty} l^4(m^2+3n^2)^2\beta^2_{mn} < \infty\}. \quad (3.3)$$

We remark that an element of H^2_{hexa} is invariant with respect to the $2\pi/3-$rotation. In view of (3.3), by letting $(m,n) = (2,0)$ in

$$\phi_m(x)\psi_n(y) + \cos\frac{l(m-3n)x}{2}\cos\frac{\sqrt{3}l(m+n)y}{2} + \cos\frac{l(m+3n)x}{2}\cos\frac{\sqrt{3}l(m-n)y}{2},$$

we see that

$$\phi_2(x)\psi_0(y) + 2\cos(lx)\cos(\sqrt{3}ly) = \phi_2(x)\psi_0(y) + 2\phi_1(x)\psi_1(y).$$

Consequently, setting

$$\beta_{m,n} = \begin{cases} 1, & \text{if } (m,n) = (2,0), \\ 0, & \text{otherwise} \end{cases}$$

in (3.3), we have $\phi_2(x)\psi_0(y) + 2\phi_1(x)\psi_1(y) \in H^2_{\mathrm{hexa}}$. This implies that a hexagonal pattern is represented by $\phi_2(x)\psi_0(y) + 2\phi_1(x)\psi_1(y)$. As in the case of (3.1), $d(2,0) = d(1,1)$ is a double eigenvalue in the sense that

$$\dim\ \mathrm{Ker}F_{(u,v)}(d(1,1)) = \dim\ \mathrm{Ker}F_{(u,v)}(d(2,0)) = 2.$$

Next we introduce the operator $\tilde{F}$ by a restriction of F to the domain of $\mathbb{R} \times H^2_{\mathrm{hexa}} \times H^2_{\mathrm{hexa}}$, namely, $\tilde{F} : \mathbb{R} \times H^2_{\mathrm{hexa}} \times H^2_{\mathrm{hexa}} \to Y$. Therefore, it follows that

$$\mathrm{Ker}\tilde{F}_{(u,v)}(d(2,0)) = \mathrm{Ker}\tilde{F}_{(u,v)}(d(1,1)) = \mathrm{span}\left\{ \begin{pmatrix} \phi_2(x) + 2\phi_1(x)\psi_1(y) \\ k_{20}(\phi_2(x) + 2\phi_1(x)\psi_1(y)) \end{pmatrix} \right\},$$

and

$$\dim\ \mathrm{Ker}\tilde{F}_{(u,v)}(d(2,0)) = \dim\ \mathrm{Ker}\tilde{F}_{(u,v)}(d(1,1)) = 1.$$

For the restriction operator $\tilde{F}$, we can apply the local bifurcation theory of [14] to obtain the following local bifurcation branch of the hexagonal pattern of (2.1).

Theorem 3.1 *Assume that $0 < f_0 < 13l^2$. There exists a positive constant δ and a neighborhood $O_{\rm hexa}$ of $(d;u,v) = (d(2,0);u^*,v^*)$ in $\mathbb{R} \times H^2_{\rm hexa} \times H^2_{\rm hexa}$ such that nonconstant solutions of (2.1) contained in $O_{\rm hexa}$ can be represented by*

$$\begin{pmatrix} u(s) \\ v(s) \end{pmatrix} = \begin{pmatrix} u^* \\ v^* \end{pmatrix} + s \begin{pmatrix} \phi_2(x) + 2\phi_1(x)\psi_1(y) \\ k_{20}(\phi_2(x) + 2\phi_1(x)\psi_1(y)) \end{pmatrix} + s^2 \begin{pmatrix} \tilde{u}(s) \\ \tilde{v}(s) \end{pmatrix} \tag{3.4}$$

and

$$d(s) = d(2,0) + s\beta(s) \tag{3.5}$$

for $s \in [-\delta, \delta]$. Here $\phi_m(x)$ and $\psi_n(y)$ are defined in (2.5), k_{20} is the positive constant in (3.2) with $(m,n) = (2,0)$, and $(\tilde{\beta}(s); \tilde{u}(s), \tilde{v}(s)) \in \mathbb{R} \times H^2_{\rm hexa} \times H^2_{\rm hexa}$ is a smooth function of s.

Proof We only have to verity

$$\tilde{F}_{(u,v)}(d(2,0)) \begin{pmatrix} \phi_2(x) + 2\phi_1(x)\psi_1(y) \\ k_{20}(\phi_2(x) + 2\phi_1(x)\psi_1(y)) \end{pmatrix} \notin {\rm Range}\tilde{F}_{(u,v)}(d(2,0)). \tag{3.6}$$

In view of (2.5), one can easily obtain

$$\tilde{F}_{(u,v)}(d(2,0)) \begin{pmatrix} \phi_2(x) + 2\phi_1(x)\psi_1(y) \\ k_{20}(\phi_2(x) + 2\phi_1(x)\psi_1(y)) \end{pmatrix} = \begin{pmatrix} 0 \\ -4l^2 k_{20}(\phi_2(x) + 2\phi_1(x)\psi_1(y)) \end{pmatrix}. \tag{3.7}$$

By the Fredholm alternative theorem, the adjoint operator $\tilde{L}^*$ of $\tilde{F}_{(u,v)}(d(2,0))$ satisfies

$${\rm Range}\tilde{F}_{(u,v)}(d(2,0)) = ({\rm Ker}\tilde{L}^*)^{\perp}. \tag{3.8}$$

Here $\tilde{L}^* : H^2_{\rm hexa} \times H^2_{\rm hexa} \to Y$ is defined by

$$\tilde{L}^* \begin{pmatrix} h \\ k \end{pmatrix} = \begin{pmatrix} (\Delta + f_0)h + g_0 k \\ f_1 h + (d(2,0)\Delta + g_1)k \end{pmatrix}.$$

After some computations, one can obtain

$${\rm Ker}\tilde{L}^* = {\rm span}\left\{ \begin{pmatrix} \phi_2(x) + 2\phi_1(x)\psi_1(y) \\ k^*_{20}\{\phi_2(x) + 2\phi_1(x)\psi_1(y)\} \end{pmatrix} \right\},$$

where k^*_{20} is the positive number defined by $k^*_{mn} = \frac{\lambda_{mn} - f_0}{g_0}$ with $(m,n) = (2,0)$. By taking the L^2 inner product of the right-hand sides of (3.7) and the base of Ker L^*, we have

$$-\iint_\Omega 4l^2 k_{20} k^*_{20} (\phi_2(x) + 2\phi_1(x)\psi_1(y))^2 {\rm d}x{\rm d}y > 0,$$

which implies

$$\tilde{F}_{(u,v)}(d(2,0)) \begin{pmatrix} \phi_2(x) + 2\phi_1(x)\psi_1(y) \\ k_{20}(\phi_2(x) + 2\phi_1(x)\psi_1(y)) \end{pmatrix} \notin ({\rm Ker}\tilde{L}^*)^{\perp}. \tag{3.9}$$

Then we can show (3.6) from (3.8) and (3.9). We can apply the local bifurcation theory to $\tilde{F}$ at $(d(2,0);u^*,v^*)$ and obtain (3.4) and (3.5). Thus the proof Theorem 3.1 is completed. □

4. Type of bifurcation diagram

We now remark that the bifurcation diagram of the hexagonal solutions represented by (3.4) and (3.5) is generically transversal with respect to the trivial branch Γ, which is different from the pitchfork bifurcation of the tripe and rectangle types. Actually, we get the following expression for $\beta(0)$.

Theorem 4.1 *Let $\beta(s)$ be the function obtained in (3.5). Then*

$$\beta(0)=\frac{f_1g_0f_{00}-2f_{01}(f_0-4l^2)g_0+(f_0-4l^2)[g_{00}f_1+2g_{01}(f_0-4l^2)]}{8l^2(f_0-4l^2)^2}, \tag{4.1}$$

where $f_{00}:=f_{uu}(u^*,v^*)$, $f_{01}:=f_{uv}(u^*,v^*)$, $g_{00}:=g_{uu}(u^*,v^*)$, $g_{01}:=g_{uv}(u^*,v^*)$.

Proof In view of (3.4) and (3.5), we set

$$\Phi(x,y)=\phi_2(x)+2\phi_1(x)\psi_1(y).$$

By substituting $(d(s);u(s),v(s))$ into (2.1), differentiating the equations of (2.1) twice with respect to s and setting $s=0$, we have

$$2\Delta\tilde{u}(0)+2f_0\tilde{u}(0)+2f_1\tilde{v}(0)+f_{00}\Phi^2+2f_{01}k_{20}\Phi^2=0, \tag{4.2}$$

$$-2\beta(0)\lambda_{20}k_{20}\Phi+2d(2,0)\Delta\tilde{v}(0)+2g_0\tilde{u}(0)+2g_1\tilde{v}(0)+g_{00}\Phi^2+2g_{01}k_{20}\Phi^2=0. \tag{4.3}$$

Taking the L^2 inner product of (4.2) and (4.3) with Φ and using Green's formula, we obtain

$$2(f_0-\lambda_{20})\langle\tilde{u}(0),\Phi\rangle+2f_1\langle\tilde{v}(0),\Phi\rangle+(f_{00}+2f_{01}k_{20})\langle\Phi^2,\Phi\rangle=0,$$

$$-2\beta(0)\lambda_{20}k_{20}\langle\Phi,\Phi\rangle+2g_0\langle\tilde{u}(0),\Phi\rangle+2(g_1-d(2,0)\lambda_{20})\langle\tilde{v}(0),\Phi\rangle+(g_{00}+2g_{01}k_{20})\langle\Phi^2,\Phi\rangle=0.$$

Since $d(2,0)$ and λ_{20} satisfy (2.8) and

$$\langle\Phi,\Phi\rangle=\frac{\sqrt{3}\pi^2}{2l^2},\quad \langle\Phi^2,\Phi\rangle=\frac{\sqrt{3}\pi^2}{2l^2},$$

we have

$$\beta(0)=\frac{f_{00}+2f_{01}k_{20}+k_{20}^*(g_{00}+2g_{01}k_{20})}{2\lambda_{20}k_{20}k_{20}^*}. \tag{4.4}$$

Substituting

$$\lambda_{20}=4l^2,\quad k_{20}=\frac{f_0-\lambda_{20}}{f_1},\quad k_{20}^*=\frac{f_0-\lambda_{20}}{g_0}$$

into (4.4), we obtain (4.1). Thus the proof of Theorem 4.1 is completed. □

Acknowledgements We thank the referees for their time and comments.

References

[1] A. M. TURING. *The chemical basis of morphogenesis*. Bull. Math. Biol., 1990, **52**(1-2): 153–197.

[2] F. M. BOISSONADE, M. CASTETS, J. DULOS, et al. *Experimental evidence of a sustained standing Turing-type nonequilibrium chemical pattern*. Phys. Rev. Lett., 1990, **64**(24): 2953–2956.

[3] I. LENGYEL, I. R. EPSTEIN. *Modeling of turing structures in the chlorite-iodide-malonic Acid-starch reaction system*. Science, 1991, **251**: 650–652.

[4] I. LENGYEL, I. R. EPSTEIN. *A chemical approach to designing Turing patterns in reaction-diffusion systems*. Proc. Natl. Acad. Sci., 1992, **89**(9): 3977–3979.

[5] J. JANG, Weiming NI, Moxun TANG. *Global bifurcation and structure of Turing patterns in the 1-D Lengyel-Epstein model*. J. Dynam. Differential Equations, 2004, **16**(2): 297–320.

[6] Fengqi YI, Junjie WEI, Junping SHI. *Diffusion-driven instability and bifurcation in the Lengyel-Epstein system*. Nonlinear Anal. Real World Appl., 2008, **9**(3): 1038–1051.

[7] Linglong DU, Mingxin WANG. *Hopf bifurcation analysis in the 1-D Lengyel-Epstein reaction-diffusion model*. J. Math. Anal. Appl., 2010, **366**(2): 473–485.

[8] Meihua WEI, Jianhua WU, Gaihui GUO. *Turing structures and stability for the 1-D Lengyel-Epstein system*. J. Math. Chem., 2012, **50**(9): 2374–2396.

[9] Jiayin JIN, Junping SHI, Junjie WEI, et al. *Bifurcations of patterned solutions in the diffusive Lengyel-Epstein system of CIMA chemical reaction*. Rocky Mountain J. Math., 2013, **43**(5): 1637–1674.

[10] T. K. CALLAHAN, E. KNOBLOCH. *Pattern formation in three-dimensional reaction-diffusion systems*. Phys. D, 1999, **132**(3): 339–362.

[11] S. L. JUDD, M. SILBER. *Simple and superlattice Turing patterns in reaction-diffusion systems: bifurcation, bistability, and parameter collapse*. Phys. D, 2000, **136**(1-2): 45–65.

[12] S. KAR, J. K. BHATTACHARJEE, D. S. RAY. *A model reaction-diffusion system under spatial perturbation:theoretical and numerical investigation*. Eur. Phys. J. B, 2005, **43**(1): 109–114.

[13] Weiming NI, Moxun TANG. *Turing patterns in the Lengyel-Epstein system for the CIMA reaction*. Trans. Amer. Math. Soc., 2005, **357**(10): 3953–3969.

[14] M. G. CRANDALL, P. H. RABINOWITZ. *Bifurcation from simple eigenvalues*. J. Funct. Anal., 1970, **8** (2): 321–340.

[15] T. NISHIDA, T. IKEDA, H. YOSHIHARA. *Pattern Formation of Heat Convection Problems*. Springer Berlin Heidelberg, 2002.

[16] Yuexia LI, Li-na ZHANG, Xiaojie ZHANG. *Bifurcation structures for the 2-D Lengyel-Epstein system*. Journal of Shandong University (Ji'nan), 2016, **51**(8): 74–83. (in Chinese)