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Common Best Proximity Points Theorems

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Abstract In this paper, an existence and uniqueness common best proximity point theorem for a pair of non-self mappings was proved. Moreover, an example is given to support our main result, which generalized some well-known results of Sadiq Basha, A.Amini-Harandi and Geraghty and so on.

Keywords common best proximity point; generally proximally dominating mappings; common fixed points

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1. Introduction and preliminaries

Fixed point theory is an important tool for solving equations $Tx = x$ for self-mappings T defined on subsets of metric spaces. Because T is not a self-mapping, the equation $Tx = x$ is unlikely to have a solution. Therefore, it is of primary importance to seek an element x which is in some sense closest to Tx. Best approximation theorems and best proximity point theorems are relevant in this perspective. A noteworthy best approximation theorem, due to [1], contends that if A is a non-void compact convex subset of a Hausdorff locally convex topological vector space X, and $T : A \to X$ is a continuous single-valued function, then there exists an element x in A such that $d(x, Tx) = d(Tx, A)$. There have been many subsequent extensions and variants of Fan's Theorem, see [2–4] and references therein.

A best proximity point theorem for non-self proximal contractions has been investigated in [5]. Analysis of several variants of contractions for the existence of a best proximity point can be found in [6–8], and research of mutually nearest and mutually furthest points problems in Banach spaces can be found in [9–13]. Best proximity point theorems for set-valued mappings have been elicited in [14–20].

Given nonempty subsets A and B of a metric space, we recall the following notations and notions, which will be used in the sequel.

$$\begin{aligned} &d(A, B) = \inf\{d(x, y) : x \in A, y \in B\},\\ &A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},\\ &B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}. \end{aligned}$$

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The main objective of this paper is to discuss a common best proximity point theorem. The common best proximity point theorem presented in this paper assures a common optimal solution at which both the real valued multiobjective functions $x \to d(x, Sx)$ and $x \to d(x, Tx)$ attains the global minimal value $d(A, B)$, thereby giving rise to a common optimal approximate solution to the fixed point equations $Sx = x$ and $Tx = x$ where the mappings $S : A \to B$ are generally proximally dominated by $T : A \to B$. Our best proximity point theorem generalizes a result due to [20, 21]. Moreover, a common fixed point theorem, due to [22], for commuting self-mappings is a special case of our common best proximity point theorem.

Now, we recall some definitions which we will use throughout the paper.

Definition 1.1 *A mapping $T : A \to B$ is said to be a proximal contraction if there exists a non-negative number $\alpha < 1$ such that, for all u_1, u_2, x_1, x_2 in A,*

$$d(u_1, Tx_1) = d(A, B) = d(u_2, Tx_2) \Rightarrow d(u_1, u_2) \le \alpha d(x_1, x_2).$$

Definition 1.2 *Given non-self mappings $T : A \to B$ and $S : A \to B$ are said to be commute proximally if they satisfy the condition that $d(u, Sx) = d(v, Tx) = d(A, B) \Rightarrow Sv = Tu$.*

Definition 1.3 ([20]) *A mapping $T : A \to B$ is said to dominate a mapping $S : A \to B$ proximally if there exists a non-negative number $\alpha < 1$ such that, for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A,*

$$d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2) \Rightarrow d(u_1, u_2) \le \alpha d(v_1, v_2).$$

Inspired by the above definition, we give the following definition.

Definition 1.4 *A mapping $T : A \to B$ is said to generally dominate a mapping $S : A \to B$ proximally if for all $u_1, u_2, v_1, v_2, x_1, x_2$ in A,*

$$\begin{aligned} d(u_1, Sx_1) = d(u_2, Sx_2) = d(A, B) = d(v_1, Tx_1) = d(v_2, Tx_2) \\ \Rightarrow \Psi(d(u_1, u_2)) \le \alpha(d(v_1, v_2))\Psi(d(v_1, v_2)), \end{aligned}$$

where α is a nondecreasing function from $[0, \infty)$ to $[0, 1)$ such that $\alpha(t_n) \to 1 \Rightarrow t_n \to 0$, and $\Psi : [0, \infty) \to [0, \infty)$ is an increasing continuous function such that $t \le \Psi(t)$ and $\Psi(0) = 0$.

Definition 1.5 ([20]) *Given non-self mappings $T : A \to B$ and $S : A \to B$, an element $x \in A$ is called a common best proximity point of the mappings if they satisfy the condition that*

$$d(x, Sx) = d(x, Tx) = d(A, B).$$

2. Main results

The following result is a best proximity point theorem for a pair of non-self mappings.

Theorem 2.1 *Let A and B be nonempty subsets of a complete metric space X. Moreover, assume that A_0 and B_0 are nonempty and A_0 is closed. Let the non-self mappings $T : A \to B$ and $S : A \to B$ satisfy the following conditions:*

(a) *T generally dominates S proximally;*

(b) *S and T commute proximally;*

(c) *S and T are continuous;*

(d) $S(A_0) \subseteq B_0$;

(e) $S(A_0) \subseteq T(A_0)$.

Then, there exists a unique element $x \in A$ *such that* $d(x, Sx) = d(x, Tx) = d(A, B)$.

Proof For convenience, use N to represent natural numbers. Let x_0 be a fixed element in A_0. Since $S(A_0) \subseteq T(A_0)$, there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. This process can be carried on. Having chosen $x_n \in A_0$, we can find an element $x_{n+1} \in A_0$ satisfying

$$Sx_n = Tx_{n+1}, \quad \forall n \in N, \tag{2.1}$$

because of the fact $S(A_0) \subseteq T(A_0)$. Since $S(A_0) \subseteq B_0$, there exists an element $u_n \in A_0$ such that

$$d(Sx_n, u_n) = d(A, B), \quad \forall n \in N. \tag{2.2}$$

Further, it follows from the choice x_n and u_n that

$$d(Sx_{n+1}, u_{n+1}) = d(A, B), \quad d(Tx_{n+1}, u_n) = d(A, B). \tag{2.3}$$

Since T generally dominates a mapping S proximally, from (2.1)–(2.3), we have

$$\Psi(d(u_{n+1}, u_n)) \le \alpha(d(u_n, u_{n-1}))\Psi(d(u_n, u_{n-1})) \le \Psi(d(u_n, u_{n-1})). \tag{2.4}$$

Since Ψ is increasing, $\{d(u_n, u_{n-1})\}$ is a non-increasing and bounded. So $\lim_{n\to\infty} d(u_n, u_{n-1})$ exists. Let $\lim_{n\to\infty} d(u_n, u_{n-1}) = \eta \ge 0$. Assume that $\eta > 0$. Then from (2.4) we obtain

$$\frac{\Psi(d(u_{n+1}, u_n))}{\Psi(d(u_n, u_{n-1}))} \le \alpha(d(u_n, u_{n-1})). \tag{2.5}$$

Since Ψ is continuous, the above inequality yields

$$\lim_{n\to\infty} \alpha(d(u_n, u_{n-1})) = 1, \tag{2.6}$$

and from condition (a), we have $\eta = 0$. Thus

$$\lim_{n\to\infty} d(u_n, u_{n-1}) = 0. \tag{2.7}$$

At the same time, from condition (a), we have

$$\alpha(d(u_0, u_1)) \ge \alpha(d(u_1, u_2)) \ge \cdots \ge \alpha(d(u_n, u_{n-1})). \tag{2.8}$$

Now we show that $\{u_n\}$ is a Cauchy sequence. In fact, by (2.4) and (2.8) we have

$$\Psi(d(u_{n+1}, u_n)) \le \delta^n \Psi(d(u_1, u_0)),$$

where $\delta = \alpha((d(u_0, u_1))) \in [0, 1)$. Then, we get

$$\begin{aligned} 0 &\le \Psi(d(u_0, u_1)) + \Psi(d(u_1, u_2)) + \cdots + \Psi(d(u_{n-1}, u_n)) \\ &\le \Psi(d(u_0, u_1)) + \delta\Psi(d(u_0, u_1)) + \cdots + \delta^{n-1}\Psi(d(u_0, u_1)) \end{aligned}$$

$$\leq \frac{1}{1-\delta}\Psi(d(u_0, u_1)),$$

which means that

$$\sum_{n=1}^{\infty}\Psi(d(u_{n-1}, u_n)) < \infty. \tag{2.9}$$

From condition (a), we have $t \leq \Psi(t)$ and then

$$\sum_{n=1}^{\infty} d(u_{n-1}, u_n) < \infty. \tag{2.10}$$

Therefore, for all $\epsilon > 0$,

$$d(u_n, u_m) \leq \sum_{i=n+1}^{m} d(u_{i-1}, u_i) < \epsilon \tag{2.11}$$

for sufficiently large $m > n \in N$. Then $\{u_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space and A_0 is closed, there exists $u \in A_0$ such that $\lim_{n\to\infty} u_n = u$. Because of the fact the mappings S and T are commuting proximally and from (2.3), we get

$$Tu_n = Su_{n-1}, \quad \forall n \in N.$$

Therefore, the continuity of the mappings S and T ensures that

$$Tu = \lim_{n\to\infty} Tu_n = \lim_{n\to\infty} Su_{n-1} = Su.$$

Since $S(A_0) \subseteq B_0$, there exists an $x \in A$ such that

$$d(x, Su) = d(A, B) = d(x, Tu). \tag{2.12}$$

As S and T commute proximally, $Sx = Tx$. Then, since $S(A_0) \subseteq B_0$, there exists a $z \in A$ such that

$$d(z, Sx) = d(A, B) = d(z, Tx). \tag{2.13}$$

By condition (a), (2.12) and (2.13), we have $\Psi(d(x, z)) \leq \alpha(d(x, z))\Psi(d(x, z))$, which implies that $x = z$. Thus, it follows that

$$d(x, Sx) = d(z, Sx) = d(A, B) = d(x, Tx) = d(z, Tx). \tag{2.14}$$

Therefore, x is a common best proximity point of the mappings S and T. Suppose that $\hat{x}$ is another common best proximity point of the mappings S and T, so that

$$d(\hat{x}, S\hat{x}) = d(A, B) = d(\hat{x}, T\hat{x}). \tag{2.15}$$

Then from condition (a), (2.14) and (2.15), we get $\Psi(d(x, \hat{x})) \leq \alpha(d(x, \hat{x}))\Psi(d(x, \hat{x}))$, which implies that $x = \hat{x}$. Therefore, we obtain the desired result. □

As a corollary, we get the following main result of [20].

Corollary 2.2 *Let A and B be nonempty subsets of a complete metric space X. Moreover, assume that A_0 and B_0 are nonempty and A_0 is closed. Let the non-self mappings $T : A \to B$ and $S : A \to B$ satisfy the following conditions:*

(a) T dominates S proximally;

(b) *S and T commute proximally;*

(c) *S and T are continuous;*

(d) *$S(A_0) \subseteq B_0$;*

(e) *$S(A_0) \subseteq T(A_0)$.*

Then, there exists a unique element $x \in A$ such that $d(x, Sx) = d(x, Tx) = d(A, B)$.

The following results in [23] and [24] are immediate consequences of Theorem 2.1, respectively.

Corollary 2.3 *Let A and B be nonempty subsets of a complete metric space X such that B is compact. Moreover, assume that A_0 and B_0 are nonempty. Let the non-self mapping $T : A_0 \to B_0$ be a proximal contraction. Then, there exists a unique element $x \in A_0$ such that $d(x, Tx) = d(A, B)$.*

Corollary 2.4 *Let X be a complete metric space and let $T : X \to X$ satisfy*

$$\Psi(d(Tx, Ty)) \leq \beta(d(x, y))\Psi(d(x, y)), \quad \forall x, y \in X,$$

where β is an increasing function from $[0, \infty)$ to $[0, 1)$ such that $\beta(t_n) \to 1 \Rightarrow t_n \to 0$, and $\Psi : [0, \infty) \to [0, \infty)$ is an increasing continuous function such that $t \leq \Psi(t)$ for each $t \geq 0$ and $\Psi(0) = 0$.

3. Illustration

Now we illustrate our common best proximity point theorem by the following example.

Example 3.1 Consider the complete metric space $X = [0, 1] \times [0, 1]$ with Euclidean metric. Let $A = \{(0, x) : 0 \leq x \leq 1\}$ and $B = \{(1, y) : 0 \leq y \leq 1\}$. Then $d(A, B) = 1$, $A_0 = A$ and $B_0 = B$. Let $T, S : A \to B$ be defined as $T(0, x) = (1, x)$, and $S(0, x) = (1, \ln(1 + x))$. Now we show that T generally dominates S proximally, where $\alpha(t) = 1 - \frac{\ln^2(1+t)}{2t}$ and $\Psi(t) = t$ for each $t > 0$. Let $\mathbf{u_1} = (0, u_1)$, $\mathbf{u_2} = (0, u_2)$, $\mathbf{v_1} = (0, v_1)$, $\mathbf{v_2} = (0, v_2)$, $\mathbf{x_1} = (0, x_1)$, $\mathbf{x_2} = (0, x_2)$ be elements in A satisfying

$$d(\mathbf{u_1}, S\mathbf{x_1}) = d(\mathbf{u_2}, S\mathbf{x_2}) = d(\mathbf{v_1}, T\mathbf{x_1}) = d(\mathbf{v_2}, T\mathbf{x_2}) = 1.$$

Then we have $x_i = v_i$ and $u_i = \ln(1 + x_i)$ for $i = 1, 2$. Hence

$$\begin{aligned} d(\mathbf{u_1}, \mathbf{u_2}) &= |u_1 - u_2| = |\ln(1 + v_1) - \ln(1 + v_2)| \\ &\leq \ln(1 + |v_1 - v_2|) \leq [1 - \frac{\ln^2(1 + |v_1 - v_2|)}{2|v_1 - v_2|}]|v_1 - v_2| \\ &= \alpha(d(\mathbf{v_1}, \mathbf{v_2}))d(\mathbf{v_1}, \mathbf{v_2}). \end{aligned}$$

Next we show that T does not dominate S proximally. On the contrary, assume that there exists $0 \leq \beta < 1$ such that

$$d(\mathbf{u_1}, \mathbf{u_2}) = |u_1 - u_2| = |\ln(1 + v_1) - \ln(1 + v_2)| < \beta d(\mathbf{v_1}, \mathbf{v_2}) = \beta|v_1 - v_2|, \quad \forall v_1, v_2 \in [0, 1].$$

Let $v_2 = 0$. We get

$$\frac{\ln(1 + v_1)}{v_1} \leq \beta < 1, \quad \forall v_1 \in (0, 1)$$

a contradiction (note that $\lim_{v_1\to 0^+}\frac{\ln(1+v_1)}{v_1}=1$).

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