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Solution Path of the Perturbed Karush-Kuhn-Tucker System for Stochastic Nonlinear Programming with Inequality Constraints

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Abstract This paper focuses on the study for the stability of stochastic nonlinear programming when the probability measure is perturbed. Under the Lipschitz continuity of the objective function and metric regularity of the feasible set-valued mapping, the outer semicontinuity of the optimal solution set and Lipschitz continuity of optimal values are guaranteed. Importantly, it is proved that, if the linear independence constraint qualification and strong second-order sufficient condition hold at a local minimum point of the original problem, there exists a Lipschitz continuous solution path satisfying the Karush-Kuhn-Tucker conditions.

Keywords Stochastic nonlinear programming; stability analysis; strong regularity; second order optimality conditions; constraint qualification

MR(2010) Subject Classification 90C31

1. Introduction

Consider the stochastic nonlinear programming problem

$$\begin{aligned} \min_{x\in\Re^n} \quad & f(x,P) \\ \text{s.t.} \quad & g_i(x,P)\le 0,\ i=1,\dots,p, \end{aligned} \tag{1.1}$$

where

$$f(x,P):=\int_\Xi F(x,\xi)\mathrm{d}P(\xi),\ \ g_i(x,P):=\int_\Xi G_i(x,\xi)\mathrm{d}P(\xi),\quad i=1,\dots,p,$$

with $F:\Re^n\times\Xi\to\Re$ and $G_i:\Re^n\times\Xi\to\Re, i=1,\dots,p$, being Carathéodory functions (continuous in x, measurable in ξ) and $\Xi\subset\Re^d$ being the support set of random variables ξ. For notation simplicity, we denote $g(x,P):=(g_1(x,P),g_2(x,P),\dots,g_p(x,P))\in\Re^p$. Then problem (1.1) can be rewritten as in a compact form:

$$\begin{aligned} \min_{x\in\Re^n} \quad & f(x,P) \\ \text{s.t.} \quad & g(x,P)\le 0. \end{aligned} \tag{1.2}$$

We define

$$\Phi(P):=\{x\in\Re^n: g(x,P)\le 0\},$$

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$$\vartheta(P) := \inf\{f(x,P) : x \in \Phi(P)\},$$
$$\mathcal{S}(P) := \{x \in \Phi(P) : f(x,P) = \vartheta(P)\},$$

as the feasible set, optimal value and optimal solution of the stochastic nonlinear programming problem (1.1).

Stability analysis of deterministic nonlinear programming has been well investigated. We list the most important ones among them as follows:

- Robinson [1] proved that if the multi-valued mapping F from $\mathcal{X}$ to $\mathcal{Y}$ is piecewise polyhedral, then F is calm at any $x^0 \in \operatorname{dom} F$, where $\mathcal{X}$ and $\mathcal{Y}$ are finite dimensional linear spaces.
- Robinson [2] showed that the strong second order sufficient condition and the LICQ imply the strong regularity of the solution to the KKT system. Interestingly, the converse is also true, see Jongen et al. [3].
- Robinson [4] showed that the second order sufficient condition and MFCQ imply the upper Lipschitz continuity of KKT solutions.
- Dontchev and Rockafellar [5] showed that the strict MFCQ and the second-order sufficient optimality conditons are equivalent to the robust isolated calmness of the KKT system.

Another important result is about the strong regularity and Aubin property for the variational inequalities over polyhedral convex sets. Namely, for the following variational inequality problem

$$S(z,w) = \{x : 0 \in z + f(w,x) + N_C(x)\}$$

where C is a polyhedral convex set, Dontchev and Rockafellar [6] showed that the strong regularity of S is equivalent to Aubin property of S around a point $(z_0, w_0, x_0) \in \operatorname{ghp} S$.

There are so many important results about perturbation analysis for deterministic optimization problems, including not only nonlinear programming but also second-order conic and semidefinite conic optimization, see for examples [7–13].

For the stochastic nonlinear programming problems, the survey paper by Römisch [14] established a systematic qualitative and quantitative analysis of optimal value functions and optimal solution sets under the perturbation of probability measure. In particular, the two-stage stochastic problems and chance constrained programming problems are discussed in [14]. Note that the survey by Römisch's mainly focuses on the optimal value function and optimal solution set of s-tochastic nonlinear programming problems, the stability analysis of Karush-Kuhn-Tucker (KKT) system is not considered. Therefore, it is worth investigating the stability of stochastic nonlinear programming problems, including not only the stability of optimal value functions and optimal solution sets, but also the stability analysis of KKT system. This is the motivation of this paper.

The rest of the paper is organized as follows. In Section 2, the outer semicontinuity of the optimal solution sets and the Lipschitz continuity of optimal value function, from [14], are given. Under the linear independence constraint qualification and the strong second-order sufficient condition, the existence and properties of the solution path are investigated in Section 3. Section 4 concludes the paper.

Throughout the paper, $\mathbf{B}$ denotes the unit closed ball in the Euclidean space $\Re^n$ and $\mathbf{B}(x,\delta)$

denotes the closed ball with center $x \in \Re^n$ and radius $\delta > 0$. For a point $x \in X \subset \Re^n$ and a set $S \subset X$, int S denotes the interior of the set S, $\mathrm{dist}(x, S) := \inf_{x' \in S} \|x - x'\|$ denotes the distance from x to S. For a closed convex cone $\mathcal{K}$ and a point $z \in \mathcal{K}$, denote by $N_{\mathcal{K}}(z)$ the normal cone of $\mathcal{K}$ at z, $T_{\mathcal{K}}(z)$ the tangent cone of $\mathcal{K}$ at z, and lin $T_{\mathcal{K}}(z)$ the largest linear space contained in $T_{\mathcal{K}}(z)$.

2. Stability results

In this section, we will recall, from [14], the results about the stability for optimal values and optimal solutions of the stochastic nonlinear programming problem under probability measure perturbation. Before this, we give some definitions. For a nonempty open subset $\mathcal{O} \subset \Re^n$, define

$$\mathcal{F}_{\mathcal{O}} := \Big\{F(x, \cdot), G_i(x, \cdot) : x \in \mathrm{cl}\,\mathcal{O}, i = 1, \dots, p\Big\},$$

$$\Phi_{\mathcal{O}}(Q) := \Big\{x \in \mathrm{cl}\,\mathcal{O} : g(x, Q) \le 0\Big\},$$

$$\mathcal{P}_{\mathcal{F}_{\mathcal{O}}}(\Xi) := \left\{ Q \in \mathcal{P}(\Xi) : \begin{array}{l} -\infty < \inf\limits_{x \in \Re^n} f(x, Q), \ \sup\limits_{x \in \mathrm{cl}\,\mathcal{O}} f(x, Q) < +\infty \\ -\infty < \inf\limits_{x \in \Re^n} g_i(x, Q), \ \sup\limits_{x \in \mathrm{cl}\,\mathcal{O}} g_i(x, Q) < +\infty, i = 1, \dots, p \end{array} \right\},$$

where $\mathrm{cl}\,\mathcal{O}$ denotes the closure of $\mathcal{O}$. Define

$$\mathbf{d}_{\mathcal{F}_{\mathcal{O}}}(P, Q) := \sup_{\psi \in \mathcal{F}_{\mathcal{O}}} \Big| \int_{\Xi} \psi(\xi) \mathrm{d}Q(\xi) - \int_{\Xi} \psi(\xi) \mathrm{d}P(\xi) \Big|, \quad \forall P, Q \in \mathcal{P}_{\mathcal{F}_{\mathcal{O}}},$$

which is called a distance with ζ-structure. By [14, Proposition 3] $f(x, Q)$ and $g_k(x, Q)$ ($k = 1, \dots, p$) are sequentially lower semicontinous when $F(x, \cdot)$ and $G_k(x, \cdot)$ are Carathéodory functions for any probability measure Q.

Just like [14, Proposition 4], we can easily obtain that the set-valued mapping $\Phi_{\mathcal{O}}(\cdot)$ is sequentially closed.

Proposition 2.1 *Let $\mathcal{O}$ be an open subset of $\Re^n$. Assume, for any $P \in \mathcal{P}(\Xi)$, that $G_i(x, \xi)$ ($i = 1, \dots, p$) are Carathéodory functions, and there exists P-integrable function $Z_i(\xi)$ such that $|G_i(x, \xi)| \le Z_i(\xi)$ for P-almost every $\xi \in \Xi$ and all x in $\mathrm{cl}\mathcal{O}$. Then the graph of the set-valued mapping $Q \to \Phi_{\mathcal{O}}(Q)$ from $(\mathcal{P}_{\mathcal{F}_{\mathcal{O}}}, \mathbf{d}_{\mathcal{F}_{\mathcal{O}}})$ into $\Re^n$ is sequentially closed.*

Now we discuss the stability of optimal value functions and optimal solution sets under the perturbation of probability measures. Define for $y \in \Re^p$,

$$\begin{aligned} \Phi(y, P) :=& \{x \in \Re^n : g(x, P) + y \le 0\}, \\ \Phi^{-1}(x, P) :=& \{y \in \Re^m : x \in \Phi(y, P)\}. \end{aligned}$$

A nonempty set $\mathcal{S} \subset \Re^n$ is called a complete local minimizing (CLM) set for

$$\min_{x \in \Re^n} \{f(x, Q) : g(x, Q) \le 0\} \tag{2.1}$$

relative to $\mathcal{O}$ if $\mathcal{O} \subset \Re^n$ is open and $\mathcal{S} = \mathcal{S}_{\mathcal{O}}(Q) \subset \mathcal{O}$, where

$$\mathcal{S}_{\mathcal{O}}(Q) := \{x \in \Phi_{\mathcal{O}}(Q) : f(x, Q) = \vartheta_{\mathcal{O}}(Q)\},$$

$$\vartheta_{\mathcal{O}}(Q) := \inf\{f(x,Q) : x \in \Phi_{\mathcal{O}}(Q)\}.$$

We obtain from [14, Theorem 5] the following result, which is about the upper semicontinuity of the solution mapping and the Lipschitz continuity of the optimal value function.

Theorem 2.2 *Let $P \in \mathcal{P}_{\mathcal{F}_{\mathcal{O}}}$ and assume that*

(a) $\mathcal{S}(P)$ is nonempty and bounded, $\mathcal{O} \subset \Re^n$ is an open bounded neighbourhood of $\mathcal{S}(P)$;

(b) the function $x \to f(x,P)$ is Lipschitz continuous on $\mathrm{cl}\,\mathcal{O}$ with Lipschitz constant L_0;

(c) the mapping $x \to \Phi^{-1}(x,P)$ is metrically regular at each $(\overline{x},0)$ with $\overline{x} \in \mathcal{S}(P)$.

Then the set-valued mapping $\mathcal{S}_{\mathcal{O}}$ from $(\mathcal{P}_{\mathcal{F}_{\mathcal{O}}}, \mathbf{d}_{\mathcal{F}_{\mathcal{O}}})$ to $\Re^n$ is upper semicontinuous at P. Furthermore, there exist constants $L > 0$ and $\delta > 0$ such that

$$|\vartheta(P) - \vartheta_{\mathcal{O}}(Q)| \leq L\mathbf{d}_{\mathcal{F}_{\mathcal{O}}}(P,Q) \tag{2.2}$$

holds and $\mathcal{S}_{\mathcal{O}}(Q)$ is a CLM set of Problem (2.1) relative to $\mathcal{O}$ for $Q \in \mathcal{P}_{\mathcal{F}_{\mathcal{O}}}$ satisfying $\mathbf{d}_{\mathcal{F}_{\mathcal{O}}}(P,Q) < \delta$.

From [14, Theorem 9] follows the following quantitative analysis about the solution mapping.

Theorem 2.3 *Let the assumptions of Theorem 2.2 be satisfied and $P \in \mathcal{P}_{\mathcal{F}_{\mathcal{O}}}$. Then there exists a constant $\widetilde{L} \geq 1$ such that*

$$\emptyset \neq \mathcal{S}_{\mathcal{O}}(Q) \subset \mathcal{S}(P) + \Psi_P(\widetilde{L}\mathbf{d}_{\mathcal{F}_{\mathcal{O}}}(P,Q))$$

holds for any $Q \in \mathcal{P}_{\mathcal{F}_{\mathcal{O}}}$ satisfying $\mathbf{d}_{\mathcal{F}_{\mathcal{O}}}(P,Q) < \delta$. Here δ is the constant in Theorem (2.1) and

$$\Psi_P(\eta) := \eta + \psi_P^{-1}(2\eta), \quad \eta > 0$$

with

$$\psi_P(\tau) := \min\{f(x,P) - \vartheta(P) : \mathbf{d}(x,\mathcal{S}(P)) \geq \tau, x \in \Phi_{\mathcal{O}}(P)\}.$$

3. The strong regularity of the KKT system

This section will discuss the strong regularity of the KKT system when the probability measure P is perturbed. We propose the following assumptions.

Assumption 3.1 *For any $\xi \in \Xi$, $\mathcal{J}_x F(x,\xi)$, $\mathcal{J}_x G_k(x,\xi)$, $\nabla^2_{xx} F(x,\xi)$, $\nabla^2_{xx} G_k(x,\xi)$ exist for any $x \in \Re^n$ and $k = 1,\ldots,p$, and for any $x \in \Re^n$ and $Q \in \mathcal{P}(\Xi)$, $f(x,Q)$, $g(x,Q)$, $\nabla_x f(x,Q)$, $\mathcal{J}_x g(x,Q)$, $\nabla^2_{xx} f(x,Q)$ and $\nabla^2_{xx} g_k(x,Q)$ are well-defined for $k = 1,\ldots,p$.*

Assumption 3.2 *For any $Q \in \mathcal{P}(\Xi)$, there exist positive valued random variables $Z(\xi)$ and $C(\xi)$ such that*

$$\sup_{Q\in\mathcal{P}(\Xi)} \int_{\Xi} Z(\xi)\mathrm{d}Q(\xi) < \infty \text{ and } \sup_{Q\in\mathcal{P}(\Xi)} \int_{\Xi} C(\xi)\mathrm{d}Q(\xi) < \infty.$$

For any $x \in \Re^n$ and for any $Q \in \mathcal{P}(\Xi)$, $F(x,\xi), G_k(x,\xi), k = 1,\ldots,p$, $\nabla_x F(x,\xi), \nabla_x G_k(x,\xi), k = 1,\ldots,p$ and $\nabla^2_{xx} F(x,\xi)$, $\nabla^2_{xx} G_k(x,\xi), k = 1,\ldots,p$ are continuous in x, and

$$\|(F(x,\xi), G(x,\xi)); (\nabla_x F(x,\xi), \mathcal{J}_x G(x,\xi)); (\nabla^2_{xx} F(x,\xi), D^2_{xx} G(x,\xi)))\| \leq Z(\xi).$$

For $x_1, x_2 \in \Re^n$ and Q-almost every $\xi \in \Xi$,

$$\|(F(x_1,\xi), G(x_1,\xi)) - (F(x_2,\xi), G(x_2,\xi))\| \le C(\xi)\|x_1 - x_2\|,$$
$$\|(\nabla_x F(x_1,\xi), \mathcal{J}_x G(x_1,\xi)) - (\nabla_x F(x_2,\xi), \mathcal{J}_x G(x_2,\xi))\| \le C(\xi)\|x_1 - x_2\|,$$
$$\|(\nabla^2_{xx} F(x_1,\xi), D^2_{xx} G(x_1,\xi)) - (\nabla^2_{xx} F(x_2,\xi), D^2_{xx} G(x_2,\xi))\| \le C(\xi)\|x_1 - x_2\|.$$

Here $\|\cdot\|$ denotes the operator norm in the corresponding spaces.

It follows from Assumptions 3.1 and 3.2 that, for any $Q \in \mathcal{P}(\Xi)$, $f(\cdot,Q)$,$g(\cdot,Q)$, $\nabla_x f(\cdot,Q)$, $\mathcal{J}_x g(\cdot,Q)$, $\nabla^2_{xx} f(\cdot,Q)$ and $D^2_{xx} g(\cdot,Q)$ are Lipschitz continuous on $\Re^n$, the Lipschitz constant is uniform with respect to $Q \in \mathcal{P}(\Xi)$, and we denote it as L_1 here. Define

$$\mathcal{F} := \mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_3$$

with

$$\mathcal{F}_1 := \Big\{F(x,\cdot), G_k(x,\cdot), x \in \Re^n, k = 1,\dots,p\Big\},$$
$$\mathcal{F}_2 := \left\{\frac{\partial F}{\partial x_i}(x,\cdot), \frac{\partial G_k}{\partial x_i}(x,\cdot), x \in \Re^n, \begin{array}{c} i = 1\dots,n \\ k = 1,\dots,p \end{array}\right\},$$
$$\mathcal{F}_3 := \left\{\frac{\partial^2 F}{\partial x_i x_j}(x,\cdot), \frac{\partial^2 G_k}{\partial x_i x_j}(x,\cdot), x \in \Re^n, \begin{array}{c} i = 1,\dots,n, \\ j = 1\dots,n, \\ k = 1,\dots,p \end{array}\right\}.$$

Define

$$\mathcal{P}_{\mathcal{F}}(\Xi) = \{Q \in \mathcal{P}(\Xi) : -\infty < \inf_{x\in\Re^n} \mathbb{E}_Q[\psi(x,\xi)], \sup_{x\in\Re^n} \mathbb{E}_Q[\psi(x,\xi)] < +\infty, \forall \psi \in \mathcal{F}$$

It is easy to prove, under Assumptions 3.1 and 3.2, for $P, Q \in \mathcal{P}_{\mathcal{F}}(\Xi)$, the following inequalities are satisfied:

$$|f(x_1,P) - f(x_2,Q)| \le \mathbf{d}_{\mathcal{F}}(P,Q) + L_1\|x_1 - x_2\|,$$
$$\|g(x_1,P) - g(x_2,Q)\| \le p\mathbf{d}_{\mathcal{F}}(P,Q) + L_1\|x_1 - x_2\|,$$
$$\|\nabla_x f(x_1,P) - \nabla_x f(x_2,Q)\| \le n\mathbf{d}_{\mathcal{F}}(P,Q) + L_1\|x_1 - x_2\|,$$
$$\|\mathcal{J}_x g(x_1,P) - \mathcal{J}_x g(x_2,Q)\| \le pn\mathbf{d}_{\mathcal{F}}(P,Q) + L_1\|x_1 - x_2\|,$$
$$\|\nabla^2_{xx} f(x_1,P) - \nabla^2_{xx} f(x_2,Q)\| \le n^2\mathbf{d}_{\mathcal{F}}(P,Q) + L_1\|x_1 - x_2\|,$$
$$\|D^2_{xx} g(x_1,P) - D^2_{xx} g(x_2,Q)\| \le pn^2\mathbf{d}_{\mathcal{F}}(P,Q) + L_1\|x_1 - x_2\|.$$

Now we turn to Problem (1.1). Let $\overline{x}$ be a local minimum point of Problem (1.1) at which Mangasarian-Fromowitz constraint qualification (MFCQ) holds, namely there exists nonzero vector $d_0 \in \Re^n$ such that

$$\nabla_x g_i(\overline{x},P)^T d_0 < 0, \quad i \in I(P,\overline{x}),$$

where $I(P,\overline{x}) = \{i : g_i(\overline{x},P) = 0\}$. Then $\overline{x}$ is a stationary point for Problem (1.1), namely there exists a Lagrange multiplier $\lambda \in \Re^p$ such that $(\overline{x},\lambda)$ satisfies the following Karush-Kuhn-Tucker (KKT) conditions:

$$\nabla_x L(P,\overline{x},\lambda) = 0, \quad 0 \le \lambda \perp g(\overline{x},P) \le 0, \tag{3.1}$$

where

$$L(P,x,\lambda) := f(x,P) + \langle \lambda, g(x,P)\rangle.$$

Define

$$\Lambda(P,\overline{x}) := \{\lambda \in \Re^p : \nabla_x L(P,\overline{x},\lambda) = 0, 0 \le \lambda \perp g(\overline{x},P) \le 0\}$$

as the set of Lagrange multipliers at $\overline{x}$. The above KKT conditions for (x,λ) can be equivalently expressed as

$$0 \in H(P,x,\lambda) + N_{\Re^n \times \Re^p_-}(x,\lambda), \tag{3.2}$$

where

$$H(P,x,\lambda) := \begin{bmatrix} \nabla_x L(P,x,\lambda) \\ g(x,P) \end{bmatrix}. \tag{3.3}$$

Let

$$F(P,x,\lambda) := \begin{bmatrix} \nabla_x L(P,x,\lambda) \\ g(x,P) - \Pi_{\Re^p_-}(g(x,P)+\lambda) \end{bmatrix}. \tag{3.4}$$

Then (x,λ) is a KKT point of Problem (1.1) if and only if $F(P,x,\lambda) = 0$. Define the critical cone of Problem (1.1) at a stationary point $x \in \Re^n$ as

$$\mathcal{C}(P,x) := \{d \in \Re^n : \mathcal{J}_x g(x,P)d \in T_{\Re^p_-}(g(x,P)), \langle \nabla_x f(x,P), d\rangle \le 0\}.$$

When probability measure P is fixed, we obtain the following results from [9] directly.

Proposition 3.3 ([9]) *Let Assumptions 3.1 and 3.2 be satisfied. Assume that $\overline{x}$ is a local minimizer of Problem (1.1) and MFCQ holds at $\overline{x}$. Then the following assertions hold.*

(i) The set of Lagrange multipliers $\Lambda(P,\overline{x})$ is nonempty and compact.

(ii) For any $d \in \mathcal{C}(P,\overline{x})$,

$$\max_{\lambda \in \Lambda(P,\overline{x})} \{\langle \nabla^2_{xx} L(P,\overline{x},\lambda)d, d\rangle\} \ge 0.$$

The linear independence constraint qualification is satisfied at $\overline{x}$ if the set of vectors $\{\nabla_x g_i(\overline{x},P) : i \in I(P,\overline{x})\}$ are linearly independent. If $\overline{x}$ is a local minimum point at which the linear independence condition holds, then $\Lambda(P,\overline{x})$ is reduced to a singleton. The following result about the second-order sufficient optimality conditions comes from [9].

Proposition 3.4 *Let Assumptions 3.1 and 3.2 be satisfied. Assume that $\overline{x}$ is a feasible point of Problem (1.1) and $\Lambda(P,\overline{x})$ is nonempty. If*

$$\sup_{\lambda \in \Lambda(P,\overline{x})} \big\{\langle \nabla^2_{xx} L(P,\overline{x},\lambda)d, d\rangle\big\} > 0, \tag{3.5}$$

for any $d \in \mathcal{C}(P,\overline{x}) \setminus \{0\}$, then the second-order growth condition holds at $\overline{x}$.

Definition 3.5 *Assume that $\overline{x}$ is a feasible point of Problem (1.1) and $\Lambda(P,\overline{x})$ is nonempty. We say that the strong second-order sufficient condition holds at $\overline{x}$ if*

$$\sup_{\lambda \in \Lambda(P,\overline{x})} \{\langle \nabla^2_{xx} L(P,\overline{x},\lambda)d, d\rangle\} > 0,$$

for any $d \in \operatorname{aff} \mathcal{C}(P, \overline{x}) \setminus \{0\}$.

Proposition 3.6 ([15]) *Let Assumptions 3.1 and 3.2 be satisfied. Assume that* $\overline{x}$ *is a local minimum point of Problem (1.1) at which MFCQ holds and* $\overline{\lambda} \in \Lambda(P, \overline{x})$. *Then the following properties are equivalent.*

(i) The strong second-order sufficient condition and the linear independence constraint qualification hold at $\overline{x}$*;*

(ii) The generalized Eq. (3.2) is strong regular at $(\overline{x}, \overline{\lambda})$*;*

(iii) Any element in $\partial_x F(P, \overline{x}, \overline{\lambda})$ *is nonsingular.*

Define for $\tau > 0$, $\Gamma(P, \tau) := \{Q \in \mathcal{P}_{\mathcal{F}} : \mathbf{d}_{\mathcal{F}}(P, Q) < \tau\}$. Now we will discuss the solution path of the following problem

$$\begin{array}{ll} \min\limits_{x} & f(x, Q) \\ \text{s.t.} & g(x, Q) \in \Re^p_-, \end{array} \tag{3.6}$$

when Q is close to P and establish the main result of this paper.

Theorem 3.7 *Let Assumptions 3.1 and 3.2 be satisfied. Assume that* $\overline{x}$ *is a local minimum point of Problem (1.1) at which the linear independence constraint qualification holds at* $\overline{x}$ *and* $\Lambda(P, \overline{x}) = \{\overline{\lambda}\}$ *and the strong second-order sufficient condition holds at* $\overline{x}$. *Then there exist* $\delta > 0$ *and* $\varepsilon > 0$, *and* $(x(\cdot), \lambda(\cdot)) : \Gamma(P, \delta) \to \boldsymbol{B}_{\varepsilon}(\overline{x}, \overline{\lambda})$ *such that* $(x(P), \lambda(P)) = (\overline{x}, \overline{\lambda})$, *and*

(i) the pair $(x(Q), \lambda(Q))$ *satisfies the KKT condition for Problem (3.6) when* $Q \in \Gamma(P, \delta)$*;*

(ii) there exists a constant $\widehat{L} > 0$ *such that*

$$\|(x(Q), \lambda(Q)) - (x(P), \lambda(P))\| \leq \widehat{L} \boldsymbol{d}_{\mathcal{F}}(P, Q) \tag{3.7}$$

for $Q \in \Gamma(P, \delta)$.

Proof Since both the linear independence constraint qualification and strong second-order sufficient condition hold at a local minimum point $\overline{x}$, we obtain from Proposition 3.6 that the generalized equation

$$\eta \in \begin{bmatrix} \nabla_x L(P, \overline{x}, \overline{\lambda}) \\ g(\overline{x}, P) \end{bmatrix} + \begin{bmatrix} \nabla^2_{xx} L(P, \overline{x}, \overline{\lambda}) & \mathcal{J}_x g(\overline{x}, P)^T \\ \mathcal{J}_x g(\overline{x}, P) & 0 \end{bmatrix} \begin{bmatrix} x - \overline{x} \\ \lambda - \overline{\lambda} \end{bmatrix} + N_{\Re^n \times \Re^p_-}(x, \lambda) \tag{3.8}$$

or

$$\eta \in H(P, \overline{x}, \overline{\lambda}) + \mathcal{J}_{x,\lambda} H(P, \overline{x}, \overline{\lambda})(x - \overline{x}, \lambda - \overline{\lambda}) + N_{\Re^n \times \Re^p_-}(x, \lambda), \tag{3.9}$$

is strong regular at $(\overline{x}, \overline{\lambda})$. That is, there is a unique Lischitz continuous solution $(\varphi(\cdot), \psi(\cdot))$: $\mathbf{B}_{\delta_1}(0) \to \mathbf{B}_{\varepsilon_1}(\overline{x}, \overline{\lambda})$ for some $\delta_1 > 0$ and $\varepsilon_1 > 0$. Let $\gamma > 0$ be the Lipschitz constant for $(\varphi(\cdot), \psi(\cdot))$ over $\mathbf{B}_{\delta_1}(0)$.

Define

$$r(Q, x, \lambda) := H(P, \overline{x}, \overline{\lambda}) + \mathcal{J}_{x,\lambda} H(P, \overline{x}, \overline{\lambda})(x - \overline{x}, \lambda - \overline{\lambda}) - H(Q, x, \lambda).$$

Then we have (x, λ) solves

$$0 \in H(Q, x, \lambda) + N_{\Re^n \times \Re^p_-}(x, \lambda) \tag{3.10}$$

if and only if

$$r(Q,x,\lambda)\in H(P,\overline{x},\overline{\lambda})+\mathcal{J}_{x,\lambda}H(P,\overline{x},\overline{\lambda})(x-\overline{x},\lambda-\overline{\lambda})+N_{\Re^n\times\Re^p_-}(x,\lambda). \tag{3.11}$$

Choose $0<\delta<\delta_1$, $0<\varepsilon<\varepsilon_1$ and $\delta_2>0$ such that

$$\begin{aligned}
&\gamma\delta_2<1/(\gamma+1),\\
&2n\delta+pn\|\overline{\lambda}\|_\infty\delta<(1-\gamma\delta_2)\varepsilon/\gamma,\\
&r(Q,x,\lambda)\in\mathbf{B}_{\delta_1}(0),\forall(x,\lambda)\in\mathbf{B}_\varepsilon(\overline{x},\overline{\lambda}),\\
&(2n^2+4pn)\delta+2pn^2\|\overline{\lambda}\|_\infty\delta+3L_1\varepsilon+L_1\|\overline{\lambda}\|_\infty\varepsilon+\\
&\quad\sqrt{n}\varepsilon\sup_{x\in\mathbf{B}_{\varepsilon_1}(\overline{x}),Q\in\Gamma(P,\delta)}\sum_{k=1}^p\|\nabla^2_{xx}g_k(x,Q)\|<\delta_2.
\end{aligned}\tag{3.12}$$

For any fixed $Q\in\Gamma(P,\delta)$, define

$$\theta_Q(\cdot):=(\varphi(r(Q,\cdot)),\psi(r(Q,\cdot))). \tag{3.13}$$

Now we prove that $\theta_Q(\cdot)$ is a contraction mapping from $\mathbf{B}_\varepsilon(\overline{x},\overline{\lambda})$ onto itself. For any $(x^1,\lambda^1),(x^2,\lambda^2)\in\mathbf{B}_\varepsilon(\overline{x},\overline{\lambda})$, from the Lipschitz continuity of $(\varphi(\cdot),\psi(\cdot))$ on $\mathbf{B}_{\delta_1}(0)$,

$$\begin{aligned}
&\|\theta_Q(x^1,\lambda^1)-\theta_Q(x^2,\lambda^2)\|\\
&\quad\le\gamma\|r(Q,x^1,\lambda^1)-r(Q,x^2,\lambda^2)\|\\
&\quad\le\gamma\Big[\sup_{\mu\in[0,1]}\|\mathcal{J}_{x,\lambda}r(Q,(1-\mu)(x^1,\lambda^1)+\mu(x^2,\lambda^2))\|\Big]\|(x^1,\lambda^1)-(x^2,\lambda^2)\|.
\end{aligned}\tag{3.14}$$

For any $(x,\lambda)=(x_\mu,\lambda_\mu):=(1-\mu)(x^1,\lambda^1)+\mu(x^2,\lambda^2)$, one has

$$\begin{aligned}
&\mathcal{J}_{x,\lambda}r(Q,x,\lambda)=\mathcal{J}_{x,\lambda}H(P,\overline{x},\overline{\lambda})-\mathcal{J}_{x,\lambda}H(Q,x,\lambda)\\
&\quad=\mathcal{J}_{x,\lambda}H(P,\overline{x},\overline{\lambda})-\mathcal{J}_{x,\lambda}H(P,x,\overline{\lambda})+\mathcal{J}_{x,\lambda}H(P,x,\overline{\lambda})-\mathcal{J}_{x,\lambda}H(Q,x,\overline{\lambda})+\\
&\quad\quad\mathcal{J}_{x,\lambda}H(Q,x,\overline{\lambda})-\mathcal{J}_{x,\lambda}H(Q,x,\lambda)\\
&\quad=\begin{bmatrix}\nabla^2_{xx}f(\overline{x},P)-\nabla^2_{xx}f(x,P)\\ \quad+\sum_{k=1}^p\overline{\lambda}_k(\nabla^2_{xx}g_k(\overline{x},P)-\nabla^2_{xx}g_k(x,P)) & \mathcal{J}_xg(\overline{x},P)^T-\mathcal{J}_xg(x,P)^T\\ -\mathcal{J}_xg(\overline{x},P)+\mathcal{J}_xg(x,P) & 0\end{bmatrix}+\\
&\quad\quad\begin{bmatrix}\nabla^2_{xx}f(x,P)-\nabla^2_{xx}f(x,Q)\\ \quad+\sum_{k=1}^p\overline{\lambda}_k(\nabla^2_{xx}g_k(x,P)-\nabla^2_{xx}g_k(x,Q)) & \mathcal{J}_xg(x,P)^T-\mathcal{J}_xg(x,Q)^T\\ -\mathcal{J}_xg(x,P)+\mathcal{J}_xg(x,Q) & 0\end{bmatrix}+\\
&\quad\quad\begin{bmatrix}\sum_{k=1}^p(\overline{\lambda}_k-\lambda_k)\nabla^2_{xx}g_k(x,Q) & 0\\ 0 & 0\end{bmatrix}.
\end{aligned}\tag{3.15}$$

Thus we obtain from (3.12) and (3.15) that

$$\|\mathcal{J}_{x,\lambda} r(Q, x_\mu, \lambda_\mu)\| \le (2n^2 + 4pn)\delta + 2pn^2\|\overline{\lambda}\|_\infty\delta + 3L_1\varepsilon + L_1\|\overline{\lambda}\|_\infty\varepsilon + \sqrt{n}\varepsilon \sup_{x\in\mathbf{B}_{\varepsilon_1}(\overline{x}), Q\in\Gamma(P,\delta)} \sum_{k=1}^{p} \|\nabla^2_{xx} g_k(x, Q)\| < \delta_2$$

for any $Q \in \Gamma(P, \delta)$, $(x_1, \lambda_1), (x_2, \lambda_2) \in \mathbf{B}_\varepsilon(\overline{x}, \overline{\lambda})$ and $\mu \in [0, 1]$. Therefore, we obtain from (3.14) that

$$\|\theta_Q(x^1, \lambda^1) - \theta_Q(x^2, \lambda^2)\| \le \gamma\delta_2\|(x^1, \lambda^1) - (x^2, \lambda^2)\|. \tag{3.16}$$

From the choice of δ_2, $\gamma\delta_2 < 1$, this implies that θ_Q is a contraction mapping. Furthermore,

$$\begin{aligned}\|\theta_Q(\overline{x}, \overline{\lambda}) - (\overline{x}, \overline{\lambda})\| &= \|(\varphi(r(Q, \overline{x}, \overline{\lambda}), \psi(r(Q, \overline{x}, \overline{\lambda})) - (\varphi(0), \psi(0))\| \\ &\le \gamma\|r(Q, \overline{x}, \overline{\lambda}) - 0\| \\ &= \gamma\|H(P, \overline{x}, \overline{\lambda}) - H(Q, \overline{x}, \overline{\lambda})\| \\ &\le \gamma(2n\delta + pn\|\overline{\lambda}\|_\infty\delta) \le (1 - \gamma\delta_2)\varepsilon.\end{aligned}$$

This implies, for $(x, \lambda) \in \mathbf{B}_\varepsilon(\overline{x}, \overline{\lambda})$, that

$$\begin{aligned}\|\theta_Q(x, \lambda) - (\overline{x}, \overline{\lambda})\| &\le \|\theta_Q(x, \lambda) - \theta_Q(\overline{x}, \overline{\lambda})\| + \|\theta_Q(\overline{x}, \overline{\lambda}) - (\overline{x}, \overline{\lambda})\| \\ &\le \gamma\delta_2\|(x, \lambda) - (\overline{x}, \overline{\lambda})\| + (1 - \gamma\delta_2)\varepsilon \\ &\le \varepsilon,\end{aligned}$$

namely θ_Q maps $\mathbf{B}_\varepsilon(\overline{x}, \overline{\lambda})$ to $\mathbf{B}_\varepsilon(\overline{x}, \overline{\lambda})$ itself. According to Banach fixed point theorem, there exists a unique mapping $(x(\cdot), \lambda(\cdot)) : \Gamma(P, \delta) \to \mathbf{B}_\varepsilon(\overline{x}, \overline{\lambda})$ satisfying

$$(x(Q), \lambda(Q)) = \theta_Q(x(Q), \lambda(Q)), \quad \forall Q \in \Gamma(P, \delta).$$

Therefore, for $Q \in \Gamma(P, \delta)$, the pair $(x(Q), \lambda(Q))$ satisfies the KKT condition for Problem (3.6) and assertion (i) is proved.

Now we prove assertion (ii). Choose $\widehat{L} := (\gamma + 1)(n + p + pn^2\|\overline{\lambda}\|_\infty)$. Then for $Q \in \Gamma(P, \delta)$, from (3.16), we obtain

$$\begin{aligned}&\|(x(Q), \lambda(Q)) - (x(P), \lambda(P))\| = \|\theta_Q(x(Q), \lambda(Q)) - \theta_P(x(P), \lambda(P))\| \\ &\le \|\theta_Q(x(Q), \lambda(Q)) - \theta_Q(x(P), \lambda(P))\| + \|\theta_Q(x(P), \lambda(P)) - \theta_P(x(P), \lambda(P))\| \\ &\le \gamma\delta_2\|(x(Q), \lambda(Q)) - (x(P), \lambda(P))\| + \\ &\quad \|(\varphi(r(Q, x(P), \lambda(P))), \psi(r(Q, x(P), \lambda(P)))) - \\ &\quad (\varphi(r(P, x(P), \lambda(P))), \psi(r(P, x(P), \lambda(P))))\| \\ &\le \gamma\delta_2\|(x(Q), \lambda(Q)) - (x(P), \lambda(P))\| + \gamma\|H(Q, x(P), \lambda(P)) - H(P, x(P), \lambda(P))\| \\ &\le \gamma\delta_2\|(x(Q), \lambda(Q)) - (x(P), \lambda(P))\| + \gamma(n + p + pn^2\|\overline{\lambda}\|_\infty)\mathbf{d}_{\mathcal{F}}(P, Q),\end{aligned}$$

which implies

$$\|(x(Q), \lambda(Q)) - (x(P), \lambda(P))\| \le \frac{\gamma(n + p + pn^2\|\overline{\lambda}\|_\infty)}{1 - \gamma\delta_2}\mathbf{d}_{\mathcal{F}}(P, Q)$$

$$< (\gamma+1)(n+p+pn^2\|\overline{\lambda}\|_\infty))\mathbf{d}_{\mathcal{F}}(P,Q) := \widehat{L}\mathbf{d}_{\mathcal{F}}(P,Q).$$

This proves assertion (ii). □

It follows from Theorem 3.7, if the linear independence constraint qualification and the strong second-order sufficient condition are satisfied for the original problem, then there exists a Lipschitz continuous solution path satisfying the Karush-Kuhn-Tucker conditions when the probability measure is perturbed.

4. Conclusions

In this paper, for stochastic nonlinear programming, we establish that if the the linear independence constraint qualification and the strong second-order sufficient condition are satisfied for the original problem, then there exists a Lipschitz continuous solution path satisfying the Karush-Kuhn-Tucker conditions when the probability measure varies in a neighborhood of the reference probability in the sense of the distance $\mathbf{d}_{\mathcal{F}}$. Besides strong regularity, there are other stability notions such as Aubin property, isolated calmness and calmness for the Karush-Kuhn-Tucker system, these stability properties for stochastic nonlinear programming are worth studying.

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