Annihilator Condition on Power Values of Commutators with Derivations

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Abstract Let $R$ be a prime ring with center $Z(R)$, $I$ a nonzero ideal of $R$, $d$ a nonzero derivation of $R$ and $0 \neq a \in R$. In the present paper, our object is to study the situation $a[d(x^k), x^k]^n \in Z(R)$ for all $x \in I$ under certain conditions, where $n \geq 1$, $k \geq 1$ are fixed integers.

Keywords prime ring; derivation; extended centroid

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1. Introduction

Let $R$ be a prime ring with center $Z(R)$. For $x, y \in R$, we set $[x, y]_1 = [x, y] = xy - yx$ and $[x, y]_n = [[x, y]_{n-1}, y]$ where $n \geq 2$ is a positive integer. By $d$ we mean a derivation of $R$. $s_4$ denotes the standard identity in four variables. In [1], a well-known result proved by Posner states that if $[d(x), x] \in Z(R)$ for all $x \in R$, then either $d = 0$ or $R$ is commutative. In [2], Lanski generalizes the Posner’s result to a Lie ideal. Lanski proved that if $L$ is a noncommutative Lie ideal of $R$ and $d \neq 0$ such that $[d(x), x] \in Z(R)$ for all $x \in L$, then either $R$ is commutative, or char $R = 2$ and $R$ satisfies $s_4$. In [3], Carini and Filippis studied more generalized situation of this result by considering power values. They proved that if $[d(u), u]^n \in Z(R)$ for all $u$ in a noncentral Lie ideal of $R$, $n \geq 1$ a fixed integer and char $R \neq 2$, then either $d = 0$ or $R$ satisfies $s_4$. In [4], Wang and You removed the restriction on characteristic and they proved that the same conclusion holds when char $R = 2$.

On the other hand, some results concerning annihilators of power values in prime and semiprime rings have been obtained in literature. In [5], Bresar proved that if $R$ is a semiprime ring, $d$ a nonzero derivation of $R$ and $a \in R$ such that $ad(x)^n = 0$, then $ad(R) = 0$ when $R$ is $(n - 1)!$-torsion free. In [6], Lee and Lin proved Bresar’s result on Lie ideals of prime rings without the $(n - 1)!$-torsion free assumption on $R$. In [7], Filippis established a similar version of Bresar’s result for multilinear polynomials in prime rings. Furthermore, Filippis studied the left annihilator of power values of commutators with derivations. In [8], he proved if char $R \neq 2$, $0 \neq d$ and $0 \neq a \in R$ such that $a[d(x), x]^n \in Z(R)$ for all $x \in L$, where $L$ is a noncentral Lie
ideal of $R$ and $n \geq 1$ a fixed integer, then $R$ satisfies $s_4$. In [9], Wang removed the assumption of $\text{char } R \neq 2$. In [10], Du and Wang proved a result on both sided ideal in prime ring. They proved that if $\text{char } R \neq 2$, $0 \neq I$ a both sided ideal of $R$ and $0 \neq d$ such that $[d(x^k), x^k]^n \in Z(R)$ for all $x \in I$, where $k, n$ are fixed positive integer, then $R$ satisfies $s_4$. For more related results concerning annihilators we refer to [11–13].

The purpose of the present paper is to study the same situation of Du and Wang with left annihilator condition.

First we recall some basic notations. We denote by $Q$ the two sided Martindale quotient ring of a prime ring $R$ and by $C$ the center of $Q$. We call $C$ the extended centroid of $R$. This $C$ is a field. It is well known that every derivation $d$ of $R$ can be uniquely extended to a derivation of $Q$, which will be also denoted by $d$. The derivation $d$ of $R$ is called a $Q$-inner induced by some $q \in Q$ if $d(x) = [q, x]$ holds for all $x \in R$. If $d$ is not $Q$-inner, then $d$ is called $Q$-outer derivation of $R$.

By Khcharchenko’s theorem [14], we have the following result:

Let $R$ be a prime ring, $d$ a derivation on $R$ and $I$ a nonzero ideal of $R$. If $I$ satisfies the differential identity $f(r_1, r_2, \ldots , r_n, d(r_1), d(r_2), \ldots , d(r_n)) = 0$ for any $r_1, r_2, \ldots , r_n \in I$, then either

(i) $I$ satisfies the generalized polynomial identity $f(r_1, r_2, \ldots , r_n, x_1, x_2, \ldots , x_n) = 0$

or (ii) $d$ is $Q$-inner.

2. Main results

We begin with lemmas.

**Lemma 2.1** Let $R = M_m(F)$ be the ring of all $m \times m$ matrices over a field $F$ of characteristic different from 2 and $m \geq 3$. Let $a$ be an invertible element in $R$. If for some $b \in R$, $([b, x^k]_{2})^n \in F \cdot a^{-1}$ for all $x \in R$, where $k (\geq 1)$, $n (\geq 1)$ are fixed integers, then $b \in F : I_n$.

**Proof** Let $a = (a_{ij})_{m \times m}$, $b = (b_{ij})_{m \times m}$. By assumption, for every $x \in R$, $([b, x^k]_{2})^n$ is zero or invertible. Write $b = (b_{11} A, B, C)^T$, where $A = (b_{12}, \ldots , b_{1m})$, $B = (b_{21}, \ldots , b_{m1})^T$ and $C = (b_{ij})_{2 \leq i, j \leq m}$. We choose $x = e_{11}$. Then $[b, e_{11}]_{2} = (0, 0, 0, 0) = (b_{11} - 2b_{11}e_{11} + e_{11}b)$. Since rank of $[b, e_{11}]_{2}$ is $\leq 2$, $([b, e_{11}]_{2})^n$ cannot be invertible, since $m \geq 3$, and so it must be zero. Therefore, $([b, e_{11}]_{2})^n = 0$ and so $([b, e_{11}]_{2})^{2n} = 0$. By simple manipulation, we have

$$0 = ([b, e_{11}]_{2})^{2n} = \begin{pmatrix} (AB)^n & 0 \\ 0 & (BA)^n \end{pmatrix}. \quad (2.1)$$

Therefore, $(AB)^n = 0$. Since $(AB) \in F$, $AB = 0$. Let $\phi$ be an inner automorphism of $R$ defined by $\phi(x) = (1 + e_{21})x(1 - e_{21})$ for all $x \in R$. Then $\phi(b)$ satisfies the same property as $b$ does, that is, either $([\phi(b), x^k]_{2})^n$ is zero or invertible for every $x \in R$. Now, we have

$$\phi(b) = \begin{pmatrix} b_{11} - b_{12} & A \\ b_{11}E - b_{12}E + B - CE & EA + C \end{pmatrix}, \quad (2.2)$$
where $E = ((1, 0, \ldots, 0)_{1 \times m-1})^T$. As above, we have

$$A(b_{11}E - b_{12}E + B + CE) = 0.$$  \hfill (2.3)

Recalling $AB = 0$, above relation implies $b_{11}b_{12} - b_{12}^2 - ACE = 0$. Now we choose $x = e_{11} + e_{21}$.

$$[b, x^k]_2 = [b, e_{11} + e_{21}]_2 = \begin{pmatrix} -b_{12} & A \\ D & EA \end{pmatrix},$$ \hfill (2.4)

where $D = B + CE - (b_{11} + 2b_{12})E$. We see in the matrix $[b, e_{11} + e_{21}]_2$ that number of distinct column vectors are 2. Hence, rank of $[b, e_{11} + e_{21}]_2$ is $\leq 2$ and so rank of $([b, e_{11} + e_{21}]_2)^n$ is also $\leq 2$. Therefore, $([b, e_{11} + e_{21}]_2)^n$ can not be invertible in $R$ for $m \geq 3$, and hence it must be zero. Therefore, we can write $([b, e_{11} + e_{21}]_2)^{2n} = 0$. Now we calculate

$$([b, x^k]_2)^2 = ([b, e_{11} + e_{21}]_2)^2 = \begin{pmatrix} b_{12}^2 + AD & 0 \\ -b_{12}D + EAD & DA + b_{12}EA \end{pmatrix}.$$ \hfill (2.5)

Now the facts $AB = 0$ and $b_{11}b_{12} - b_{12}^2 - ACE = 0$ together imply $AD = -3b_{12}^2$. Thus, we have

$$([b, x^k]_2)^2 = ([b, e_{11} + e_{21}]_2)^2 = \begin{pmatrix} -2b_{12}^2 & 0 \\ -b_{12}D - 3b_{12}^2E & DA + b_{12}EA \end{pmatrix},$$ \hfill (2.6)

and hence

$$0 = ([b, x^k]_2)^{2n} = ([b, e_{11} + e_{21}]_2)^{2n} = \begin{pmatrix} -2b_{12}^2 & 0 \\ U & (DA + b_{12}EA)^n \end{pmatrix},$$ \hfill (2.7)

where $U$ is an $(m - 1) \times 1$ matrix. This gives $(-2b_{12}^2)^n = 0$, implying $b_{12} = 0$. Since for any $F$-automorphism $\varphi$, $b$ and $b^\varphi$ satisfies the same properties, we can write $(b^\varphi)^{12} = 0$. Therefore, $0 = ((1 - e_{12})b(1 + e_{12}))_{12}$ for any $i \neq 1, 2$. This implies $b_{ii} = 0$ for all $i \neq 1, 2$. Since $b_{12} = 0$, all the entries in 1st row of the matrix $b$ are zeros, except $b_{11}$. Hence, we can write,

$$0 = ((1 - e_{12})b(1 + e_{12}))_{1j}$$

for any $j \neq 1$ and $t \neq 1$. This implies $b_{jt} = 0$ for all $j \neq t$. Thus, the matrix $b$ is diagonal. Let $b = \sum_{i=1}^m b_{ii}e_{ii}$. Then for $s \neq t$, we have $(1 + e_{ss})b(1 - e_{ss}) = \sum_{i=1}^m b_{ii}e_{ii} + (b_{ss} - b_{tt})e_{ts}$ is diagonal. Hence, $b_{ss} = b_{tt}$ and so $b$ is a scalar matrix, that is, $b \in F \cdot I_m$. \hfill \Box

**Lemma 2.2** ([15]) Let $R$ be a noncommutative simple algebra, finite-dimensional over its center $Z$. If $g(x_1, \ldots, x_t) \in R + Z\{x_j\}$, the free product over $Z$, is an identity for $R$ that is homogeneous in $\{x_1, \ldots, x_t\}$ of degree $d$, then for some field $F$ and $n > 1$, $R \subseteq M_n(F)$ and $g(x_1, \ldots, x_t)$ is an identity for $M_n(F)$.

**Theorem 2.3** Let $R$ be a prime ring of characteristic different from 2 with center $Z(R)$, $I$ a nonzero ideal of $R$, $d$ a nonzero derivation of $R$ and $0 \neq a \in R$. Suppose that there exists $x \in I$ such that $a[d(x^k), x^k]^n \neq 0$. If $a[d(x^k), x^k]^n \in Z(R)$ for all $x \in I$, where $n \geq 1$, $k \geq 1$ are fixed integers, then $R$ satisfies $s_4$, the standard identity in four variables.

**Proof** Suppose that $R$ does not satisfy $s_4$. By our assumption, we have

$$a[d(x^k), x^k]^n \in Z(R),$$ \hfill (2.8)
for all $x \in I$. Since there exists $r \in I$ such that $a[d(r^k), r^k]^n \neq 0$, $a[d(x^k), x^k]^n$ is a central differential identity for $I$. It follows from [16, Theorem 1] that $R$ is a prime PI-ring and so $RC(= Q)$ is a finite-dimensional central simple $C$-algebra by Posner’s theorem for prime PI-ring. Now we divide the proof in the following two cases:

**Case 1** Let $d$ be inner derivation of $R$ induced by $p \in Q$. Then

$$[a[p, x^k]^2]^n, x_3] = 0, \tag{2.9}$$

for all $x \in I$ and so for all $x \in Q$, since $I$ and $Q$ satisfy same GPI [17]. Since $a[d(r^k), r^k]^n \neq 0$ for some $r \in I$, (2.9) is a nontrivial GPI for $Q$. Also, since $Q$ is a finite-dimensional central simple $C$-algebra, Lemma 2.2 is applicable. By Lemma 2.2, there exists a suitable field $F$ such that $Q \subseteq M_k(F)$, the ring of all $k \times k$ matrices over $F$, and moreover $M_k(F)$ satisfies (2.9). Since by assumption, $R$ does not satisfy $s_4$, $k \geq 3$. Therefore, we have

$$a[p, x^k]^2 \in Z(M_k(F))$$

for all $x \in M_k(F)$. Since $I \subseteq Q \subseteq M_k(F)$, there exists $r \in M_k(F)$, such that $a[p, r^k]^2 \neq 0$. Then $a$ is invertible and so $(p, x^k)^2 \in F \cdot a^{-1}$ for all $x \in M_k(F)$. By Lemma 2.1, $p \in Z(R)$ implying $d = 0$, a contradiction.

**Case 2** Let $d$ be outer derivation of $R$. We rewrite the relation (2.8) as

$$a[\sum_{i=0}^{k-1} x^i d(x) x^{k-i-1}, x^k]^n \in Z(R). \tag{2.10}$$

By Kharchenko’s theorem [14], we have that $I$ satisfies

$$a[\sum_{i=0}^{k-1} x^i y x^{k-i-1}, x^k]^n \in Z(R). \tag{2.11}$$

Since we assumed that $R$ does not satisfy $s_4$, $R$ cannot be commutative. Therefore, we may choose $b \in R$ such that $b \notin Z(R)$. Replacing $y$ with $[b, x]$ in (2.11), we obtain that for all $x \in I$

$$[a[[b, x^k], x^k]^n, x_3] = 0. \tag{2.12}$$

Then by the same argument as given in case-I, $b \in Z(R)$, a contradiction. □

The following example demonstrates that in the hypothesis the condition $a[d(r^k), r^k]^n \neq 0$ for some $r \in I$ cannot be omitted.

**Example 2.4** Let $R_1$ be any ring not satisfying $s_4$ and $R_2 = \{(a, b) | a, b \in F\}$, where $F$ is a field. Set $R = R_1 \bigoplus R_2$, we define a map $d : R \to R$ by $d(r, s) = (0, t)$ with $t = (0, a)$ for all $r \in R_1$ and $s = (0, 0) \in R_2$. It is easy to check $d$ is a nonzero derivation of $R$. Now let $I = \{0\} \times \{(0, a) | a \in F\}$ be a nonzero ideal of $R$. It is straightforward to check that $d$ satisfies the property $a[d(x^k), x^k]^n = 0$ for all $x \in I$, however $R$ does not satisfy $s_4$.

Now to prove our next theorem we need the following lemma.

**Lemma 2.5** Let $n$ be a fixed positive integer, $R$ be an $n!$-torsion free ring with center $Z(R)$.
Suppose \( y_1, y_2, \ldots, y_n \in R \) satisfy \( \lambda y_1 + \lambda^2 y_2 + \cdots + \lambda^n y_n \in Z(R) \) for \( \lambda = 1, 2, \ldots, n \). Then \( y_i \in Z(R) \) for all \( i \).

**Proof** The proof of this lemma is analogous to the proof of Lemma 1 in [18]. \( \square \)

Now we prove our next theorem.

**Theorem 2.6** Let \( n \geq 1 \), \( k \geq 1 \) be fixed integers, \( R \) a noncommutative \( 2(nk - 1) \)-torsion free prime ring with center \( Z(R) \), \( 0 \neq I \) an ideal of \( R \), \( 0 \neq a \in R \) and \( d \) a derivation of \( R \). If \( a[d(x^k), x^kn] \in Z(R) \) for all \( x \in I \), then either \( d = 0 \) or \( R \) satisfies \( s_4 \).

**Proof** By [19], since \( I, R \) and \( Q \) satisfies the same differential identities, we have

\[
a[d(x^k), x^kn] \in C,
\]

for all \( x \in Q \). Since \( 1 \in Q \), we may replace \( x \) with \( x + 1 \). By this replacement, we obtain

\[
a[d((x + 1)^k), (x + 1)^kn] \in C,
\]

for all \( x \in Q \). We have the facts \((x + 1)^k = x^k + \binom{k}{1}x^{k-1} + \binom{k}{2}x^{k-2} + \cdots + 1 \) and \( d(1) = 0 \). Using these facts, (2.14) implies that

\[
a\left[d(x^k) + \binom{k}{1}d(x^{k-1}) + \cdots + \binom{k}{k - 1}d(x)\right]d(x^{k-1}) + \cdots + \binom{k}{k - 1}x\right]^n \in C,
\]

that is

\[
a\left[d(x^k), x^k\right] + \binom{k}{1}\left[d(x^k), x^{k-1}\right] + \cdots + \binom{k}{k - 1}\left[d(x^k), x\right]\right]^n \in C,
\]

for all \( x \in Q \). Now expanding the expression completely and then using (2.13), the above expression can be rewritten as

\[
a[f_{2kn-1}(x)] + a[f_{2kn-2}(x)] + \cdots + a[f_{2n}(x)] \in C,
\]

where \( f_n(x) \) denotes a suitable homogeneous function of degree \( n \) in \( x \). Putting \( x = \lambda x \), where \( \lambda \in C \), in (2.17), we get

\[
\lambda^{2n-1}\{\lambda^{2kn-2n}a[f_{2kn-1}(x)] + \lambda^{2kn-2n-1}a[f_{2kn-2}(x)] + \cdots + \lambda a[f_{2n}(x)]\} \in C.
\]

Since \( \lambda \in C \) is invertible in \( C \), above relation yields that

\[
\lambda^{2kn-2n}a[f_{2kn-1}(x)] + \lambda^{2kn-2n-1}a[f_{2kn-2}(x)] + \cdots + \lambda a[f_{2n}(x)] \in C.
\]

Putting \( \lambda = 1, 2, \ldots, 2kn - 2n \) and then using Lemma 2.5, we have \( a[f_{2n}(x)] \in C \) for all \( x \in Q \), since \( R \) is \((2kn - 2n)\)-torsion free. Now, \( a[f_{2n}(x)] \in C \) is \( a\{(\binom{k}{k - 1})[d(x), x]\}^n \in C \) for all \( x \in Q \) i.e., \( ak^{2n}[d(x), x]^n \in C \) for all \( x \in Q \). Since \( R \) is \((2kn - k)\)-torsion free, \( a[d(x), x]^n \in C \) for all \( x \in Q \). This implies that either \( d = 0 \) or \( R \) satisfies \( s_4 \) (see [8,9]). \( \square \)

We conclude with an example in a prime ring \( R \) satisfying the differential identity in above theorem.

**Example 2.7** Let \( R = M_2(F) \) be a \( 2 \times 2 \) matrix ring over a field \( F \). Then for any \( 0 \neq a \in Z(R) \)
and any derivation $d$ of $R$, we have $a[d(x^k), x^k]^{2n} \in \mathbb{Z}(R)$ for all $x \in R$, where $k$ and $n$ are any positive integers.

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