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Initial Bounds for a Subclass of Analytic and Bi-Univalent Functions Defined by Chebyshev Polynomials and q-Differential Operator

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Abstract In this paper, we investigate the coefficient estimate and Fekete-Szegö inequality of a subclass of analytic and bi-univalent functions defined by Chebyshev polynomials and q-differential operator. The results presented in this paper improve or generalize the recent works of other authors.

Keywords analytic functions; bi-univalent functions; coefficient estimates; Fekete-Szegö inequality; Chebyshev polynomials; q-differential operator

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1. Introduction

Let $\mathcal{R} = (-\infty, +\infty)$ be the set of real numbers, $\mathcal{N} := \{1, 2, 3, \ldots\} = \mathcal{N}_0 \backslash \{0\}$ be the set of positive integers.

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we denote the family of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, which is defined by

$$f^{-1}(f(z)) = z, \quad z \in \mathcal{U}$$

and

$$f(f^{-1}(\omega)) = \omega, \quad |\omega| < r_0(f), r_0(f) \geq \frac{1}{4}.$$

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The inverse function $g = f^{-1}$ is given by

$$f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 - (5a_2^3 - 5a_2a_3 + a_4)\omega^4 + \cdots . \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both f and f^{-1} are univalent in $\mathcal{U}$. Let Σ denote the class of all bi-univalent functions in $\mathcal{U}$ given by (1.1). The class of bi-univalent functions was first introduced and studied by Lewin [1] and was showed that $|a_2| < 1.51$. Brannan and Clunie [2] improved Lewin's results to $|a_2| \leq \sqrt{2}$ and later Netanyahu [3] proved that $max|a_2| = 4/3$. Recently, many authors investigated bounds for various subclasses of bi-univalent functions [4–12].

Nowadays, area of q-calculus has attracted the attention of researchers. Ismail et al. [13] first introduced the class of generalized complex functions via q-calculus on some subclasses of analytic functions. Recently many newsworthy results related to bi-univalent and q-calculus are studied by various authors [4, 14, 15].

Kamble et al. [4] defined Sălăgean q-differential operator [16] using q-differential operator as follows:

$$D_q^n f(z) = z + \sum_{k=2}^{\infty} [k]_q^n a_k z^k, \quad n \in \mathcal{N}_0, \ z \in \mathcal{U}. \tag{1.3}$$

We note that $q \to 1^-$,

$$D^n f(z) = z + \sum_{k=2}^{\infty} k^n a_k z^k, \quad n \in \mathcal{N}_0, \ z \in \mathcal{U}. \tag{1.4}$$

The Chebyshev polynomials are a sequence of orthogonal polynomials that are related to De Moivre's formula and which can be defined recursively. They have abundant properties, which make them useful in many areas in applied mathematics, numerical analysis and approximation theory. There are four kinds of Chebyshev polynomials, see for details Doha [17] and Mason [18]. The Chebyshev polynomials of degree n of the second kind, which are denoted $U_n(t)$, are defined for $t \in [-1, 1]$ by the following three-terms recurrence relation:

$$U_0(t) = 1, U_1(t) = 2t, U_{n+1}(t) := 2tU_n(t) - U_{n-1}(t).$$

The first few of the Chebyshev polynomials of the second kind are

$$U_1(t) = 2t, U_2(t) = 4t^2 - 1, U_3(t) = 8t^3 - 4t, U_4(t) = 16t^4 - 12t^2 + 1, \ldots . \tag{1.5}$$

The generating function for the Chebyshev polynomials of the second kind, $U_n(t)$ is given by:

$$H(z, t) = \frac{1}{1 - 2tz + z^2} = \sum_{n=0}^{\infty} U_n(t) z^n, \quad z \in \mathcal{U}.$$

Using q-differential operator and Chebyshev polynomials, we define the following new subclass.

Definition 1.1 *For $\lambda \geq 1, \mu \geq 0, 0 < q < 1, n \in \mathcal{N}_0$ and $t \in (1/2, 1)$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma^{q,\mu}(n, \lambda, t)$ if the following subordinations hold for all $z, \omega \in \mathcal{U}$:*

$$(1 - \lambda)\left(\frac{D_q^n f(z)}{z}\right)^\mu + \lambda (D_q^n f(z))' \left(\frac{D_q^n f(z)}{z}\right)^{\mu - 1} \prec H(z, t) := \frac{1}{1 - 2tz + z^2}$$

and

$$(1-\lambda)(\frac{D_q^n g(\omega)}{\omega})^{\mu}+\lambda(D_q^n g(\omega))'(\frac{D_q^n g(\omega)}{\omega})^{\mu-1}\prec H(\omega,t):=\frac{1}{1-2t\omega+\omega^2},$$

where the function $g=f^{-1}$ is given by (1.2).

The following special cases of Definition 1.1 are worthy of note:

(1) For $0<q<1, n\in\mathcal{N}_0$ and $t\in(1/2,1)$, a function $f\in\Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{q,1}(n,1,t)=\mathcal{B}_{\Sigma}^{q}(n,t)$ if the following subordinations hold for all $z,\omega\in\mathcal{U}$:

$$(D_q^n f(z))'\prec H(z,t):=\frac{1}{1-2tz+z^2}$$

and

$$(D_q^n g(\omega))'\prec H(\omega,t):=\frac{1}{1-2t\omega+\omega^2},$$

where the function $g=f^{-1}$ is given by (1.2).

Remark 1.2 For $n=0$ and $q\to 1^-$ in $\mathcal{B}_{\Sigma}^{q}(n,t)$, the class $\mathcal{B}_{\Sigma}^{q}(n,t)$ reduces to $\mathcal{B}_{\Sigma}(t)$ studied by Altinkaya and Yalçin [7].

(2) For $\lambda\geq 1, 0<q<1, n\in\mathcal{N}_0$ and $t\in(1/2,1)$, a function $f\in\Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{q,1}(n,\lambda,t)=\mathcal{B}_{\Sigma}^{q}(n,\lambda,t)$ if the following subordinations hold for all $z,\omega\in\mathcal{U}$:

$$(1-\lambda)\frac{D_q^n f(z)}{z}+\lambda(D_q^n f(z))'\prec H(z,t):=\frac{1}{1-2tz+z^2}$$

and

$$(1-\lambda)\frac{D_q^n g(\omega)}{\omega}+\lambda(D_q^n g(\omega))'\prec H(\omega,t):=\frac{1}{1-2t\omega+\omega^2},$$

where the function $g=f^{-1}$ is given by (1.2).

Remark 1.3 For $n=0$ and $q\to 1^-$ in $\mathcal{B}_{\Sigma}^{q}(n,\lambda,t)$, the class $\mathcal{B}_{\Sigma}^{q}(n,\lambda,t)$ reduces to $\mathcal{B}_{\Sigma}(\lambda,t)$ studied by Bulut and Magesh [8].

(3) For $\mu\geq 0, 0<q<1, n\in\mathcal{N}_0$ and $t\in(1/2,1)$, a function $f\in\Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_{\Sigma}^{q,\mu}(n,t)$ if the following subordinations hold for all $z,\omega\in\mathcal{U}$:

$$(D_q^n f(z))'(\frac{D_q^n f(z)}{z})^{\mu-1}\prec H(z,t):=\frac{1}{1-2tz+z^2}$$

and

$$(D_q^n g(\omega))'(\frac{D_q^n g(\omega)}{\omega})^{\mu-1}\prec H(\omega,t):=\frac{1}{1-2t\omega+\omega^2},$$

where the function $g=f^{-1}$ is given by (1.2).

Remark 1.4 (i) For $n=\mu=0$ and $q\to 1^-$ in $\mathcal{B}_{\Sigma}^{q,\mu}(n,t)$, the class $\mathcal{B}_{\Sigma}^{q,\mu}(n,t)$ reduces to $\mathcal{S}_{\Sigma}^{*}(t)$ studied by Magesh and Bulut [9].

(ii) For $n=0$ and $q\to 1^-$ in $\mathcal{B}_{\Sigma}^{q,\mu}(n,t)$, the class $\mathcal{B}_{\Sigma}^{q,\mu}(n,t)$ reduces to $\mathcal{B}_{\Sigma}^{\mu}(t)$ studied by Altinkaya and Yaçin [10].

We note that $n=\lambda=1, \mu=0$ and $q\to 1^-$, the class $\mathcal{B}_{\Sigma}^{q,\mu}(n,\lambda,t)$ reduces to $\mathcal{K}_{\Sigma}(t)$ studied by Murugusundaramoothy et al. ([5], also see [11]).

(4) For $t \in (1/2, 1)$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{K}_\Sigma(t)$ if the following subordinations hold for all $z, \omega \in \mathcal{U}$:

$$1 + \frac{zf''(z)}{f'(z)} \prec H(z,t) := \frac{1}{1 - 2tz + z^2}$$

and

$$1 + \frac{\omega g''(\omega)}{g'(\omega)} \prec H(\omega,t) := \frac{1}{1 - 2t\omega + \omega^2},$$

where the function $g = f^{-1}$ is given by (1.2).

For $n = \mu = 1$ and $q \to 1^-$, the class $\mathcal{B}_\Sigma^{q;\mu}(n,t)$ reduces to the following subclass of Σ.

(5) For $\lambda \geq 1$ and $t \in (1/2, 1)$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{L}_\Sigma(\lambda, t)$, if the following subordinations hold for all $z, \omega \in \mathcal{U}$:

$$f'(z) + \lambda z f''(z) \prec H(z,t) := \frac{1}{1 - 2tz + z^2}$$

and

$$g'(\omega) + \lambda \omega g''(\omega) \prec H(\omega,t) := \frac{1}{1 - 2t\omega + \omega^2},$$

where the function $g = f^{-1}$ is given by (1.2).

Remark 1.5 For $\lambda = 1$, the class $\mathcal{L}_\Sigma(\lambda, t)$ reduces to the class $\mathcal{L}_\Sigma(t)$ studied by Murugusundaramoothy et al. [11].

(6) For $\lambda \geq 1, \mu \geq 0, 0 < q < 1, n \in \mathcal{N}_0$ and $t \in (1/2, 1)$, a function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma^\mu(\lambda, t)$, if the following subordinations hold for all $z, \omega \in \mathcal{U}$:

$$(1-\lambda)(\frac{f(z)}{z})^\mu + \lambda(f(z))'(\frac{f(z)}{z})^{\mu-1} \prec H(z,t) := \frac{1}{1 - 2tz + z^2}$$

and

$$(1-\lambda)(\frac{g(\omega)}{\omega})^\mu + \lambda(g(\omega))'(\frac{g(\omega)}{\omega})^{\mu-1} \prec H(\omega,t) := \frac{1}{1 - 2t\omega + \omega^2},$$

where the function $g = f^{-1}$ is given by (1.2).

Remark 1.6 (i) In [12], Bulut et al. investigate the estimates of $|a_2|$ and $|a_3|$ and get Fekete-szegö inequalities of the class $\mathcal{B}_\Sigma^\mu(\lambda, t)$.

(ii) Orhan et al. [6] obtained an upper bound estimate for the second Hankel determinant of the subclass $\mathcal{B}_\Sigma^\mu(\lambda, t)$ of analytic bi-univalent function.

In order to derive our main results, we shall need the following lemma.

Lemma 1.7 ([19]) *Let $u(z)$ be an analytic function with $u(0) = 0$, $|u(z)| < 1$ and let*

$$u(z) = c_1 z + c_2 z^2 + \cdots, \quad z \in \mathcal{U}.$$

Then $|c_1| \leq 1$ and $|c_n| \leq 1 - |c_1|^2$ $(n \geq 2)$.

2. Coefficient estimates

In this section, we give our main results.

Theorem 2.1 *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}^{q,\mu}(n,\lambda,t)$. Then*

$$|a_2|\le \min\{\frac{2t}{(\mu+\lambda)[2]_q^n},\sqrt{\frac{8t^2+4t-2}{(\mu+2\lambda)|2[3]_q^n+(\mu-1)[2]_q^{2n}|}},\Omega_1\}, \tag{2.1}$$

$$|a_3|\le \begin{cases} \frac{2t}{(\mu+2\lambda)[3]_q^n}, & t\le \frac{(\lambda+\mu)^2[2]_q^{2n}}{2(\mu+2\lambda)[3]_q^n},\\ \min\{\frac{4t^2}{(\mu+\lambda)^2[2]_q^{2n}},\Omega_2,\Omega_3\}, & t> \frac{(\lambda+\mu)^2[2]_q^{2n}}{2(\mu+2\lambda)[3]_q^n}, \end{cases} \tag{2.2}$$

where

$$\Omega_1=\frac{2t\sqrt{2t}}{\sqrt{|(2[3]_q^n+(\mu-1)[2]_q^{2n})(\mu+2\lambda)2t^2-(4t^2-1)(\mu+\lambda)^2[2]_q^{2n}|+2t(\mu+\lambda)^2[2]_q^{2n}}},$$

$$\Omega_2=(1-\frac{(\lambda+\mu)^2[2]_q^{2n}}{2t(\mu+2\lambda)[3]_q^n})\times\frac{8t^2+4t-2}{(\mu+2\lambda)|2[3]_q^n+(\mu-1)[2]_q^{2n}|}+\frac{2t}{(\mu+2\lambda)[3]_q^n},$$

$$\begin{aligned}\Omega_3=&(1-\frac{(\lambda+\mu)^2[2]_q^{2n}}{2t(\mu+2\lambda)[3]_q^n})\times\\ &\frac{8t^3}{|(2[3]_q^n+(\mu-1)[2]_q^{2n})(\mu+2\lambda)2t^2-(4t^2-1)(\mu+\lambda)^2[2]_q^{2n}|+2t(\mu+\lambda)^2[2]_q^{2n}}+\frac{2t}{(\mu+2\lambda)[3]_q^n}.\end{aligned}$$

Proof Let $f\in\mathcal{B}_{\Sigma}^{q,\mu}(n,\lambda,t)$ and $g=f^{-1}$. Then there are analytic functions $u,v:U\to U$, with $u(0)=v(0)=0$ satisfying

$$(1-\lambda)(\frac{D_q^nf(z)}{z})^{\mu}+\lambda(D_q^nf(z))'(\frac{D_q^nf(z)}{z})^{\mu-1}=1+U_1(t)u(z)+U_2(t)u^2(z)+\cdots, \tag{2.3}$$

$$(1-\lambda)(\frac{D_q^ng(\omega)}{\omega})^{\mu}+\lambda(D_q^ng(\omega))'(\frac{D_q^ng(\omega)}{\omega})^{\mu-1}=1+U_1(t)v(\omega)+U_2(t)v^2(\omega)+\cdots. \tag{2.4}$$

By definition of the functions $u(z)$ and $v(\omega)$

$$u(z)=c_1z+c_2z^2+c_3z^3\cdots, \tag{2.5}$$

$$v(\omega)=d_1\omega+d_2\omega^2+d_3\omega^3\cdots. \tag{2.6}$$

From (2.3)–(2.6), we get

$$(1-\lambda)(\frac{D_q^nf(z)}{z})^{\mu}+\lambda(D_q^nf(z))'(\frac{D_q^nf(z)}{z})^{\mu-1}=1+U_1(t)c_1z+[U_1(t)c_2+U_2(t)c_1^2]z^2+\cdots \tag{2.7}$$

and

$$(1-\lambda)(\frac{D_q^ng(\omega)}{\omega})^{\mu}+\lambda(D_q^ng(\omega))'(\frac{D_q^ng(\omega)}{\omega})^{\mu-1}=1+U_1(t)d_1\omega+[U_1(t)d_2+U_2(t)d_1^2]\omega^2+\cdots. \tag{2.8}$$

Equating the coefficients in (2.7) and (2.8), we have

$$(\lambda+\mu)[2]_q^na_2=U_1(t)c_1, \tag{2.9}$$

$$(\mu-1)(\lambda+\frac{\mu}{2})[2]_q^{2n}a_2^2+(\mu+2\lambda)[3]_q^na_3=U_1(t)c_2+U_2(t)c_1^2, \tag{2.10}$$

$$-(\lambda+\mu)[2]_q^na_2=U_1(t)d_1, \tag{2.11}$$

$$-(\mu+2\lambda)[3]_q^na_3+(4[3]_q^n+(\mu-1)[2]_q^{2n})(\lambda+\frac{\mu}{2})a_2^2=U_1(t)d_2+U_2(t)d_1^2. \tag{2.12}$$

From (2.9) and (2.11), we obtain

$$c_1 = -d_1, \tag{2.13}$$

$$a_2^2 = \frac{U_1^2(t)(c_1^2 + d_1^2)}{2(\mu+\lambda)^2[2]_q^{2n}}. \tag{2.14}$$

Applying Lemma 1.7 and (1.5), we have

$$|a_2| \leq \frac{2t}{(\mu+\lambda)[2]_q^n}. \tag{2.15}$$

Adding (2.10) and (2.12), we get

$$(2[3]_q^n + (\mu-1)[2]_q^{2n})(\mu+2\lambda)a_2^2 = U_1(t)(c_2+d_2) + U_2(t)(c_1^2+d_1^2). \tag{2.16}$$

Using Lemma 1.7 for the coefficients c_1, c_2, d_1 and d_2, we get

$$|a_2| \leq \sqrt{\frac{8t^2+4t-2}{(\mu+2\lambda)|2[3]_q^n + (\mu-1)[2]_q^{2n}|}}. \tag{2.17}$$

Substituting (2.13) and (2.14) into (2.16), we obtian

$$c_1^2 = \frac{U_1(t)(\mu+\lambda)^2[2]_q^{2n}(c_2+d_2)}{(2[3]_q^n + (\mu-1)[2]_q^{2n})(\mu+2\lambda)U_1^2(t) - 2U_2(t)(\mu+\lambda)^2[2]_q^{2n}}. \tag{2.18}$$

Applying (2.13) and (2.18) in (2.14), we get

$$a_2^2 = \frac{U_1^3(t)(c_2+d_2)}{(2[3]_q^n + (\mu-1)[2]_q^{2n})(\mu+2\lambda)U_1^2(t) - 2U_2(t)(\mu+\lambda)^2[2]_q^{2n}}. \tag{2.19}$$

Then, in view of Lemma 1.7 and (2.9), we have

$$|a_2| \leq \frac{2t\sqrt{2t}}{\sqrt{|(2[3]_q^n + (\mu-1)[2]_q^{2n})(\mu+2\lambda)2t^2 - (4t^2-1)(\mu+\lambda)^2[2]_q^{2n}| + 2t(\mu+\lambda)^2[2]_q^{2n}}}. \tag{2.20}$$

Therefore, from (2.15), (2.17) and (2.20), we get assertion (2.1).

By subtracting (2.12) from (2.10), we have

$$2(\mu+2\lambda)[3]_q^n a_3 - 2(\mu+2\lambda)[3]_q^n a_2^2 = U_1(t)(c_2-d_2) + U_2(t)(c_1^2-d_1^2).$$

Further, in view of (2.13), we get

$$a_3 = a_2^2 + \frac{U_1(t)(c_2-d_2)}{2(\mu+2\lambda)[3]_q^n}.$$

Then, in view of (1.5), (2.9) and (2.13), applying Lemma 1.7 for the coefficients c_2 and d_2, we get

$$\begin{aligned} |a_3| &\leq |a_2|^2 + \frac{t}{(\mu+2\lambda)[3]_q^n}(|c_2|+|d_2|) \leq |a_2|^2 + \frac{2t}{(\mu+2\lambda)[3]_q^n}(1-|c_1|^2) \\ &\leq (1 - \frac{(\lambda+\mu)^2[2]_q^{2n}}{2t(\mu+2\lambda)[3]_q^n})|a_2|^2 + \frac{2t}{(\mu+2\lambda)[3]_q^n}. \end{aligned} \tag{2.21}$$

Hence, from (2.15), (2.17), (2.20) and (2.21), we get assertion (2.2).

Thus, this completes the proof of Theorem 2.1. □

Now, we are ready to find the sharp bounds of Fekete-Szegö functional $|a_3-\eta a_2^2|$ defined for $\mathcal{B}_{\Sigma}^{q,\mu}(n,\lambda,t)$.

Theorem 2.2 *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}^{q,\mu}(n,\lambda,t)$ and $\eta\in\mathcal{R}$. Then*

$$|a_3-\eta a_2^2|\le\begin{cases}\frac{2t}{(\mu+2\lambda)[3]_q^n}, & 0\le|h(\eta)|\le\frac{1}{2(\mu+2\lambda)[3]_q^n},\\ \frac{8|1-\eta|t^3}{|(2[3]_q^n+(\mu-1)[2]_q^{2n})(\mu+2\lambda)2t^2-(4t^2-1)(\mu+\lambda)^2[2]_q^{2n}|}, & |h(\eta)|\ge\frac{1}{2(\mu+2\lambda)[3]_q^n},\end{cases}$$

where

$$h(\eta)=\frac{2(1-\eta)t^2}{(2[3]_q^n+(\mu-1)[2]_q^{2n})(\mu+2\lambda)2t^2-(4t^2-1)(\mu+\lambda)^2[2]_q^{2n}}.$$

Proof By using the equalities (2.19) and (2.21), we have

$$a_3-\eta a_2^2=U_1(t)[(h(\eta)+\frac{1}{2(\mu+2\lambda)[3]_q^n})c_2+(h(\eta)-\frac{1}{2(\mu+2\lambda)[3]_q^n})d_2],$$

where

$$h(\eta)=\frac{(1-\eta)U_1^2(t)}{(2[3]_q^n+(\mu-1)[2]_q^{2n})(\mu+2\lambda)U_1^2(t)-2U_2(t)(\mu+\lambda)^2[2]_q^{2n}}.$$

So, we conclude that

$$|a_3-\eta a_2^2|\le\begin{cases}\frac{2t}{(\mu+2\lambda)[3]_q^n}, & 0\le|h(\eta)|\le\frac{1}{2(\mu+2\lambda)[3]_q^n},\\ 4|h(\eta)|t, & |h(\eta)|\ge\frac{1}{2(\mu+2\lambda)[3]_q^n},\end{cases}$$

which completes the proof. □

3. Corollaries and consequences

Now, we would like to draw attention to some remarkable results obtained for some values of q, μ, n, λ and t in Theorems 2.1 and 2.2.

Setting $\mu=\lambda=1$, $n=0$ and $q\to1^-$ in Theorem 2.1, we have the following corollary.

Corollary 3.1 *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}(t)$. Then*

$$|a_2|\le\min\{t,\sqrt{\frac{4t^2+2t-1}{3}},\frac{t\sqrt{2t}}{\sqrt{1-t^2+2t}}\}=\frac{t\sqrt{2t}}{\sqrt{1-t^2+2t}},$$

$$|a_3|\le\begin{cases}\frac{2t}{3}, & \frac{1}{2}<t\le\frac{2}{3},\\ (1-\frac{2}{3t})\frac{2t^3}{1-t^2+2t}+\frac{2t}{3}, & \frac{2}{3}<t<1.\end{cases}$$

Remark 3.2 The estimates for $|a_2|$ and $|a_3|$ given by Corollary 3.1 improve the estimates given by Altinkaya and Yalç in [7, Corollary 8].

Setting $\mu=1,n=0$ and $q\to1^-$ in Theorem 2.1, we have the following corollary.

Corollary 3.3 *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_{\Sigma}(\lambda,t)$. Then*

$$|a_2|\le\min\{\frac{2t}{1+\lambda},\sqrt{\frac{4t^2+2t-1}{1+2\lambda}},\frac{2t\sqrt{2t}}{\sqrt{|(1+\lambda)^2-4t^2\lambda^2|+2t(1+\lambda)^2}}\},$$

$$|a_3|\le\begin{cases}\frac{2t}{1+2\lambda}, & \frac{1}{2}<t\le\frac{(1+\lambda)^2}{2(1+2\lambda)},\\ \min\{\frac{4t^2}{(1+\lambda)^2},(1-\frac{(1+\lambda)^2}{2t(1+2\lambda)})\frac{4t^2+2t-1}{1+2\lambda}+\frac{2t}{1+2\lambda},\Omega_3\}, & \frac{(1+\lambda)^2}{2(1+2\lambda)}<t<1,\end{cases}$$

where

$$\Omega_3 = (1 - \frac{(1+\lambda)^2}{2t(1+2\lambda)}) \times \frac{8t^3}{|(1+\lambda)^2 - 4t^2\lambda^2| + 2t(1+\lambda)^2} + \frac{2t}{1+2\lambda}.$$

Remark 3.4 The estimates of the coefficients $|a_2|$ and $|a_3|$ of Corollary 3.3 are the improvement of the estimates obtained in [8, Theorem 1].

Setting $\mu = n = 0, \lambda = 1$ and $q \to 1^-$ in Theorem 2.1, we have the following corollary.

Corollary 3.5 *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}^*_\Sigma(t)$. Then*

$$|a_2| \leq \min\{2t, \sqrt{4t^2+2t-1}, \frac{2t\sqrt{2t}}{\sqrt{1+2t}}\} = \frac{2t\sqrt{2t}}{\sqrt{1+2t}},$$

$$|a_3| \leq \min\{4t^2, (1-\frac{1}{4t})(4t^2+2t-1)+t, (1-\frac{1}{4t})\frac{8t^3}{1+2t}+t\} = (1-\frac{1}{4t})\frac{8t^3}{1+2t}+t.$$

Remark 3.6 Corollary 3.5 provides an improvement of the estimates for $|a_2|$ and $|a_3|$ obtained by Magesh and Bulut [9, Corollary 2].

Setting $n = \mu = 1$ and $q \to 1^-$, the Theorem 2.1 reduces to the following corollary.

Corollary 3.7 *Let $f(z)$ given by (1.1) be in the class $\mathcal{L}_\Sigma(\lambda, t)$. Then*

$$|a_2| \leq \min\{\frac{t}{1+\lambda}, \sqrt{\frac{4t^2+2t-1}{3(1+2\lambda)}}, \frac{t\sqrt{2t}}{\sqrt{|3(1+2\lambda)t^2 - (4t^2-1)(1+\lambda)^2| + 2t(1+\lambda)^2}}\},$$

$$|a_3| \leq \begin{cases} \frac{2t}{3(1+2\lambda)}, & t \leq \frac{2(1+\lambda)^2}{3(1+2\lambda)}, \\ \min\{\frac{t^2}{(1+\lambda)^2}, (1-\frac{2(1+\lambda)^2}{3t(1+2\lambda)}) \times \frac{4t^2+2t-1}{3(1+2\lambda)} + \frac{2t}{3(1+2\lambda)}, \Omega_1\}, & t > \frac{2(1+\lambda)^2}{3(1+2\lambda)}, \end{cases}$$

where

$$\Omega_1 = (1 - \frac{2(1+\lambda)^2}{3t(1+2\lambda)}) \times \frac{2t^3}{|3(1+2\lambda)t^2 - (4t^2-1)(1+\lambda)^2| + 2t(1+\lambda)^2} + \frac{2t}{3(1+2\lambda)}.$$

Setting $n = \mu = \lambda = 1$ and $q \to 1^-$, the Theorem 2.1 reduces to the following corollary.

Corollary 3.8 *Let $f(z)$ given by (1.1) be in the class $\mathcal{L}_\Sigma(t)$. Then*

$$|a_2| \leq \min\{\frac{t}{2}, \sqrt{\frac{4t^2+2t-1}{9}}, \frac{t\sqrt{2t}}{\sqrt{|-7t^2+4|+8t}}\} = \frac{t\sqrt{2t}}{\sqrt{7t^2+8t-4}},$$

$$|a_3| \leq \begin{cases} \frac{2t}{9}, & t \leq \frac{8}{9}, \\ \frac{32t^3-8t}{9(7t^2+8t-4)}, & t > \frac{8}{9}. \end{cases}$$

Remark 3.9 The estimates for $|a_2|$ and $|a_3|$ given by Corollary 3.8 are more accurate than the bounds given by Corollary 5.5 in Murugusundaramoothy er al. [11].

Setting $n = \lambda = 1, \mu = 0$ and $q \to 1^-$ in Theorem 1, the Theorem 1 reduces to the following corollary.

Corollary 3.10 *For $t \in (\frac{\sqrt{2}}{2}, 1)$, let $f(z)$ given by (1.1) be in the class $\mathcal{K}_\Sigma(t)$. Then*

$$|a_2| \leq \min\{t, \sqrt{\frac{4t^2+2t-1}{2}}, \frac{t\sqrt{2t}}{\sqrt{2t^2+2t-1}}\} = \frac{t\sqrt{2t}}{\sqrt{2t^2+2t-1}},$$

$$|a_3|\le \min\{t^2,(1-\frac{1}{3t})\frac{4t^2+2t-1}{2}+\frac{t}{3},(1-\frac{1}{3t})\frac{2t^3}{2t^2+2t-1}+\frac{t}{3}\}=(1-\frac{1}{3t})\frac{2t^3}{2t^2+2t-1}+\frac{t}{3}.$$

Remark 3.11 The estimates for $|a_2|$ and $|a_3|$ given by Corollary 3.10 are smaller than the improvement of the estimates obtained in [11, Corollary 5.4].

Setting $n=0$ and $q\to 1^-$ in Theorem 2.1, we have the following corollary.

Corollary 3.12 *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma^\mu(\lambda,t)$. Then*

$$|a_2|\le \min\{\frac{2t}{\mu+\lambda},\sqrt{\frac{8t^2+4t-2}{(\mu+2\lambda)(\mu+1)}},\Omega_1\},$$

$$|a_3|\le \begin{cases}\frac{2t}{\mu+2\lambda}, & t\le \frac{(\lambda+\mu)^2}{2(\mu+2\lambda)},\\ \min\{\frac{4t^2}{(\mu+\lambda)^2},\Omega_2,\Omega_3\}, & t>\frac{(\lambda+\mu)^2}{2(\mu+2\lambda)},\end{cases}$$

where

$$\Omega_1=\frac{2t\sqrt{2t}}{\sqrt{|2(\mu+2\lambda)(\mu+1)t^2-(4t^2-1)(\mu+\lambda)^2|+2t(\mu+\lambda)^2}},$$

$$\Omega_2=(1-\frac{(\lambda+\mu)^2}{2t(\mu+2\lambda)})\times\frac{8t^2+4t-2}{(\mu+2\lambda)(\mu+1)}+\frac{2t}{(\mu+2\lambda)},$$

$$\Omega_3=(1-\frac{(\lambda+\mu)^2}{2t(\mu+2\lambda)})\times\frac{8t^3}{|2(\mu+2\lambda)(\mu+1)t^2-(4t^2-1)(\mu+\lambda)^2|+2t(\mu+\lambda)^2}+\frac{2t}{(\mu+2\lambda)}.$$

Remark 3.13 Corollary 3.12 provides an improvement of the estimates for $|a_2|$ and $|a_3|$ obtained by Bulut et al. [12, Theorem 1]

Setting $\eta=0$ in Theorem 2.2, we get the following corollary.

Corollary 3.14 *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma^{q,\mu}(n,\lambda,t)$. Then*

$$|a_3-a_2^2|\le\frac{2t}{(\mu+2\lambda)[3]_q^n}.$$

Setting $\mu=\lambda=1,n=0$ and $q\to 1^-$ in Theorem 2.2, we have the following corollary.

Corollary 3.15 ([8]) *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(t)$ and $\eta\in\mathcal{R}$. Then*

$$|a_3-\eta a_2^2|\le\begin{cases}\frac{2t}{3}, & |1-\eta|\le\frac{1-t^2}{3t^2},\\ \frac{2|1-\eta|t^3}{1-t^2}, & |1-\eta|\ge\frac{1-t^2}{3t^2}.\end{cases}$$

Setting $\mu=1,n=0$ and $q\to 1^-$ in Theorem 2.2, we have the following corollary.

Corollary 3.16 ([8]) *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\lambda,t)$ and $\eta\in\mathcal{R}$. Then*

$$|a_3-\eta a_2^2|\le\begin{cases}\frac{2t}{1+2\lambda}, & |1-\eta|\le\frac{|(1+\lambda)^2-4t^2\lambda^2|}{4(1+2\lambda)t^2},\\ \frac{8|1-\eta|t^3}{|(1+\lambda)^2-4t^2\lambda^2|}, & |1-\eta|\ge\frac{|(1+\lambda)^2-4t^2\lambda^2|}{4(1+2\lambda)t^2}.\end{cases}$$

Setting $\mu=n=0,\lambda=1$ and $q\to 1^-$ in Theorem 2.2, we have the following corollary.

Corollary 3.17 ([9]) *Let $f(z)$ given by (1.1) be in the class $\mathcal{S}_\Sigma^*(t)$ and $\eta\in\mathcal{R}$. Then*

$$|a_3-\eta a_2^2|\le\begin{cases}t, & |1-\eta|\le\frac{1}{8t^2},\\ 8|1-\eta|t^3, & |1-\eta|\ge\frac{1}{8t^2}.\end{cases}$$

Setting $n=\mu=1$ and $q\to 1^-$, the Theorem 2.2 reduces to the following corollary.

Corollary 3.18 *Let $f(z)$ given by (1.1) be in the class $\mathcal{L}_\Sigma(\lambda,t)$ and $\eta\in\mathcal{R}$. Then*

$$|a_3-\eta a_2^2|\le\begin{cases}\frac{2t}{3(1+2\lambda)}, & |1-\eta|\le\frac{|3(1+2\lambda)t^2-(4t^2-1)(1+\lambda)^2|}{3(1+2\lambda)t^2},\\ \frac{2|1-\eta|t^3}{|3(1+2\lambda)t^2-(4t^2-1)(1+\lambda)^2|}, & |1-\eta|\ge\frac{|3(1+2\lambda)t^2-(4t^2-1)(1+\lambda)^2|}{3(1+2\lambda)t^2}.\end{cases}$$

Setting $n=\mu=\lambda=1$ and $q\to 1^-$, the Theorem 2.2 reduces to the following corollary.

Corollary 3.19 *Let $f(z)$ given by (1.1) be in the class $\mathcal{L}_\Sigma(t)$ and $\eta\in\mathcal{R}$. Then*

$$|a_3-\eta a_2^2|\le\begin{cases}\frac{2t}{9}, & |1-\eta|\le\frac{|-7t^2+4|}{9t^2},\\ \frac{2|1-\eta|t^3}{|-7t^2+4|}, & |1-\eta|\ge\frac{|-7t^2+4|}{9t^2}.\end{cases}$$

Setting $n=\lambda=1,\mu=0$ and $q\to 1^-$, the Theorem 2.2 reduces to the following corollary.

Corollary 3.20 *For $t\in(\frac{\sqrt{2}}{2},1)$, let $f(z)$ given by (1.1) be in the class $\mathcal{K}_\Sigma(t)$ and $\eta\in\mathcal{R}$. Then*

$$|a_3-\eta a_2^2|\le\begin{cases}\frac{t}{3}, & |1-\eta|\le\frac{2t^2-1}{6t^2},\\ \frac{2|1-\eta|t^3}{2t^2-1}, & |1-\eta)|\ge\frac{2t^2-1}{6t^2}.\end{cases}$$

Setting $n=0$ and $q\to 1^-$ in Theorem 2.2, we have the following corollary.

Corollary 3.21 ([12]) *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma^\mu(\lambda,t)$ and $\eta\in\mathcal{R}$. Then*

$$|a_3-\eta a_2^2|\le\begin{cases}\frac{2t}{\mu+2\lambda}, & |1-\eta|\le\frac{|(\mu+\lambda)^2-2[2(\lambda+\mu)^2-(2\lambda+\mu)(\mu+1)]t^2|}{4(\mu+2\lambda)t^2},\\ \frac{8|1-\eta|t^3}{|(\mu+\lambda)^2-2t^2[2(\mu+\lambda)^2-(\mu+2\lambda)(\mu+1)]|}, & |1-\eta|\ge\frac{|(\mu+\lambda)^2-2[2(\lambda+\mu)^2-(2\lambda+\mu)(\mu+1)]t^2|}{4(\mu+2\lambda)t^2}.\end{cases}$$

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