

Mixed-Type Reverse Order Laws Associated to $\{1, 3, 4\}$ -Inverse

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Abstract In this paper, we study the mixed-type reverse order laws to $\{1, 3, 4\}$ -inverses for closed range operators A , B and AB . It is shown that $B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3\}$ if and only if $R(A^*AB) \subseteq R(B)$. For every $A^{(134)} \in A\{1, 3, 4\}$, it has $(A^{(134)}AB)\{1, 3, 4\}A\{1, 3, 4\} = (AB)\{1, 3, 4\}$ if and only if $R(AA^*AB) \subseteq R(AB)$. As an application of our results, some new characterizations of the mixed-type reverse order laws associated to the Moore-Penrose inverse and the $\{1, 3, 4\}$ -inverse are established.

Keywords $\{1, 3, 4\}$ -inverse; reverse order law; generalized inverse; block-operator matrix

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1. Introduction

Let \mathcal{H} and \mathcal{K} be complex Hilbert spaces and $\mathcal{B}(\mathcal{H}, \mathcal{K})$ be the set of all bounded linear operators from \mathcal{H} into \mathcal{K} , and abbreviate $\mathcal{B}(\mathcal{H}, \mathcal{K})$ to $\mathcal{B}(\mathcal{H})$ if $\mathcal{H} = \mathcal{K}$. For $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $N(A)$ and $R(A)$ are the null space and the range of A , respectively. A generalized inverse of A is an operator $G \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying some of the following four equations, which is said to be the Moore-Penrose conditions:

$$(1) AGA = A, (2) GAG = G, (3) (AG)^* = AG, (4) (GA)^* = GA.$$

Let $A\{i, j, \dots, l\}$ denote the set of all operators $G \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ which satisfy equation $(i), (j), \dots, (l)$ from the above equations. An operator $G \in A\{i, j, \dots, l\}$ is called an $\{i, j, \dots, l\}$ -inverse of A , denoted by $A^{(ij\dots l)}$. The unique $\{1, 2, 3, 4\}$ -inverse of A is denoted by A^\dagger , which is called the Moore-Penrose inverse of A . As is well known, A is the Moore-Penrose invertible if and only if $R(A)$ is closed.

The reverse order law for many types of generalized inverses has been the subject of intensive research since 1960s, and many interesting results have been obtained. For the Moore-Penrose inverse, Greville gave a classical result

$$(AB)^\dagger = B^\dagger A^\dagger \iff R(A^*AB) \subseteq R(B), \quad R(BB^*A^*) \subseteq R(A^*)$$

for any complex matrices A and B in [1]. This result was extended to bounded linear operators on Hilbert spaces by Izumino [2] and Bouldin [3]. Following these, reverse order laws for different

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types of generalized inverses have been studied [4–12]. The mixed-type reverse-order laws for AB like

$$(A^\dagger AB)^\dagger A^\dagger = (AB)^\dagger, \quad B^\dagger(ABB^\dagger)^\dagger = (AB)^\dagger$$

have been considered in [2] and [13]. Many scholars also discussed Mixed-type reverse order laws for $(AB)^{(13)}$, $(AB)^{(123)}$, $(AB)^{(124)}$ and $(AB)^{(134)}$ (see [14–18]). In [17], the author considered the equivalent condition for $B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3\}$ in matrix algebra. By the block operator matrix technique, the author studied the equivalent condition for $B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3, 4\}$ in [12]. Using the similar space decompositions method in [12], Liu, et al. established the necessary and sufficient conditions for the mixed-type reverse-order laws for $B\{1, 3, 4\}(ABB^{(134)})\{1, 3, 4\} \subseteq (AB)\{1, 3, 4\}$ (see [14]). However, this method is not suitable for studying $(A^{(134)}AB)\{1, 3, 4\}A\{1, 3, 4\} = (AB)\{1, 3, 4\}$.

In this paper, we shall improve the space decompositions method in [11, 12] and study the mixed-type reverse order laws associated to $\{1, 3, 4\}$ -inverse. In Section 2, some preliminaries are given and the $\{1, 3, 4\}$ -inverse for a class of triangular matrix is obtained. In Section 3, we derive the necessary and sufficient condition for $B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3\}$ when $R(A)$, $R(B)$ and $R(AB)$ are closed. A new equivalent condition for $B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3, 4\}$ is obtained. Moreover, the necessary and sufficient condition for $(A^{(134)}AB)\{1, 3, 4\}A\{1, 3, 4\} = (AB)\{1, 3, 4\}$ is given. As an application of our results, the mixed-type reverse order laws associated to the Moore-Penrose inverse are considered.

In this section, we mainly discuss representations for generalized inverses of triangular operator matrices. Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ have closed range. It is well known that A , as an operator from $\mathcal{H} = R(A^*) \oplus N(A)$ into $\mathcal{K} = R(A) \oplus N(A^*)$, has the diagonal matrix form $A = A_1 \oplus 0$, where $A_1 \in \mathcal{B}(R(A^*), R(A))$ is invertible. In this case, the Moore-Penrose inverse A^\dagger of A can be represented by $A^\dagger = A_1^{-1} \oplus 0$. In general, we have the following results.

Lemma 2.1 *For a given operator $C \in \mathcal{B}(\mathcal{H})$, let $M_0 = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$. Then the following statements hold.*

(1) ([14]) *If C is invertible, then the $\{1, 3\}$ -inverse $M_0^{(13)}$ and the $\{1, 3, 4\}$ -inverse $M_0^{(134)}$ of M_0 can be represented by*

$$M_0^{(13)} = \begin{pmatrix} C^{-1} & 0 \\ G_{21} & G_{22} \end{pmatrix} \quad \text{and} \quad M_0^{(134)} = \begin{pmatrix} C^{-1} & 0 \\ 0 & G_{22} \end{pmatrix},$$

respectively, where $G_{21} \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $G_{22} \in \mathcal{B}(\mathcal{K})$ are arbitrary.

(2) *If C^* is surjective, then the $\{1, 3, 4\}$ -inverse $M_0^{(134)}$ of M_0 can be represented by*

$$M_0^{(134)} = \begin{pmatrix} C^\dagger & 0 \\ G_{21} & G_{22} \end{pmatrix}, \tag{2.1}$$

where $G_{22} \in \mathcal{B}(\mathcal{K})$ is arbitrary. $G_{21} = X(I_{\mathcal{H}} - CC^\dagger)$ for arbitrary $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.

Proof We only need to prove the statement (2). Since C^* is surjective, M_0 can be represented

by

$$M_0 = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \rightarrow \begin{pmatrix} R(C) \\ N(C^*) \\ \mathcal{K} \end{pmatrix},$$

where $C = \begin{pmatrix} C_1 \\ 0 \end{pmatrix}$ such that C_1 is invertible. By the statement (1), the $\{1, 3, 4\}$ -inverse $M_0^{(134)}$ of M_0 has the matrix form as

$$M_0^{(134)} = \begin{pmatrix} C_1^{-1} & 0 & 0 \\ 0 & G_{21}'' & G_{22} \end{pmatrix} : \begin{pmatrix} R(C) \\ N(C^*) \\ \mathcal{K} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix},$$

in which $G_{21}'' \in \mathcal{B}(N(C^*), \mathcal{K}), G_{22} \in \mathcal{B}(\mathcal{K})$ are arbitrary. Here $(C_1^{-1} \ 0) = C^\dagger$ and

$$G_{21} = \begin{pmatrix} 0 & G_{21}'' \end{pmatrix} = X(I_{\mathcal{H}} - CC^\dagger), \quad \forall X \in \mathcal{B}(\mathcal{H}, \mathcal{K}).$$

Therefore, (2.1) holds. The proof is completed. \square

Moreover, for given operators $C \in \mathcal{B}(\mathcal{H})$ and $D \in \mathcal{B}(\mathcal{K})$, denote by $M_F = \begin{pmatrix} C & 0 \\ F & D \end{pmatrix}$, where $F \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. Then we have the following results.

Lemma 2.2 *Let $C \in \mathcal{B}(\mathcal{H})$ and $D \in \mathcal{B}(\mathcal{K})$ be given such that C is invertible and D^* is surjective. Then the $\{1, 3, 4\}$ -inverse $M_F^{(134)}$ is unique and can be formulated by*

$$M_F^{(134)} = M_F^+ = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix},$$

where

$$\begin{cases} G_{11} = C^{-1} (I_{\mathcal{H}} + C^{*-1} F^* (I_{\mathcal{K}} - DD^\dagger) FC^{-1})^{-1}, \\ G_{12} = C^{-1} (I_{\mathcal{H}} + C^{*-1} F^* (I_{\mathcal{K}} - DD^\dagger) FC^{-1})^{-1} C^{*-1} F^* (I_{\mathcal{K}} - DD^\dagger), \\ G_{21} = -D^\dagger FC^{-1} (I_{\mathcal{H}} + C^{*-1} F^* (I_{\mathcal{K}} - DD^\dagger) FC^{-1})^{-1}, \\ G_{22} = D^\dagger - D^\dagger FG_{12}. \end{cases} \quad (2.2)$$

Moreover, $G_{12} = 0$ if and only if $R(F) \subseteq R(D)$.

Proof Since the range of M_F^* is surjective, by Lemma 2.1(2), we have the $\{1, 3, 4\}$ -inverse $M_F^{(134)}$ of M_F must be equal to the Moore-Penrose inverse M_F^+ . Set $\mathcal{K} = R(D) \oplus N(D^*)$. Then M_F has the matrix form

$$M_F = \begin{pmatrix} C & 0 \\ F_1 & D_1 \\ F_2 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H} \\ R(D) \\ N(D^*) \end{pmatrix},$$

where C, D_1 are invertible. Let M_F^+ have the matrix form

$$M_F^+ = \begin{pmatrix} G_{11} & G'_{12} & G''_{12} \\ G_{21} & G'_{22} & G''_{22} \end{pmatrix} : \begin{pmatrix} \mathcal{H} \\ R(D) \\ N(D^*) \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H} \\ \mathcal{K} \end{pmatrix},$$

where $(G'_{12} \ G''_{12}) = G_{12}$ and $(G'_{22} \ G''_{22}) = G_{22}$. Set $A = \begin{pmatrix} C & 0 \\ F_1 & D_1 \end{pmatrix}$ and $B = (F_2 \ 0)$, then $A^{-1} = \begin{pmatrix} C^{-1} & 0 \\ -D_1^{-1}F_1C^{-1} & D_1^{-1} \end{pmatrix}$ and $BA^{-1} = (F_2C^{-1} \ 0)$. We have

$$\begin{aligned} M_F^\dagger &= \left[\begin{pmatrix} A^* & B^* \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \right]^\dagger \begin{pmatrix} A^* & B^* \end{pmatrix} \\ &= [A^*A + B^*B]^\dagger \begin{pmatrix} A^* & B^* \end{pmatrix} \\ &= A^{-1} [I_{\mathcal{H} \oplus R(D)} + (BA^{-1})^*(BA^{-1})]^\dagger \begin{pmatrix} I_{\mathcal{H} \oplus R(D)} & (BA^{-1})^* \end{pmatrix} \\ &= \begin{pmatrix} C^{-1} & 0 \\ -D_1^{-1}F_1C^{-1} & D_1^{-1} \end{pmatrix} \begin{pmatrix} (I_{\mathcal{H}} + C^{*-1}F_2^*F_2C^{-1})^{-1} & 0 \\ 0 & I_{R(D)} \end{pmatrix} \begin{pmatrix} I_{\mathcal{H}} & 0 & C^{-1*}F_2^* \\ 0 & I_{R(D)} & 0 \end{pmatrix} \\ &= \begin{pmatrix} C^{-1}(I_{\mathcal{H}} + C^{*-1}F_2^*F_2C^{-1})^{-1} & 0 & C^{-1}(I_{\mathcal{H}} + C^{*-1}F_2^*F_2C^{-1})^{-1}C^{*-1}F_2^* \\ -D_1^{-1}F_1C^{-1}(I_{\mathcal{H}} + C^{*-1}F_2^*F_2C^{-1})^{-1} & D_1^{-1} & -D_1^{-1}F_1C^{-1}(I_{\mathcal{H}} + C^{*-1}F_2^*F_2C^{-1})^{-1}C^{*-1}F_2^* \end{pmatrix}. \end{aligned}$$

Therefore,

$$G_{11} = C^{-1} \left(I_{\mathcal{H}} + C^{*-1}F^*(I_{\mathcal{K}} - DD^\dagger)FC^{-1} \right)^{-1},$$

$$\begin{aligned} G_{12} &= \begin{pmatrix} 0 & C^{-1}(I_{\mathcal{H}} + C^{*-1}F_2^*F_2C^{-1})^{-1}C^{*-1}F_2^* \end{pmatrix} \\ &= C^{-1} \left(I_{\mathcal{H}} + C^{*-1}F^*(I_{\mathcal{K}} - DD^\dagger)FC^{-1} \right)^{-1} C^{*-1}F^*(I_{\mathcal{K}} - DD^\dagger), \end{aligned}$$

$$G_{21} = -D_1^{-1}F_1C^{-1} \left(I_{\mathcal{H}} + C^{*-1}F_2^*F_2C^{-1} \right)^{-1} = -D^\dagger FC^{-1} \left(I_{\mathcal{H}} + C^{*-1}F^*(I_{\mathcal{K}} - DD^\dagger)FC^{-1} \right)^{-1}$$

and

$$\begin{aligned} G_{22} &= \begin{pmatrix} G'_{22} & G''_{22} \end{pmatrix} = \begin{pmatrix} D_1^{-1} & -D_1^{-1}F_1C^{-1}(I_{\mathcal{H}} + C^{*-1}F_2^*F_2C^{-1})^{-1}C^{*-1}F_2^* \end{pmatrix} \\ &= \begin{pmatrix} D_1^{-1} & 0 \end{pmatrix} - \begin{pmatrix} 0 & D_1^{-1}F_1C^{-1}(I_{\mathcal{H}} + C^{*-1}F_2^*F_2C^{-1})^{-1}C^{*-1}F_2^* \end{pmatrix} \\ &= D^\dagger - D^\dagger FG_{12}. \end{aligned}$$

This shows that equalities in (2.2) hold.

Moreover, $G_{12} = 0$ if and only if $F^*(I_{\mathcal{K}} - DD^\dagger) = 0$, which is equivalent to $R(F) \subseteq R(D)$. The proof is completed. \square

3. Mixed-reverse order laws associated to $\{1, 3, 4\}$ -inverse

Let \mathcal{H}, \mathcal{K} and \mathcal{J} be complex Hilbert spaces. Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{J})$, $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and AB have closed ranges. In this section, we will give necessary and sufficient conditions for

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3\}$$

and

$$(A^{(134)}AB)\{1, 3, 4\}A\{1, 3, 4\} = (AB)\{1, 3, 4\}.$$

Firstly, we give some space decompositions. Denote

$$\begin{cases} \mathcal{H}_1 = R(A^*) \ominus (R(A^*) \cap N(B^*)), \\ \mathcal{H}_2 = N(A) \ominus (N(A) \cap N(B^*)), \\ \mathcal{H}_3 = R(A^*) \cap N(B^*), \\ \mathcal{H}_4 = N(A) \cap N(B^*), \end{cases} \quad \begin{cases} \mathcal{K}_1 = R(B^*A^*), \\ \mathcal{K}_2 = R(B^*) \ominus R(B^*A^*), \\ \mathcal{K}_3 = N(B), \end{cases} \quad \begin{cases} \mathcal{J}_1 = (A^*)^\dagger \mathcal{H}_3, \\ \mathcal{J}_2 = R(A) \ominus (A^*)^\dagger \mathcal{H}_3, \\ \mathcal{J}_3 = N(A^*). \end{cases}$$

Then

$$\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4, \quad \mathcal{K} = \mathcal{K}_1 \oplus \mathcal{K}_2 \oplus \mathcal{K}_3 \quad \text{and} \quad \mathcal{J} = \mathcal{J}_1 \oplus \mathcal{J}_2 \oplus \mathcal{J}_3.$$

Lemma 3.1 *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{J})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Suppose that $AB \neq \{0\}$. The following statements hold.*

(1) *If $\mathcal{H}_3 = R(A^*) \cap N(B^*) \neq \{0\}$, then A and B have the following matrix forms*

$$A = \begin{pmatrix} 0 & 0 & A_{13} & 0 \\ A_{21} & 0 & A_{23} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \end{pmatrix} \tag{3.1}$$

and

$$B = \begin{pmatrix} B_{11} & 0 & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix} \tag{3.2}$$

such that A_{13}, A_{21}, B_{11} are invertible and B_{22}^* is surjective.

(2) *If $\mathcal{H}_3 = R(A^*) \cap N(B^*) = \{0\}$, then A and B have the following matrix forms*

$$A = \begin{pmatrix} A_{21} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_4 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{J}_2 \\ \mathcal{J}_3 \end{pmatrix}, \tag{3.3}$$

$$B = \begin{pmatrix} B_{11} & 0 & 0 \\ B_{21} & B_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_4 \end{pmatrix} \tag{3.4}$$

such that A_{21}, B_{11} are invertible and B_{22}^* is surjective.

Proof (1) According to space decompositions of \mathcal{H} and \mathcal{J} , it is clear that A^* has matrix form as follows,

$$A^* = \begin{pmatrix} A_{11}^* & A_{21}^* & 0 \\ 0 & 0 & 0 \\ A_{13}^* & A_{23}^* & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix}.$$

Since $A^* \mathcal{J}_1 = A^* A^*{}^\dagger \mathcal{H}_3 = \mathcal{H}_3$, it is obvious that $A_{11}^* = 0$ and A_{13}^* is surjective. Moreover, $N(A^*) = \mathcal{J}_3$, this implies that A_{13}^* is injective. So A_{13}^* is invertible. This infers A_{21}^* is surjective,

since $R(A^*) = \mathcal{H}_1 \oplus \mathcal{H}_3$. In fact, A_{21}^* is also injective. Otherwise there is a non-zero element $x \in \mathcal{J}_2$ such that $A_{21}^*x = 0$. Then there exists an element $y \in \mathcal{J}_1$ such that $A_{13}^*y = -A_{23}^*x$ since A_{13}^* is invertible. This follows that $A^*(y \oplus x) = 0$ which is a contradiction with $N(A^*) = \mathcal{J}_3$. Therefore, A_{21}^* is invertible. And then A has the matrix form (3.1) where A_{13} and A_{21} are invertible.

By space decompositions \mathcal{H} and \mathcal{K} , it is elementary that

$$B^* = \begin{pmatrix} B_{11}^* & B_{21}^* & 0 & 0 \\ 0 & B_{22}^* & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix},$$

where B_{11}^* is invertible and B_{22}^* is surjective. So B has the matrix form (3.2). Similarly, the statement (2) holds. The proof is completed. \square

Theorem 3.2 *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{J})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Then*

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3\} \iff R(A^*AB) \subset R(B).$$

Proof The result naturally holds when $AB = 0$, since $(AB)\{1, 3\} = \mathcal{B}(\mathcal{J}, \mathcal{K})$ and $R(A^*AB) = \{0\}$ in this case. Assume that $AB \neq 0$. We divide the proof into two cases.

Case 1. $\mathcal{H}_3 \neq \{0\}$. By Lemma 3.1, A, B can be represented by (3.1) and (3.2), respectively, where operators A_{13}, A_{21}, B_{11} are invertible and B_{22}^* is surjective. This implies that

$$AB = \begin{pmatrix} 0 & 0 & 0 \\ A_{21}B_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \end{pmatrix}, \tag{3.5}$$

and

$$A^*AB = \begin{pmatrix} A_{21}^*A_{21}B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ A_{23}^*A_{21}B_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix}. \tag{3.6}$$

For any $A^{(134)} \in A\{1, 3, 4\}$, by Lemma 2.1(1), $A^{(134)}$ has the matrix form

$$A^{(134)} = \begin{pmatrix} -A_{21}^{-1}A_{23}A_{13}^{-1} & A_{21}^{-1} & 0 \\ 0 & 0 & G_1 \\ A_{13}^{-1} & 0 & 0 \\ 0 & 0 & G_2 \end{pmatrix} : \begin{pmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix}, \tag{3.7}$$

where $G_1 \in \mathcal{B}(\mathcal{J}_3, \mathcal{H}_2)$ and $G_2 \in \mathcal{B}(\mathcal{J}_3, \mathcal{H}_4)$. Combining Lemma 2.1(2) with Lemma 2.2, we

conclude that the $\{1, 3, 4\}$ -inverse $B^{(134)}$ of B has the matrix form as follows:

$$B^{(134)} = \begin{pmatrix} G_{11} & G_{12} & 0 & 0 \\ G_{21} & G_{22} & 0 & 0 \\ G_{31} & G_{32} & G_{33} & G_{34} \end{pmatrix} : \begin{pmatrix} \mathcal{H}_1 \\ \mathcal{H}_2 \\ \mathcal{H}_3 \\ \mathcal{H}_4 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix}, \tag{3.8}$$

where $G_{33} \in \mathcal{B}(\mathcal{H}_3, \mathcal{K}_3)$, $G_{43} \in \mathcal{B}(\mathcal{H}_4, \mathcal{K}_3)$ and

$$\begin{cases} G_{11} = B_{11}^{-1}(I_{\mathcal{K}_1} + B_{11}^{*-1}B_{21}^*(I_{\mathcal{H}_2} - B_{22}B_{22}^\dagger)B_{21}B_{11}^{-1})^{-1}, \\ G_{12} = B_{11}^{-1}(I_{\mathcal{K}_1} + B_{11}^{*-1}B_{21}^*(I_{\mathcal{H}_2} - B_{22}B_{22}^\dagger)B_{21}B_{11}^{-1})^{-1}B_{11}^{-1}B_{21}^*(I_{\mathcal{H}_2} - B_{22}B_{22}^\dagger), \\ G_{21} = -B_{22}^\dagger B_{21}G_{11}, \\ G_{22} = B_{22}^\dagger - B_{22}^\dagger B_{21}G_{12}. \end{cases} \tag{3.9}$$

Then it follows from matrix forms of (3.7) and (3.8) that

$$B^{(134)}A^{(134)} = \begin{pmatrix} -G_{11}A_{21}^{-1}A_{23}A_{13}^{-1} & G_{11}A_{21}^{-1} & G_{12}G_1 \\ -G_{21}A_{21}^{-1}A_{23}A_{13}^{-1} & G_{21}A_{21}^{-1} & G_{22}G_1 \\ -G_{31}A_{21}^{-1}A_{23}A_{13}^{-1} + G_{32}A_{13}^{-1} & G_{31}A_{21}^{-1} & G_{32}G_1 + G_{34}G_2 \end{pmatrix}. \tag{3.10}$$

Using Lemma 2.1(1), we get

$$(AB)^{(13)} = \begin{pmatrix} 0 & B_{11}^{-1}A_{21}^{-1} & 0 \\ M_{21} & M_{22} & M_{23} \\ M_{31} & M_{32} & M_{33} \end{pmatrix}, \tag{3.11}$$

where $M_{ij} \in \mathcal{B}(\mathcal{K}_j, \mathcal{J}_i)$, $i \in \{2, 3\}$, $j \in \{1, 2, 3\}$.

Assume that $B\{1, 3, 4\}A\{1, 3, 4\} \subseteq AB\{1, 3\}$, it follows from (3.10) and (3.11) that

$$G_{11} = B_{11}^{-1}, \quad A_{23} = 0, \quad G_{12}G_1 = 0,$$

since M_{ij} , $i \in \{2, 3\}$, $j \in \{1, 2, 3\}$ are arbitrary. Combining $G_{11} = B_{11}^{-1}$ with the equality $G_{11} = B_{11}^{-1} - G_{12}B_{21}B_{11}^{-1}$ in (3.9), we can obtain that $G_{12}B_{21} = 0$, and consequently $R(G_{12}^*) \subseteq N(B_{21}^*)$. From the relation $G_{12}B_{22} = 0$ in (3.9), we can infer that $R(G_{12}^*) \subseteq N(B_{22}^*)$. This shows $R(G_{12}^*) \subseteq N(B_{22}^*) \cap N(B_{21}^*)$ and so $G_{12} = 0$ by the definition of \mathcal{H}_2 . From Lemma 2.2 again, it is obvious that $R(B_{21}) \subseteq R(B_{22})$. This infers that $N(B_{22}^*) \subseteq N(B_{21}^*)$ and so B_{22}^* is injective since $N(B_{22}^*) \cap N(B_{21}^*) = \{0\}$. Therefore, B_{22} is invertible. Combining (3.6) with (3.2), we have $R(A^*AB) = \mathcal{H}_1 \subseteq R(B)$ since $A_{23} = 0$ and B_{22} is invertible.

On the contrary, suppose that $R(A^*AB) \subseteq R(B)$. It is evident from the formula (3.6) that $A_{23}^*A_{21}B_{11} = 0$, and so $A_{23} = 0$. This infers $R(A^*AB) = \mathcal{H}_1 = R(B_{11}) \subseteq R\begin{pmatrix} B_{11} & 0 \\ B_{21} & B_{22} \end{pmatrix}$. It follows that $R(B_{21}) \subseteq R(B_{22})$ and then $N(B_{22}^*) \subseteq N(B_{21}^*)$. Thus B_{22}^* is injective by the definition of \mathcal{H}_2 and so B_{22} is invertible. From Lemma 2.1 again, $G_{11} = B_{11}^{-1}$, $G_{12} = 0$ in (3.8). Combining formulae (3.10) with (3.11), we infer that

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq AB\{1, 3\}$$

by the arbitrariness of M_{2i} , M_{3i} , $i \in \{1, 2, 3\}$.

Case 2. $\mathcal{H}_3 = \{0\}$. In this case, A , B can be represented by (3.3) and (3.4), respectively, by Lemma 3.1. Similar to Case 1, it is clear that $B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3\}$ if and only if

$R(A^*AB) \subseteq R(B)$. The proof is completed. \square

In view of the relationship of $\{1, 3\}$ -inverse and $\{1, 4\}$ -inverse, we have the following result.

Corollary 3.3 *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{J})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Then*

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 4\} \iff R(BB^*A^*) \subseteq R(A^*).$$

Proof For any $E \in A\{1, 3, 4\}$ and $F \in B\{1, 3, 4\}$, then $E^* \in A^*\{1, 3, 4\}$ and $F^* \in B^*\{1, 3, 4\}$. It is clear that $FE \in (AB)\{1, 4\}$ if and only if $E^*F^* \in (B^*A^*)\{1, 3\}$ by Moore-Penrose equations (1), (3) and (4). Therefore, $B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 4\}$ if and only if $A^*\{1, 3, 4\}B^*\{1, 3, 4\} \subseteq (B^*A^*)\{1, 3\}$. Moreover, from Theorem 3.2, we have that

$$A^*\{1, 3, 4\}B^*\{1, 3, 4\} \subseteq (B^*A^*)\{1, 3\} \iff R(BB^*A^*) \subseteq R(A^*).$$

Hence,

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 4\} \iff R(BB^*A^*) \subseteq R(A^*).$$

The proof is completed. \square

Corollary 3.4 *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{J})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Then the following statements are equivalent,*

- (1) $B\{1, 3, 4\}A\{1, 3, 4\} \subseteq (AB)\{1, 3, 4\}$;
- (2) $R(A^*AB) \subseteq R(B)$ and $R(BB^*A^*) \subseteq R(A^*)$;
- (3) $BB^\dagger A^*AB = A^*AB$ and $ABB^*A^\dagger A = ABB^*$.

Proof From Theorem 3.2 and Corollary 3.3, it is easy to get that statements (1) and (2) are equivalent. It follows from matrix forms of A^*AB , BB^*A^* , A^* and B that

$$R(A^*AB) \subseteq R(B), R(BB^*A^*) \subseteq R(A^*) \iff A_{23} = 0, B_{21} = 0.$$

While $A_{23} = 0, B_{21} = 0$ is equivalent to $BB^\dagger A^*AB = A^*AB, ABB^*A^\dagger A = ABB^*$. \square

The reverse order law for $\{1, 3, 4\}$ -inverse was considered in [12]. Under the premise condition in Corollary 3.4, it was given therein

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq AB\{1, 3, 4\}$$

if and only if

$$R(A^*AB) = R(B) \ominus (R(B) \cap N(A)) \text{ and } B^*(R(B) \cap N(A)) = B^\dagger(R(B) \cap N(A)),$$

which is equivalent to the statement (2) in Corollary 3.4 from Lemma 3.1 and matrix forms of A^*AB and BB^*A^* . Moreover, the statement (3) was also obtained by Cvetković -Ilić in [5] for

$$B\{1, 3, 4\}A\{1, 3, 4\} \subseteq AB\{1, 3, 4\}$$

under the condition $A, B, AB, A(I - BB^\dagger)$ and $(I - A^\dagger A)B$ are generalized invertible. Here, Corollary 3.4 is a refinement of the related result in [5].

Theorem 3.5 *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{J})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that all ranges $R(A)$, $R(B)$ and $R(AB)$ are closed. Then the following statements are equivalent:*

- (1) $(A^{(134)}AB)\{1, 3, 4\}A\{1, 3, 4\} = (AB)\{1, 3, 4\}$ for any $A^{(134)} \in A\{1, 3, 4\}$;
- (2) $R(AA^*AB) \subseteq R(AB)$.

Proof If $AB = 0$, the conclusion holds. Suppose that $AB \neq 0$. In the case of $\mathcal{H}_3 \neq \{0\}$, A , B and $A^{(134)}$ have the matrix forms (3.1), (3.2) and (3.7), respectively. Direct computation yields

$$A^{(134)}AB = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

which gives by Lemma 2.1(1) that

$$(AB)^{(134)} = \begin{pmatrix} 0 & B_{11}^{-1}A_{21}^{-1} & 0 \\ M_1 & 0 & M_2 \\ M_3 & 0 & M_4 \end{pmatrix}, \quad (A^{(134)}AB)^{(134)} = \begin{pmatrix} B_{11}^{-1} & 0 & 0 & 0 \\ 0 & F_1 & F_2 & F_3 \\ 0 & F_4 & F_5 & F_6 \end{pmatrix}, \quad (3.12)$$

where $M_i, i = 1, 2, 3, 4, F_k, k = 1, 2, \dots, 6$ are arbitrary. Then

$$(A^{(134)}AB)^{(134)}A^{(134)} = \begin{pmatrix} -B_{11}^{-1}A_{21}^{-1}A_{23}A_{13}^{-1} & B_{11}^{-1}A_{21}^{-1} & 0 \\ F_2A_{13}^{-1} & 0 & F_1G_1 + F_3G_2 \\ F_5A_{13}^{-1} & 0 & F_3G_1 + F_6G_2 \end{pmatrix}. \quad (3.13)$$

Compare (3.12) and (3.13), we get

$$(A^{(134)}AB)\{1, 3, 4\}A\{1, 3, 4\} = (AB)\{1, 3, 4\} \iff A_{23} = 0$$

by the arbitrariness of $G_1, G_2, M_i, i = 1, 2, 3, 4$ and $F_k, k = 1, 2, \dots, 6$. Moreover,

$$AA^*AB = \begin{pmatrix} A_{13}A_{23}^*A_{21}B_{11} & 0 & 0 \\ A_{21}A_{21}^*A_{21}B_{11} + A_{23}A_{23}^*A_{21}B_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \begin{pmatrix} \mathcal{K}_1 \\ \mathcal{K}_2 \\ \mathcal{K}_3 \end{pmatrix} \rightarrow \begin{pmatrix} \mathcal{J}_1 \\ \mathcal{J}_2 \\ \mathcal{J}_3 \end{pmatrix}.$$

This shows that

$$R(AA^*AB) \subseteq R(AB) \iff A_{23} = 0,$$

since A_{13}, A_{21} and B_{11} are all invertible. Therefore,

$$(A^{(134)}AB)\{1, 3, 4\}A\{1, 3, 4\} = (AB)\{1, 3, 4\} \iff R(AA^*AB) \subseteq R(AB).$$

In the similar way, the conclusion also holds in the case of $\mathcal{H}_3 = \{0\}$. The proof is completed. \square

Next, we will study the mixed-type reverse order laws associated to the Moore-Penrose inverse and the $\{1, 3, 4\}$ -inverse.

Theorem 3.6 Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that all ranges $R(A), R(B)$ and $R(AB)$ are closed. Then the following statements are equivalent:

- (1) $(A^\dagger AB)^\dagger A^\dagger = (AB)^\dagger$;
- (2) $(A^\dagger AB)^\dagger A^\dagger \in (AB)\{1, 3, 4\}$;
- (3) $R(AA^*AB) \subseteq R(AB)$.

Proof By Theorem 3.5, (1) \Rightarrow (2) and (3) \Rightarrow (2) hold. Next, we prove the implications (2) \Rightarrow (1)

and (2) \Rightarrow (3). It needs only to consider the case when $AB \neq 0$ and $\mathcal{H}_3 \neq \{0\}$. Here A, B, AB and $(AB)^{(134)}$ have matrix forms as in (3.1), (3.2), (3.5) and (3.12), respectively. It is easy to know that Moore-Penrose inverses A^\dagger and $(AB)^\dagger$ have matrix forms

$$A^\dagger = \begin{pmatrix} -A_{21}^{-1}A_{23}A_{13}^{-1} & A_{21}^{-1} & 0 \\ 0 & 0 & 0 \\ A_{13}^{-1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } (AB)^\dagger = \begin{pmatrix} 0 & B_{11}^{-1}A_{21}^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{3.14}$$

respectively. By direct computation, we have

$$A^\dagger AB = \begin{pmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } (A^\dagger AB)^\dagger = \begin{pmatrix} B_{11}^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.15}$$

Therefore,

$$(A^\dagger AB)^\dagger A^\dagger = \begin{pmatrix} -B_{11}^{-1}A_{21}^{-1}A_{23}A_{13}^{-1} & B_{11}^{-1}A_{21}^{-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.16}$$

Suppose that the statement (2) of this theorem holds. Comparing (3.16) and the matrix form of $(AB)^{(134)}$ in (3.12), we have $B_{11}^{-1}A_{21}^{-1}A_{23}A_{13}^{-1} = 0$ and so $A_{23} = 0$. Thus combining (3.16) with the matrix form of $(AB)^\dagger$ in (3.14), we have $(A^\dagger AB)^\dagger A^\dagger = (AB)^\dagger$. So the statement (1) holds. Moreover, it follows from $A_{23} = 0$ that AA^*AB has the matrix form as follows:

$$AA^*AB = \begin{pmatrix} 0 & 0 & 0 \\ A_{21}A_{21}^*A_{21}B_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

This shows that $R(AA^*AB) \subseteq R(AB)$. Therefore, the statement (3) holds. The proof is completed. \square

Theorem 3.7 *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $B \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be such that all ranges $R(A), R(B)$ and $R(AB)$ are closed. Then the following statements are equivalent:*

- (1) $B^\dagger A\{1, 3, 4\} \subseteq (AB)\{1, 3, 4\}$;
- (2) $B^\dagger A^\dagger \in (AB)\{1, 3, 4\}$;
- (3) $R(BB^*A^*) \subseteq R(A^*)$ and $R(A^*AB) \subseteq R(B)$.

Proof By Corollary 3.4, (1) \Rightarrow (2) and (3) \Rightarrow (1) hold. We need only to prove (2) \Rightarrow (3) when $AB \neq 0$ and the case $\mathcal{H}_3 \neq \{0\}$, since the case $\mathcal{H}_3 = \{0\}$ is similar. Here A, A^\dagger and B have the matrix form as in (3.1), (3.14) and (3.2), respectively. The Moore-Penrose inverse B^\dagger can be represented by

$$B^\dagger = \begin{pmatrix} G_{11} & G_{12} & 0 & 0 \\ G_{21} & G_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{3.17}$$

where $G_{ij}, i \in \{1, 2\}, j \in \{1, 2\}$ satisfy the system of equations (3.9) by Lemma 2.2. Combining (3.14) with (3.17), we get

$$B^\dagger A^\dagger = \begin{pmatrix} -G_{11}A_{21}^{-1}A_{23}A_{13}^{-1} & G_{11}A_{21}^{-1} & 0 \\ -G_{21}A_{21}^{-1}A_{23}A_{13}^{-1} & G_{21}A_{21}^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.18)$$

If $B^\dagger A^\dagger \in (AB)\{1, 3, 4\}$ holds, then from (3.18) and (3.12), we obtain $-G_{11}A_{21}^{-1}A_{23}A_{13}^{-1} = 0$, $G_{11}A_{21}^{-1} = B_{11}^{-1}A_{21}^{-1}$ and $G_{21}A_{21}^{-1} = 0$. This shows that $A_{23} = 0$, $G_{11} = B_{11}^{-1}$ and $G_{21} = 0$. Using the same technique in the proof of theorem 3.2, we know that $R(A^*AB) \subseteq R(B)$ and B_{22} is invertible. Notice that $G_{21} = -B_{22}^{-1}B_{21}B_{11}^{-1}$, it follows that $B_{21} = 0$. This means that $R(BB^*A^*) \subseteq R(A^*)$ by (3.1) and (3.2). The proof is completed. \square

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