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On Some Generalized Countably Compact Spaces

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Abstract We first give alternative expressions of some generalized countably compact spaces such as quasi-γ spaces, quasi-Nagata spaces, $M^{\#}$-spaces and wM-spaces with g-functions. Then by means of these expressions, we present some characterizations of the corresponding spaces with real-valued functions.

Keywords real-valued functions; g-functions; quasi-γ spaces; quasi-Nagata spaces; wN-spaces; $M^{\#}$-spaces; wM-spaces

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1. Introduction

Throughout, a space always means a Hausdorff topological space unless otherwise stated.

Let X be a space. Denote by $\mathcal{C}_X$ ($\mathcal{S}_X$) the family of all compact (sequentially compact) subsets of X. τ and τ^c denote the topology of X and the families of all closed subsets of X, respectively. $\mathcal{F}_0(X)$ denotes the family of all decreasing sequences of closed subsets of X with empty intersection. The set of all positive integers is denoted by $\mathbb{N}$ while $\langle x_n \rangle$ denotes a sequence.

A real-valued function f on a space X is called lower (upper) semi-continuous [1] if for any real number r, the set $\{x \in X : f(x) > r\}$ ($\{x \in X : f(x) < r\}$) is open. We write $L(X)$ ($U(X)$) for the set of all lower (upper) semi-continuous functions from X into the unit interval $[0, 1]$.

A g-function for a space X is a map $g : \mathbb{N} \times X \to \tau$ such that for each $x \in X$ and $n \in \mathbb{N}$, $x \in g(n, x)$ and $g(n + 1, x) \subset g(n, x)$. For a subset $A \subset X$, let $g(n, A) = \cup\{g(n, x) : x \in A\}$. Consider the following conditions.

(q) If $x_n \in g(n, x)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(quasi-γ) If $x_n \in g(n, y_n)$ for all $n \in \mathbb{N}$ and $y_n \to x$, then $\langle x_n \rangle$ has a cluster point.

(β) If $x \in g(n, x_n)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(quasi-Nagata) If $y_n \in g(n, x_n)$ for all $n \in \mathbb{N}$ and $y_n \to x$, then $\langle x_n \rangle$ has a cluster point.

($k\beta$) For each $K \in \mathcal{C}_X$, if $K \cap g(n, x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

(wN) If $g(n, x) \cap g(n, x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

A space that has a g-function satisfying condition (q) ((quasi-γ), (β), (quasi-Nagata), ($k\beta$), (wN)) is called a q-space [2] (quasi-γ space [3], β-space [4], quasi-Nagata space [5], $k\beta$-space [6], wN-space [7]). The g-function satisfying condition (q) is called a q-function. The others are

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defined analogously. β-spaces were also called monotonically countably metacompact spaces in [8]. $k\beta$-spaces were also called monotonically countably mesocompact spaces in [9] and k-MCM spaces in [10].

It is known that a space X is countably compact if and only if every sequence in X has a cluster point. Thus for a countably compact space X, if we let $g(n,x) = X$ for each $x \in X$ and $n \in \mathbb{N}$, then we get a g-function for X which clearly satisfies all the conditions listed above. Thus all these spaces can be viewed as generalizations of countably compact spaces. On the other hand, they are also natural generalizations of some corresponding generalized metric spaces. Actually, if we replace '$\langle x_n \rangle$ has a cluster point' in condition (q) ((quasi-γ), (β), (quasi-Nagata), (wN)) with 'x is a cluster point of $\langle x_n \rangle$', then we get the g-function for first countable spaces (γ-spaces, semi-stratifiable spaces, k-semi-stratifiable spaces, Nagata-spaces). In [11], it was shown that most of generalized metric spaces such as γ-spaces, Nagata-spaces, semi-metrizable spaces and quasi-metrizable spaces can be characterized with real-valued functions. A natural question is that, as generalizations of the corresponding generalized metric spaces, whether the generalized countably compact spaces mentioned above can also be characterized with real-valued functions. With the question in mind, in this paper, we shall show that many classes of generalized countably compact spaces such as the spaces mentioned above as well as $M^{\#}$-spaces, wM-spaces can be characterized analogously to the corresponding generalized metric spaces.

For undefined terminologies, we refer the readers to [1].

2. Alternative expressions of some corresponding spaces

In this section, we give alternative expressions of some corresponding spaces with g-functions which will be used in Section 3.

Lemma 2.1 *If $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x_n \in F_n$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has no cluster point.*

Proof Since $\langle F_n \rangle$ is decreasing and $x_n \in F_n$, we have that $\{x_m : m \geq n\} \subset F_n$ for each $n \in \mathbb{N}$. Thus $\overline{\{x_m : m \geq n\}} \subset F_n$ because F_n is closed. It follows that $\bigcap_{n\in\mathbb{N}} \overline{\{x_m : m \geq n\}} \subset \bigcap_{n\in\mathbb{N}} F_n = \emptyset$. This implies that $\langle x_n \rangle$ has no cluster point. $\square$

Proposition 2.2 *g is a q-function for a space X if and only if for each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $F_n \cap g(n,x) = \emptyset$ for some $n \in \mathbb{N}$.*

Proof Let g be a q-function for X, $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$. Assume that $F_n \cap g(n,x) \neq \emptyset$ for each $n \in \mathbb{N}$ and choose $x_n \in F_n \cap g(n,x)$. Since g is a q-function, $\langle x_n \rangle$ has a cluster point, a contradiction to Lemma 2.1.

Conversely, suppose that $x_n \in g(n,x)$ and let $F_n = \overline{\{x_m : m \geq n\}}$ for each $n \in \mathbb{N}$. Then $F_n \cap g(n,x) \neq \emptyset$ for each $n \in \mathbb{N}$. By the condition, $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$ which implies that $\langle x_n \rangle$ has a cluster point. Therefore, g is a q-function. $\square$

Proposition 2.3 *For a space X, the following are equivalent.*

(a) *g is a quasi-γ function for X;*

(b) *For each $S \in \mathcal{S}_X$, if $x_n \in g(n,S)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point;*

(c) *For each $S \in \mathcal{S}_X$ and $\langle F_n \rangle \in \mathcal{F}_0(X)$, $F_n \cap g(n,S) = \emptyset$ for some $n \in \mathbb{N}$.*

Proof (a) $\Rightarrow$ (b). Let g be a quasi-γ function for X and $S \in \mathcal{S}_X$. Suppose that $x_n \in g(n,S)$ for each $n \in \mathbb{N}$. Then there exists $y_n \in S$ such that $x_n \in g(n,y_n)$ for each $n \in \mathbb{N}$. Since $S \in \mathcal{S}_X$, $\langle y_n \rangle$ has a convergent subsequence $\langle y_{n_k} \rangle$ which clearly also converges in X. Since $x_{n_k} \in g(k, y_{n_k})$ and g is a quasi-γ function, $\langle x_{n_k} \rangle$ has a cluster point which is clearly also a cluster point of $\langle x_n \rangle$.

(b) $\Rightarrow$ (c). Let g be the g-function in (b) and $\langle F_n \rangle \in \mathcal{F}_0(X)$. Let $S \in \mathcal{S}_X$ and suppose that $F_n \cap g(n,S) \neq \emptyset$ for each $n \in \mathbb{N}$. Choose $x_n \in F_n \cap g(n,S)$ for each $n \in \mathbb{N}$. By (b), $\langle x_n \rangle$ has a cluster point, a contradiction to Lemma 2.1.

(c) $\Rightarrow$ (a). Let g be the g-function in (c). Suppose that $x_n \in g(n,y_n)$ for all $n \in \mathbb{N}$ and $y_n \to x$. Let $S = \{y_n : n \in \mathbb{N}\} \cup \{x\}$ and let $F_n = \overline{\{x_m : m \geq n\}}$ for each $n \in \mathbb{N}$. Then $S \in \mathcal{S}_X$ and $F_n \cap g(n,S) \neq \emptyset$ for each $n \in \mathbb{N}$. By (c), $\bigcap_{n\in\mathbb{N}} F_n \neq \emptyset$ which implies that $\langle x_n \rangle$ has a cluster point. Therefore, g is a quasi-γ function. $\square$

Proposition 2.4 *g is a β-function for a space X if and only if for each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $x \notin g(n, F_n)$ for some $n \in \mathbb{N}$.*

Proof Similar to the proof of Proposition 2.2. $\square$

Proposition 2.5 *For a space X, the following are equivalent.*

(a) *g is a quasi-Nagata function for X;*

(b) *For each $S \in \mathcal{S}_X$, if $S \cap g(n,x_n) \neq \emptyset$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point;*

(c) *For each $S \in \mathcal{S}_X$ and $\langle F_n \rangle \in \mathcal{F}_0(X)$, $S \cap g(n,F_n) = \emptyset$ for some $n \in \mathbb{N}$.*

Proof Similar to the proof of Proposition 2.3. $\square$

Since $k\beta$-function can be obtained by replacing $S \in \mathcal{S}_X$ in (b) of Proposition 2.5 with $K \in \mathcal{C}_X$, with a similar argument, we have the following.

Proposition 2.6 *g is a $k\beta$-function for a space X if and only if for each $K \in \mathcal{C}_X$ and $\langle F_n \rangle \in \mathcal{F}_0(X)$, $K \cap g(n,F_n) = \emptyset$ for some $n \in \mathbb{N}$.*

A space X is called an $M^{\#}$-space [12] if there exists a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ of closure preserving closed covers of X such that if $x_n \in \mathrm{st}(x, \mathcal{F}_n)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

Proposition 2.7 *For a space X, the following are equivalent.*

(a) *X is an $M^{\#}$-space;*

(b) *There exists a g-function g for X such that (1) if $g(n,x) \cap g(n,x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point; (2) if $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$;*

(c) *There exists a g-function g for X such that (1) for each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $g(n,x) \cap g(n,F_n) = \emptyset$ for some $n \in \mathbb{N}$; (2) if $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$.*

Proof (a) $\Rightarrow$ (b). Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a sequence of closure preserving closed covers of X satisfying the condition of an $M^{\#}$-space. For each $x \in X$ and $n \in \mathbb{N}$, put $h(n,x) = X \setminus \cup\{F \in \mathcal{F}_n : x \notin F\}$ and $g(n,x) = \cap_{i \le n} h(i,x)$. Then g is a g-function for X and it is clear that g satisfies (2). Suppose that $y_n \in g(n,x) \cap g(n,x_n) \subset h(n,x) \cap h(n,x_n)$ for each $n \in \mathbb{N}$. Since $\mathcal{F}_n$ covers X, there is $F_n \in \mathcal{F}_n$ such that $y_n \in F_n$. Thus $x, x_n \in F_n$ from which it follows that $x_n \in st(x, \mathcal{F}_n)$ for each $n \in \mathbb{N}$. Therefore, $\langle x_n \rangle$ has a cluster point.

(b) ⇒ (c). Let g be the g-function in (b), $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$. Assume that $g(n,x) \cap g(n,F_n) \neq \emptyset$ for each $n \in \mathbb{N}$. Then there exists $x_n \in F_n$ such that $g(n,x) \cap g(n,x_n) \neq \emptyset$ for each $n \in \mathbb{N}$. By (1) of (b), $\langle x_n \rangle$ has a cluster point, a contradiction to Lemma 2.1.

(c) ⇒ (b). Let g be the g-function in (c). Suppose that $g(n,x) \cap g(n,x_n) \neq \emptyset$ for all $n \in \mathbb{N}$. Let $F_n = \overline{\{x_m : m \ge n\}}$ for each $n \in \mathbb{N}$. Then $g(n,x) \cap g(n,F_n) \neq \emptyset$ for each $n \in \mathbb{N}$. Thus $\bigcap_{n \in \mathbb{N}} F_n \neq \emptyset$ which implies that $\langle x_n \rangle$ has a cluster point.

(b) ⇒ (a). Let g be the g-function in (b). For each $x \in X$ and $n \in \mathbb{N}$, let $G_n(x) = \cup\{g(n,y) : y \in X, x \notin g(n,y)\}$. For each $n \in \mathbb{N}$, let $\mathcal{F}_n = \{X \setminus G_n(x) : x \in X\}$. Then $\mathcal{F}_n$ is a closed cover of X.

To show that $\mathcal{F}_n$ is closure preserving, let $A \subset X$. We show that $\cap\{G_n(x) : x \in A\}$ is open. Let $y \in \cap\{G_n(x) : x \in A\}$. Then for each $x \in A$, $y \in G_n(x)$ and thus there exists $z \in X$ such that $y \in g(n,z)$ and $x \notin g(n,z)$. By (2) of (b), we have that $g(n,y) \subset g(n,z)$ and thus $x \notin g(n,y)$. This implies that $g(n,y) \subset G_n(x)$ and thus $g(n,y) \subset \cap\{G_n(x) : x \in A\}$. It follows that $\cap\{G_n(x) : x \in A\}$ is open. Therefore, $\cup\{X \setminus G_n(x) : x \in A\}$ is closed which implies that $\mathcal{F}_n$ is closure preserving.

Now, suppose that $x_n \in st(x, \mathcal{F}_n)$ for each $n \in \mathbb{N}$. Then there exists $y_n \in X$ such that $x_n, x \in X \setminus G_n(y_n)$. Thus for each $y \in X$, if $y_n \notin g(n,y)$, then $x_n, x \notin g(n,y)$. It follows that $y_n \in g(n,x_n)$ and $y_n \in g(n,x)$ and thus $g(n,x) \cap g(n,x_n) \neq \emptyset$. By (1) of (b), $\langle x_n \rangle$ has a cluster point. Therefore, X is an $M^{\#}$-space. □

A cover $\mathcal{P}$ of a space X is called a quasi-(mod k)-network [13] if there is a closed cover $\mathcal{H}$ of X by countably compact subsets such that whenever $H \subset U$ with $H \in \mathcal{H}$ and $U \in \tau$, then $H \subset P \subset U$ for some $P \in \mathcal{P}$. X is called a $\Sigma^{\#}$-space [13] if it has a σ-closure-preserving closed quasi-(mod k)-network.

Lemma 2.8 ([14]) *X is a $\Sigma^{\#}$-space if and only if there exists a g-function g for X such that*

(1) If $x \in g(n,x_n)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point;

(2) If $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$.

The g-function in the above lemma is called a $\Sigma^{\#}$-function. We see that a $\Sigma^{\#}$-function is precisely a β-function which satisfies an additional condition. Thus by Proposition 2.4, we have the following.

Proposition 2.9 *g is a $\Sigma^{\#}$-function for X if and only if*

(1) For each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $x \notin g(n,F_n)$ for some $n \in \mathbb{N}$;

(2) $y \in g(n,x)$, then $g(n,y) \subset g(n,x)$.

A space X is called a wM-space [15] if there exists a sequence $\{\mathcal{G}_n\}_{n \in \mathbb{N}}$ of open covers of X

such that if $x_n \in st^2(x, \mathcal{G}_n)$ for each $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point. Notice that without loss of generality, we may assume that $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ for each $n \in \mathbb{N}$.

Proposition 2.10 *For a space X, the following are equivalent.*

(a) *X is a wM-space.*

(b) *There exists a g-function g for X such that*

(1) *If $g(n,x) \cap g(n,x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point;*

(2) *For each $x, y \in X$ and $n \in \mathbb{N}$, $y \in g(n,x)$ if and only if $x \in g(n,y)$.*

(c) *There exists a g-function g for X such that*

(1) *For each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $g(n,x) \cap g(n,F_n) = \emptyset$ for some $n \in \mathbb{N}$;*

(2) *For each $x, y \in X$ and $n \in \mathbb{N}$, $y \in g(n,x)$ if and only if $x \in g(n,y)$.*

(d) *There exists a g-function g for X such that*

(1) *For each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $x \notin \overline{g(n,F_n)}$ for some $n \in \mathbb{N}$;*

(2) *For each $x, y \in X$ and $n \in \mathbb{N}$, $y \in g(n,x)$ if and only if $x \in g(n,y)$.*

Proof (a) $\Rightarrow$ (b). Let $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ be a sequence of open covers of X satisfying the condition of a wM-space and $\mathcal{G}_{n+1} \prec \mathcal{G}_n$ for each $n \in \mathbb{N}$. For each $x \in X$ and $n \in \mathbb{N}$, let $g(n,x) = st(x, \mathcal{G}_n)$. Then g is a g-function for X and it is clear that g satisfies (2). Suppose that $g(n,x) \cap g(n,x_n) \neq \emptyset$ for each $n \in \mathbb{N}$. Then $x_n \in st^2(x, \mathcal{G}_n)$ for each $n \in \mathbb{N}$ and thus $\langle x_n \rangle$ has a cluster point.

(b) $\Rightarrow$ (c) is similar to the proof of (b) $\Rightarrow$ (c) of Proposition 2.7.

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (a). Let g be the g-function in (d). For each $n \in \mathbb{N}$, let $\mathcal{G}_n = \{g(n,x), x \in X\}$. Then $\{\mathcal{G}_n\}_{n\in\mathbb{N}}$ is a sequences of open covers of X.

Claim 1 If $x_n \in g(n,x)$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

Proof of Claim 1 For each $n \in \mathbb{N}$, let $F_n = \overline{\{x_m : m \geq n\}}$. Assume that $\langle x_n \rangle$ has no cluster point. Then $\langle F_n \rangle \in \mathcal{F}_0(X)$. By (1), $x \notin \overline{g(k,F_k)} \supset g(k,x_k)$ for some $k \in \mathbb{N}$. By (2), $x_k \notin g(k,x)$, a contradiction.

Claim 2 If $g(n,x) \cap g(n,x_n) \neq \emptyset$ for all $n \in \mathbb{N}$, then $\langle x_n \rangle$ has a cluster point.

Proof of Claim 2 Choose $y_n \in g(n,x) \cap g(n,x_n)$ for each $n \in \mathbb{N}$. By Claim 1, $\langle y_n \rangle$ has a cluster point p. For each $n \in \mathbb{N}$, let $F_n = \overline{\{x_m : m \geq n\}}$. Assume that $\langle x_n \rangle$ has no cluster point. Then $\langle F_n \rangle \in \mathcal{F}_0(X)$. By (1), $p \notin \overline{g(j,F_j)}$ for some $j \in \mathbb{N}$. Since p is a cluster point of $\langle y_n \rangle$, there exists $i \geq j$ such that $y_i \notin \overline{g(j,F_j)} \supset g(i,F_i) \supset g(i,x_i)$, a contradiction.

Now, suppose that $x_n \in st^2(x, \mathcal{G}_n)$ for each $n \in \mathbb{N}$. Then there exist $y_n, z_n, w_n \in X$ such that $x \in g(n,z_n)$, $w_n \in g(n,y_n) \cap g(n,z_n)$ and $x_n \in g(n,y_n)$ for each $n \in \mathbb{N}$. By (2), $z_n \in g(n,x)$ and $z_n \in g(n,w_n)$ from which it follows that $g(n,x) \cap g(n,w_n) \neq \emptyset$ for all $n \in \mathbb{N}$. By Claim 2, $\langle w_n \rangle$ has a cluster point p. Then there is a subsequence $\langle w_{n_k} \rangle$ of $\langle w_n \rangle$ such that $w_{n_k} \in g(k,p)$ for all $k \in \mathbb{N}$. Since $w_{n_k} \in g(k, y_{n_k})$, we have that $g(k,p) \cap g(k,y_{n_k}) \neq \emptyset$ for all $k \in \mathbb{N}$. By Claim 2, $\langle y_{n_k} \rangle$ has a cluster point q which is also a cluster point of $\langle y_n \rangle$. Then there is a subsequence

$\langle y_{m_j}\rangle$ of $\langle y_n\rangle$ such that $y_{m_j} \in g(j,q)$ for all $j \in \mathbb{N}$. Since $x_n \in g(n,y_n)$ for each $n \in \mathbb{N}$, by (2), $y_{m_j} \in g(j,x_{m_j})$ for each $j \in \mathbb{N}$. It follows that $g(j,q) \cap g(j,x_{m_j}) \neq \emptyset$ for all $j \in \mathbb{N}$. By Claim 2, $\langle x_{m_j}\rangle$ has a cluster point which is also a cluster point of $\langle x_n\rangle$. Therefore, X is a wM-space. □

3. Main results

In this section, we present characterizations of some generalized countably compact spaces such as q-spaces, quasi-Nagata spaces, quasi-γ spaces, wN-spaces, $M^{\#}$-spaces and wM-spaces with real-valued functions. To shorten the expressions of the corresponding results, we introduce the following notations.

Let $\mathcal{A}$ be a family of subsets of X, $\mathcal{F}$ a family of real-valued functions on X and $f : \mathcal{A} \to \mathcal{F}$. For $A \in \mathcal{A}$, we write f_A instead of $f(A)$. For a singleton $\{x\}$, we write f_x instead of $f_{\{x\}}$. Consider the following conditions.

(c_A) $A \subset f_A^{-1}(0)$.

(m_A) If $A_1 \subset A_2$, then $f_{A_1} \geq f_{A_2}$.

($i_{A\langle F_n\rangle}$) For each $\langle F_n\rangle \in \mathcal{F}_0(X)$, there is $m \in \mathbb{N}$ such that $\inf\{f_A(x) : x \in F_m\} > 0$.

($i_{\langle F_n\rangle A}$) For each $\langle F_n\rangle \in \mathcal{F}_0(X)$, there is $m \in \mathbb{N}$ such that $\inf\{f_{F_m}(x) : x \in A\} > 0$.

Theorem 3.1 *X is a q-space if and only if for each $x \in X$, there exists $f_x \in U(X)$ satisfying ($c_{\{x\}}$) and ($i_{\{x\}\langle F_n\rangle}$).*

Proof Let g be the g-function in Proposition 2.2. For each $x \in X$, let

$$f_x = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{g(n,x)}.$$

Then $f_x \in U(X)$ and $f_x(x) = 0$.

Let $\langle F_n\rangle \in \mathcal{F}_0(X)$. By Proposition 2.2, there is $m \in \mathbb{N}$ such that $F_m \cap g(n,x) = \emptyset$ for all $n > m$. Thus for each $y \in F_m$,

$$f_x(y) = 1 - \sum_{n=1}^{m} \frac{1}{2^n} \chi_{g(n,x)}(y) \geq 1 - \sum_{n=1}^{m} \frac{1}{2^n} = \frac{1}{2^m}.$$

This implies that $\inf\{f_x(y) : y \in F_m\} > 0$.

Conversely, for each $x \in X$ and $n \in \mathbb{N}$, let $g(n,x) = \{y \in X : f_x(y) < \frac{1}{n}\}$. Then $g(n,x)$ is open, $x \in g(n,x)$ and $g(n+1,x) \subset g(n,x)$ which implies that g is a g-function for X. Let $\langle F_n\rangle \in \mathcal{F}_0(X)$ and $x \in X$. By ($i_{\{x\}\langle F_n\rangle}$), there exists $m \in \mathbb{N}$ such that $\inf\{f_x(y) : y \in F_m\} > 0$. Then there exists $k \geq m$ such that $f_x(y) > \frac{1}{k}$ for each $y \in F_m$. Thus for each $y \in F_k$, $f_x(y) > \frac{1}{k}$ which implies that $F_k \cap g(k,x) = \emptyset$. By Proposition 2.2, X is a q-space. □

Theorem 3.2 *X is a quasi-γ space if and only if for each $S \in \mathcal{S}_X$, there exists $f_S \in U(X)$ satisfying (c_S), (m_S) and ($i_{S\langle F_n\rangle}$).*

Proof Let g be the g-function in Proposition 2.3 (c). For each $S \in \mathcal{S}_X$, let

$$f_S = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{g(n,S)}.$$

Then $f_S \in U(X)$ satisfies (c_S) and (m_S).

Let $\langle F_n \rangle \in \mathcal{F}_0(X)$. By Proposition 2.3 (c), there is $m \in \mathbb{N}$ such that $F_m \cap g(n,S) = \emptyset$ for all $n > m$. Thus for each $x \in F_m$,

$$f_S(x) = 1 - \sum_{n=1}^{m} \frac{1}{2^n} \chi_{g(n,S)}(x) \geq 1 - \sum_{n=1}^{m} \frac{1}{2^n} = \frac{1}{2^m}.$$

This implies that $\inf\{f_S(x) : x \in F_m\} > 0$.

Conversely, define a g-function g for X by letting $g(n,x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. Let $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $S \in \mathcal{S}_X$. By $(i_{S\langle F_n \rangle})$, there exists $k \in \mathbb{N}$ such that $f_S(x) > \frac{1}{k}$ for each $x \in F_k$. Thus for each $y \in S$, $f_y(x) \geq f_S(x) > \frac{1}{k}$ which implies that $x \notin g(k,S)$. It follows that $F_k \cap g(k,S) = \emptyset$. By Proposition 2.3 (c), X is a quasi-γ space. $\square$

Theorem 3.3 *X is a β-space if and only if for each $F \in \tau^c$, there exists $f_F \in U(X)$ satisfying (c_F), (m_F) and $(i_{\langle F_n \rangle \{x\}})$.*

Proof Let g be the g-function in Proposition 2.4. For each $F \in \tau^c$, let

$$f_F = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{g(n,F)}.$$

Then $f_F \in U(X)$ satisfies (c_F) and (m_F).

Let $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$. By Proposition 2.4, there is $m \in \mathbb{N}$ such that $x \notin g(n, F_m)$ for all $n > m$. Thus,

$$f_{F_m}(x) = 1 - \sum_{n=1}^{m} \frac{1}{2^n} \chi_{g(n,F_m)}(x) \geq 1 - \sum_{n=1}^{m} \frac{1}{2^n} = \frac{1}{2^m} > 0.$$

Conversely, define a g-function g for X by letting $g(n,x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. Let $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$. By $(i_{\langle F_n \rangle \{x\}})$, there exist $m \in \mathbb{N}$ and $k \geq m$ such that $f_{F_m}(x) > \frac{1}{k}$. Thus for each $y \in F_k$, $f_y(x) \geq f_{F_k}(x) \geq f_{F_m}(x) > \frac{1}{k}$ which implies that $x \notin g(k, F_k)$. By Proposition 2.4, X is a β-space. $\square$

Theorem 3.4 *X is a quasi-Nagata space if and only if for each $F \in \tau^c$, there exists $f_F \in U(X)$ satisfying (c_F), (m_F) and $(i_{\langle F_n \rangle S})$ with $S \in \mathcal{S}_X$.*

Proof Let g be the g-function in Proposition 2.5 (c). For each $F \in \tau^c$, let

$$f_F = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{g(n,F)}.$$

Then $f_F \in U(X)$ satisfies (c_F) and (m_F).

Let $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $S \in \mathcal{S}_X$. By Proposition 2.5 (c), there is $m \in \mathbb{N}$ such that $S \cap$

$g(n, F_m) = \emptyset$ for all $n > m$. Thus for each $x \in S$,

$$f_{F_m}(x) = 1 - \sum_{n=1}^{m} \frac{1}{2^n} \chi_{g(n,F_m)}(x) \geq 1 - \sum_{n=1}^{m} \frac{1}{2^n} = \frac{1}{2^m}.$$

This implies that $\inf\{f_{F_m}(x) : x \in S\} > 0$.

Conversely, define a g-function g for X by letting $g(n, x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. Let $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $S \in \mathcal{S}_X$. By $(i_{\langle F_n \rangle S})$, there exist $m \in \mathbb{N}$ and $k \geq m$ such that $f_{F_m}(x) > \frac{1}{k}$ for each $x \in S$. Thus for each $y \in F_k$, $f_y(x) \geq f_{F_k}(x) \geq f_{F_m}(x) > \frac{1}{k}$ which implies that $x \notin g(k, F_k)$. It follows that $S \cap g(k, F_k) = \emptyset$. By Proposition 2.5 (c), X is a quasi-Nagata space. □

Theorem 3.5 *X is a $k\beta$-space if and only if for each $F \in \tau^c$, there exists $f_F \in U(X)$ satisfying (c_F), (m_F) and $(i_{\langle F_n \rangle K})$ with $K \in \mathcal{C}_X$.*

Proof Similar to the proof of Theorem 3.4 by using Proposition 2.6. □

For the definition of an MCP-space i.e., monotonically countably paracompact space [8].

Lemma 3.6 ([16]) *X is an MCP-space if and only if there exists a g-function g for X such that for each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $x \notin \overline{g(n, F_n)}$ for some $n \in \mathbb{N}$.*

In the following some theorems, the following notation is used: $(\star)$ For each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, there is an open neighborhood V of x and $m \in \mathbb{N}$ such that $\inf\{f_{F_m}(y) : y \in V\} > 0$.

Theorem 3.7 *For a space X, the following are equivalent.*

(a) X is an MCP-space;

(b) For each $F \in \tau^c$, there exist $h_F \in L(X)$ and $f_F \in U(X)$ with $h_F \leq f_F$ such that f_F satisfies (c_F), (m_F) and h_F satisfies $(i_{\langle F_n \rangle \{x\}})$;

(c) For each $F \in \tau^c$, there exists $f_F \in U(X)$ satisfying (c_F), (m_F) and $(\star)$.

Proof (a) ⇒ (b). Let g be the g-function in Lemma 3.6. For each $F \in \tau^c$, let

$$h_F = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{\overline{g(n,F)}}, \quad f_F = 1 - \sum_{n=1}^{\infty} \frac{1}{2^n} \chi_{g(n,F)}.$$

Then $h_F \in L(X)$, $f_F \in U(X)$ and $h_F \leq f_F$. It is clear that f_F satisfies (c_F) and (m_F). With a similar argument to the proof of the necessity of Theorem 3.3, one readily shows that h_F satisfies $(i_{\langle F_n \rangle \{x\}})$.

(b) ⇒ (c). Assume (b). Let $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$. Then there is $m \in \mathbb{N}$ and $r > 0$ such that $h_{F_m}(x) > r$. Let $V = \{x \in X : h_{F_m}(x) > r\}$. Then V is an open neighborhood of x. For each $y \in V$, $f_{F_m}(y) \geq h_{F_m}(y) > r$. This implies that $\inf\{f_{F_m}(y) : y \in V\} > 0$.

(c) ⇒ (a). Define a g-function g for X by letting $g(n, x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. Let $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$. By $(\star)$, there exist an open neighborhood V of x and $m \in \mathbb{N}$ and $k \geq m$ such that $f_{F_m}(y) > \frac{1}{k}$ for each $y \in V$. Thus for each $z \in F_k$, $f_z(y) \geq f_{F_k}(y) \geq f_{F_m}(y) > \frac{1}{k}$ which implies that $x \notin \overline{g(k, F_k)}$. By Lemma 3.6, X is an MCP-space. □

Lemma 3.8 ([16]) *X is a wN-space if and only if there exists a g-function g for X such that for each $\langle F_n\rangle \in \mathcal{F}_0(X)$ and $x \in X$, $g(n,x) \cap g(n,F_n) = \emptyset$ for some $n \in \mathbb{N}$.*

It is clear that the g-function in Lemma 3.8 satisfies both the condition in Proposition 2.2 and that in Lemma 3.6. Thus combining Theorems 3.1 and 3.7, we have the following.

Theorem 3.9 *For a space X, the following are equivalent.*

(a) X is a wN-space;

(b) For each $F \in \tau^c$, there exist $h_F \in L(X)$ and $f_F \in U(X)$ with $h_F \le f_F$ such that f_F satisfies (c_F), (m_F) and $(i_{\{x\}\langle F_n\rangle})$, h_F satisfies $(i_{\langle F_n\rangle\{x\}})$;

(c) For each $F \in \tau^c$, there exists $f_F \in U(X)$ satisfying (c_F), (m_F), $(i_{\{x\}\langle F_n\rangle})$ and $(\star)$.

Proof (a) $\Rightarrow$ (b). Let g be the g-function in Lemma 3.8. For each $F \in \tau^c$, define $h_F \in L(X)$, $f_F \in U(X)$ as that in the proof of (a) $\Rightarrow$ (b) of Theorem 3.7. Since g satisfies the conditions in Lemma 3.6 and Proposition 2.2, with similar arguments to the proofs of (a) $\Rightarrow$ (b) of Theorem 3.7 and the necessity of Theorem 3.1, one readily shows that h_F, f_F satisfy all the conditions.

(b) $\Rightarrow$ (c) is similar to the proof of (b) $\Rightarrow$ (c) of Theorem 3.7.

(c) $\Rightarrow$ (a). Assume (c). By Theorem 3.1, X is a q-space. By Theorem 3.7, X is an MCP-space. Therefore X is a wN-space [8]. $\square$

In the following two theorems, the following notation is used: (U) For each $x, y, z \in X$, $f_x(z) \le \max\{f_x(y), f_y(z)\}$.

Theorem 3.10 *For a space X, the following are equivalent.*

(a) X is an $M^{\#}$-space;

(b) For each $F \in \tau^c$, there exist $h_F \in L(X)$ and $f_F \in U(X)$ with $h_F \le f_F$ such that f_F satisfies (c_F), (m_F), (U) and $(i_{\{x\}\langle F_n\rangle})$, h_F satisfies $(i_{\langle F_n\rangle\{x\}})$;

(c) For each $F \in \tau^c$, there exists $f_F \in U(X)$ satisfying (c_F), (m_F), (U), $(i_{\{x\}\langle F_n\rangle})$ and $(\star)$.

Proof (a) $\Rightarrow$ (b). Let g be the g-function in Proposition 2.7 (c). For each $F \in \tau^c$, define $h_F \in L(X)$, $f_F \in U(X)$ as that in the proof of (a) $\Rightarrow$ (b) of Theorem 3.9. Then f_F satisfies (c_F), (m_F) and $(i_{\{x\}\langle F_n\rangle})$, h_F satisfies $(i_{\langle F_n\rangle\{x\}})$. That f_F satisfies (U) has been shown in [11]. We sketch the proof as follows.

For each pair x, y of distinct points of X, let $m_{xy} = \min\{n \in \mathbb{N} : y \notin g(n,x)\}$. Then $f_x(y) = \frac{1}{2^{m_{xy}-1}}$.

Let x, y, z be distinct points of X. Assume that $\max\{f_x(y), f_y(z)\} < f_x(z)$. Then $m_{xz} < m_{xy}$ and $m_{xz} < m_{yz}$. From $m_{xz} < m_{xy}$ it follows that $y \in g(m_{xz}, x)$ and thus $g(m_{xz}, y) \subset g(m_{xz}, x)$. From $m_{xz} < m_{yz}$ it follows that $z \in g(m_{xz}, y)$. Thus $z \in g(m_{xz}, x)$, a contradiction.

(b) $\Rightarrow$ (c) is similar to the proof of (b) $\Rightarrow$ (c) of Theorem 3.9.

(c) $\Rightarrow$ (a). Assume (c). Define a g-function g for X by letting $g(n,x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. Let $\langle F_n\rangle \in \mathcal{F}_0(X)$ and $x \in X$. Assume that $g(n,x) \cap g(n,F_n) \ne \emptyset$ for each $n \in \mathbb{N}$ and choose $x_n \in g(n,x) \cap g(n,F_n)$. By $(i_{\{x\}\langle F_n\rangle})$ and the proof of the sufficiency of Theorem 3.1, g is a q-function for X. Since $x_n \in g(n,x)$ for each $n \in \mathbb{N}$, $\langle x_n\rangle$ has a cluster point p. By $(\star)$ and the proof of (c) $\Rightarrow$ (a) of Theorem 3.7, $p \notin \overline{g(k, F_k)}$ for some $k \in \mathbb{N}$. Since p is a cluster point of $\langle x_n \rangle$, there exists $i \geq k$ such that $x_i \notin \overline{g(k, F_k)} \supset g(i, F_i)$, a contradiction. By (U), we have that if $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$. By Proposition 2.7 (c), X is an $M^{\#}$-space. □

Theorem 3.11 *X is a $\Sigma^{\#}$-space if and only if for each $F \in \tau^c$, there exists $f_F \in U(X)$ satisfying (c_F), (m_F), (U) and $(i_{\langle F_n \rangle \{x\}})$.*

Proof Let g be the g-function in Proposition 2.9. For each $F \in \tau^c$, define $f_F \in U(X)$ as that in the proof of the sufficiency of Theorem 3.3. Then f_F satisfies (c_F), (m_F) and $(i_{\langle F_n \rangle \{x\}})$. That f_F satisfies (U) has been shown in (a) $\Rightarrow$ (b) of Theorem 3.10.

Conversely, define a g-function g for X by letting $g(n, x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. By the proof of the sufficiency of Theorem 3.3, we have that for each $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$, $x \notin g(n, F_n)$ for some $n \in \mathbb{N}$. By (U), we have that if $y \in g(n, x)$, then $g(n, y) \subset g(n, x)$. By Proposition 2.9, X is a $\Sigma^{\#}$-space. □

In the following theorem, the following notation is used: (S) For each $x, y \in X$, $f_x(y) = f_y(x)$.

Theorem 3.12 *For a space X, the following are equivalent.*

(a) X is a wM-space;

(b) For each $F \in \tau^c$, there exist $h_F \in L(X)$ and $f_F \in U(X)$ with $h_F \leq f_F$ such that f_F satisfies (c_F), (m_F) and (S), h_F satisfies $(i_{\langle F_n \rangle \{x\}})$;

(c) For each $F \in \tau^c$, there exists $f_F \in U(X)$ satisfying (c_F), (m_F), (S) and $(\star)$.

Proof (a) $\Rightarrow$ (b). Let g be the g-function in Proposition 2.10 (d). For each $F \in \tau^c$, define $h_F \in L(X)$, $f_F \in U(X)$ as that in the proof of (a) $\Rightarrow$ (b) of Theorem 3.7. Then f_F satisfies (c_F) and (m_F), h_F satisfies $(i_{\langle F_n \rangle \{x\}})$. Let $x, y \in X$. Then for each $n \in \mathbb{N}$, $y \in g(n, x)$ if and only if $x \in g(n, y)$. It follows that $\chi_{g(n,x)}(y) = \chi_{g(n,y)}(x)$ for each $n \in \mathbb{N}$ and thus $f_x(y) = f_y(x)$.

(b) $\Rightarrow$ (c) is similar to the proof of (b) $\Rightarrow$ (c) of Theorem 3.7.

(c) $\Rightarrow$ (a). Assume (c). Define a g-function g for X by letting $g(n, x) = \{y \in X : f_x(y) < \frac{1}{n}\}$ for each $x \in X$ and $n \in \mathbb{N}$. Let $\langle F_n \rangle \in \mathcal{F}_0(X)$ and $x \in X$. By $(\star)$ and the proof of (c) $\Rightarrow$ (a) of Theorem 3.7, $p \notin \overline{g(k, F_k)}$ for some $k \in \mathbb{N}$. By (S), we have that $y \in g(n, x)$ if and only if $x \in g(n, y)$. By Proposition 2.10 (d), X is a wM-space. □

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