Journal of Mathematical Research with Applications
Nov., 2019, Vol. 39, No. 6, pp. 575–580
DOI:10.3770/j.issn:2095-2651.2019.06.004
Http://jmre.dlut.edu.cn

Some New Properties of Morgan-Voyce Polynomials

Yanni PEI, Yi WANG*
School of Mathematical Sciences, Dalian University of Technology, Liaoning 116024*, P. R. China*

Dedicated to the Memory of Professor L. C. HSU on the Occasion of His 100th Birthday

Abstract We show that zeros of Morgan-Voyce polynomials are dense in the closed interval $[-4, 0]$. We show also that coefficients of Morgan-Voyce polynomials are approximately normally distributed and that the coefficient arrays are totally positive matrices.

Keywords Morgan-Voyce polynomial; asymptotically normal distribution; totally positive matrix

MR(2010) Subject Classification 05A15; 26C10; 60F05; 15B99

1. Introduction

Morgan-Voyce polynomials $b_n(x)$ and $B_n(x)$, introduced by A. M. Morgan-Voyce in his study of electrical ladder networks of resistors, are defined by the recurrence relations

$$\begin{cases} b_n(x) = xB_{n-1}(x) + b_{n-1}(x); \\ B_n(x) = (x+1)B_{n-1}(x) + b_{n-1}(x), \end{cases}$$

for $n \geq 1$, with $b_0(x) = B_0(x) = 1$. Alternative recurrences are

$$\begin{cases} b_n(x) = (x+2)b_{n-1}(x) - b_{n-2}(x); \\ b_0(x) = 1, b_1(x) = x+1, \end{cases} \tag{1.1}$$

$$\begin{cases} B_n(x) = (x+2)B_{n-1}(x) - B_{n-2}(x); \\ B_0(x) = 1, B_1(x) = x+2. \end{cases} \tag{1.2}$$

Morgan-Voyce polynomials have many fascinating and interesting analytic properties [1, 2], as well as [3, Chapter 41] and references therein. The polynomials can be given explicitly by the sums

$$b_n(x) = \sum_{k=0}^{n} b(n,k)x^k, \quad b(n,k) = \binom{n+k}{n-k}, \tag{1.3}$$

$$B_n(x) = \sum_{k=0}^{n} B(n,k)x^k, \quad B(n,k) = \binom{n+k+1}{n-k}. \tag{1.4}$$

It is known that zeros of all Morgan-Voyce polynomials $b_n(x)$ (resp., $B_n(x)$) are real and in the open interval $(-4, 0)$. In the next section we show that these zeros are dense in the closed

Received August 24, 2019; Accepted October 12, 2019
Supported by the National Natural Science Foundation of China (Grant No. 11771065).
* Corresponding author
E-mail address: peiyanni@hotmail.com (Yanni PEI); wangyi@dlut.edu.cn (Yi WANG)

interval $[-4, 0]$. In Section 3, we show that coefficients $b(n, k)$ (resp., $B(n, k)$) of Morgan-Voyce polynomials, just like the binomial coefficients $\binom{n}{k}$, are approximately normally distributed. In Section 4, we show that the coefficient array $(b(n, k))_{n,k\geq 0}$ (resp., $(B(n, k))_{n,k\geq 0}$), just like the Pascal triangle $\left(\binom{n}{k}\right)_{n,k\geq 0}$, is a totally positive matrix. Finally in Section 5, we propose a couple of problems for further work.

2. Zeros of Morgan-Voyce polynomials

It is known [2] that $b_n(x)$ and $B_n(x)$ have only real zeros:

$$b_n(x) = \prod_{k=1}^{n}(x + r_{n,k}), \quad r_{n,k} = 4\sin^2\frac{(2k-1)\pi}{4n+2}, \tag{2.1}$$

$$B_n(x) = \prod_{k=1}^{n}(x + R_{n,k}), \quad R_{n,k} = 4\sin^2\frac{k\pi}{2n+2}. \tag{2.2}$$

Clearly, zeros of all $b_n(x)$ (resp., $B_n(x)$) are in the open interval $(-4, 0)$. In this section we show that these zeros turn out to be dense in the closed interval $[-4, 0]$.

Let $(f_n(x))_{n\geq 0}$ be a sequence of complex polynomials. We say that the complex number x is a limit of zeros of the sequence $(f_n(x))_{n\geq 0}$ if there is a sequence $(z_n)_{n\geq 0}$ such that $f_n(z_n) = 0$ and $z_n \to x$ as $n \to +\infty$. Suppose now that $(f_n(x))_{n\geq 0}$ is a sequence of polynomials satisfying the recursion $f_{n+k}(x) = -\sum_{j=1}^{k} c_j(x)f_{n+k-j}(x)$ where $c_j(x)$ are polynomials in x. Let $\lambda_j(x)$ be all roots of the associated characteristic equation $\lambda^k + \sum_{j=1}^{k} c_j(x)\lambda^{k-j} = 0$. It is well known that if $\lambda_j(x)$ are distinct, then

$$f_n(x) = \sum_{j=1}^{k} \alpha_j(x)\lambda_j^n(x), \tag{2.3}$$

where $\alpha_j(x)$ are determined from the initial conditions.

Lemma 2.1 ([4, Theorem]) *Under the non-degeneracy requirements that in (2.3) no $\alpha_j(x)$ is identically zero and that for no pair $i \neq j$ is $\lambda_i(x) \equiv \omega\lambda_j(x)$ for some $\omega \in \mathbb{C}$ of unit modulus, then x is a limit of zeros of $(f_n(x))_{n\geq 0}$ if and only if either*

(1) Two or more of the $\lambda_i(x)$ are of equal modulus, and strictly greater (in modulus) than the others; or

(2) For some j, $\lambda_j(x)$ has modulus strictly greater than all the other $\lambda_i(x)$ have, and $\alpha_j(x) = 0$.

Theorem 2.2 *Zeros of $(b_n(x))_{n\geq 0}$ (resp., $(B_n(x))_{n\geq 0}$) are dense in the closed interval $[-4, 0]$.*

Proof We prove the result only for $B_n(x)$ since the proof for $b_n(x)$ is similar. We present a stronger result: each $x \in [-4, 0]$ is a limit of zeros of the sequence $(B_n(x))_{n\geq 0}$.

By the recurrence relation (1.2) we may obtain the Binet form

$$B_n(x) = \frac{\lambda_1^{n+1} - \lambda_2^{n+1}}{\lambda_1 - \lambda_2}, \tag{2.4}$$

where $\lambda_{1,2}(x) = \frac{x+2\pm\sqrt{x^2+4x}}{2}$ are two roots of the characteristic equation $\lambda^2 - (x+2)\lambda + 1 = 0$.

The non-degeneracy conditions of Lemma 2.1 are clearly satisfied from (2.4). So the limits of zeros of $(B_n(x))_{n\geq 0}$ are those real numbers x for which $|\lambda_1(x)| = |\lambda_2(x)|$, i.e.,

$$|\frac{x+2+\sqrt{x^2+4x}}{2}| = |\frac{x+2-\sqrt{x^2+4x}}{2}|.$$

In other words, $\sqrt{x^2+4x}$ must be purely imaginary (allowing 0 to be purely imaginary). Thus $x^2+4x \leq 0$, i.e., $-4 \leq x \leq 0$, which is what we wanted to show. □

3. Asymptotic normality

Let $a(n,k)$ be a double-indexed sequence of nonnegative numbers and let $p(n,k) = \frac{a(n,k)}{\sum_{j=0}^{n} a(n,j)}$ denote the normalized probabilities. Following Bender [5], we say that the sequence $a(n,k)$ is asymptotically normal by a central limit theorem, if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \Big| \sum_{k\leq \mu_n + x\sigma_n} p(n,k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-t^2/2} \mathrm{d}t \Big| = 0, \tag{3.1}$$

where μ_n and σ_n^2 are the mean and variance of $a(n,k)$, respectively. We say that $a(n,k)$ is asymptotically normal by a local limit theorem on $\mathbb{R}$ if

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \big| \sigma_n p(n, \lfloor \mu_n + x\sigma_n \rfloor) - \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \big| = 0. \tag{3.2}$$

In this case,

$$a(n,k) \sim \frac{e^{-x^2/2} \sum_{j=0}^{n} a(n,j)}{\sigma_n \sqrt{2\pi}} \text{ as } n \to \infty, \tag{3.3}$$

where $k = \mu_n + x\sigma_n$ and $x = O(1)$. Clearly, the validity of (3.2) implies that of (3.1).

Many well-known combinatorial sequences enjoy central and local limit theorems. For example, the famous de Moivre-Laplace theorem states that the binomial coefficients $\binom{n}{k}$ are asymptotically normal (by central and local limit theorems). Other examples include the signless Stirling numbers $c(n,k)$ of the first kind, the Stirling numbers $S(n,k)$ of the second kind, and the Eulerian numbers $A(n,k)$. See [6] for an excellent survey about asymptotic normality of combinatorial sequences. A standard approach to demonstrating asymptotic normality is the following criterion (see [5, Theorem 2] for instance and [6, Example 3.4.2] for historical remarks).

Lemma 3.1 *Suppose that $A_n(x) = \sum_{k=0}^{n} a(n,k)x^k$ have only real zeros and $A_n(x) = \prod_{i=1}^{n}(x + r_i)$, where all $a(n,k)$ and r_i are nonnegative. Let*

$$\mu_n = \sum_{i=1}^{n} \frac{1}{1+r_i}, \quad \sigma_n^2 = \sum_{i=1}^{n} \frac{r_i}{(1+r_i)^2}.$$

Then if $\sigma_n^2 \to +\infty$, the numbers $a(n,k)$ are asymptotically normal (by central and local limit theorems) with the mean μ_n and variance σ_n^2.

Theorem 3.2 *The numbers $b(n,k) = \binom{n+k}{n-k}$ (resp., $B(n,k)$) are asymptotically normal (by central and local limit theorems) with the mean $\mu_n = n/\sqrt{5}$ and variance $\sigma_n^2 \sim (2n)/(5\sqrt{5})$.*

Proof We prove the result only for $B(n,k)$ since the proof for $b(n,k)$ is similar. By (2.2) we have

$$\mu_n = \sum_{k=1}^{n} \frac{1}{1+4\sin^2\frac{k\pi}{2n+2}} \to \frac{2n}{\pi}\int_0^{\pi/2} \frac{1}{1+4\sin^2\theta}\mathrm{d}\theta = \frac{n}{\sqrt{5}},$$

$$\sigma_n^2 = \sum_{k=1}^{n} \frac{4\sin^2\frac{k\pi}{2n+2}}{(1+4\sin^2\frac{k\pi}{2n+2})^2} \to \frac{2n}{\pi}\int_0^{\pi/2} \frac{4\sin^2\theta}{(1+4\sin^2\theta)^2}\mathrm{d}\theta = \frac{2n}{5\sqrt{5}}.$$

Thus the statement follows from Lemma 3.1. □

4. Total positivity

Following Karlin [7], a (finite or infinite) matrix is called totally positive (TP for short) if all its minors are nonnegative. Let $(a_n)_{n\geq 0}$ be an infinite sequence of nonnegative numbers (we identify a finite sequence $a_0, a_1, \ldots, a_n$ with the infinite sequence $a_0, a_1, \ldots, a_n, 0, 0, \ldots$). Define its Toeplitz matrix

$$[a_{i-j}] = \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ a_2 & a_1 & a_0 & \\ \vdots & & & \ddots \end{bmatrix}.$$

We say that the sequence is a Pólya frequency (PF for short) sequence if the corresponding Toeplitz matrix is TP. A fundamental characterization for PF sequences is due to Schoenberg and Edrei, which states that a sequence $(a_n)_{n\geq 0}$ is PF if and only if its generating function

$$\sum_{n\geq 0} a_n x^n = a x^k e^{\gamma x} \frac{\prod_{j\geq 0}(1+\alpha_j x)}{\prod_{j\geq 0}(1-\beta_j x)},$$

where $a > 0, k \in \mathbb{N}, \alpha_j, \beta_j, \gamma \geq 0$, and $\sum_{j\geq 0}(\alpha_j + \beta_j) < +\infty$ (see [7, p.412] for instance). In this case, we say also that the corresponding generating function is PF.

Let $d(x)$ and $h(x)$ be two formal power series. Denote by $R = (d(x), h(x))$ an infinite matrix whose generating function of the kth column is $h^k(x)d(x)$ for $k = 0, 1, 2, \ldots$. We say that R is a (proper) *Riordan array* when $d(0) = 1, h(0) = 0$ and $h'(0) \neq 0$. Riordan arrays play an important unifying role in enumerative combinatorics and many well-known combinatorial matrices are Riordan arrays [8]. For example, the Pascal triangle $P = \left[\binom{n}{k}\right]$ is a Riordan array and $P = \left(\frac{1}{1-x}, \frac{x}{1-x}\right)$.

Lemma 4.1 ([9]) *If both $d(x)$ and $h(x)$ are PF, then the Riordan array $R = (d(x), h(x))$ is TP.*

Now consider coefficient arrays for Morgan-Voyce polynomials

$$\mathfrak{b} = \left[\binom{n+k}{n-k}\right]_{n,k\geq 0} = \begin{pmatrix} 1 & & & & \\ 1 & 1 & & & \\ 1 & 3 & 1 & & \\ 1 & 6 & 5 & 1 & \\ \vdots & & & & \ddots \end{pmatrix},$$

$$\mathfrak{B} = \left[\binom{n+k+1}{n-k}\right]_{n,k\geq 0} = \begin{pmatrix} 1 & & & & \\ 2 & 1 & & & \\ 3 & 4 & 1 & & \\ 4 & 10 & 6 & 1 & \\ \vdots & & & & \ddots \end{pmatrix}.$$

We have the following result.

Theorem 4.2 *Both $\mathfrak{b}$ and $\mathfrak{B}$ are totally positive matrices.*

Proof The generating function of the kth column of $\mathfrak{b}$ is

$$\sum_{n\geq k}\binom{n+k}{n-k}x^n = x^k\sum_{m\geq 0}\binom{m+2k}{m}x^m = \frac{x^k}{(1-x)^{2k+1}}.$$

Hence $\mathfrak{b}$ is a Riordan array and $\mathfrak{b} = (\frac{1}{1-x}, \frac{x}{(1-x)^2})$.

Similarly, $\mathfrak{B}$ is the Riordan array $(\frac{1}{(1-x)^2}, \frac{x}{(1-x)^2})$. It immediately follows from Lemma 4.1 that both $\mathfrak{b}$ and $\mathfrak{B}$ are TP. □

Remark 4.3 *The columns of $\mathfrak{b}$ and $\mathfrak{B}$ correspond with odd and even columns of P, respectively.*

5. Remarks

Let $(a_n)_{n\leq 0}$ be a sequence of real numbers. We say that $(a_n(q))_{n\geq 0}$ is a log-convex sequence (LCX for short) if $a_{n-1}a_{n+1} \geq a_n^2$ for $n \geq 1$. We say that $(a_n)_{n\geq 0}$ is a Stieltjes moment sequence (SM for short) if the Hankel matrix $[a_{i+j}]_{i,j\geq 0}$ is TP. Clearly, SM implies LCX (see [10] for instance). It is known [11] that the central binomial coefficients $\binom{2n}{n}$ and the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$ form Stieltjes moment sequences, respectively.

Let $f(q)$ and $g(q)$ be two real polynomials in q. We say that $f(q)$ is *q-nonnegative* if all coefficients of $f(q)$ are nonnegative. Denote $f(q) \geq_q g(q)$ if $f(q) - g(q)$ is q-nonnegative. Let $A(q) = [a_{n,k}(q)]_{n,k\geq 0}$ be a matrix whose entries are all real polynomials in q. We say that $A(q)$ is a q-totally positive matrix (q-TP for short) if all minors are q-nonnegative. Let $(f_n(q))_{n\leq 0}$ be a sequence of real polynomials in q. We say that $(f_n(q))_{n\geq 0}$ is a q-log-convex sequence (q-LCX for short) if $f_{n-1}(q)f_{n+1}(q) \geq_q f_n^2(q)$ for $n \geq 1$. We say that $(f_n(q))_{n\geq 0}$ is a strongly q-log-convex sequence (q-SLCX for short) if $f_{m-1}(q)f_{n+1}(q) \geq_q f_m(q)f_n(q)$ for $n \geq m \geq 1$. By the definition, q-SLCX implies q-LCX. If the Hankel matrix $[f_{i+j}(q)]_{i,j\geq 0}$ is q-TP, then we say that $(f_n(q))_{n\geq 0}$ is a q-Stieltjes moment sequence (q-SM for short). It is known [11] that q-SM implies q-SLCX.

Clearly, Morgan-Voyce polynomials $b_n(q)$ form a q-LCX sequence since $b_{n-1}(q)b_{n+1}(q) - b_n^2(q) = q$. It is not difficult to show that $(b_n(q))_{n\geq 0}$ is also q-SLCX. Actually, Wang and Zhu [11] showed that $(b_n(q))_{n\geq 0}$ is q-SM. The q-central Delannoy numbers and the q-Schröder numbers are defined as

$$D_n(q) = \sum_{k=0}^{n}\binom{n+k}{n-k}\binom{2k}{k}q^{n-k},$$

$$r_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} \frac{1}{k+1} \binom{2k}{k} q^{n-k},$$

respectively. It is known [11] that both $(D_n(q))_{n\geq 0}$ and $(r_n(q))_{n\geq 0}$ are q-SM.

Let

$$z_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} x_k q^{n-k}, \quad n = 0, 1, 2, \ldots.$$

Suppose that $(x_k)_{k\geq 0}$ is LCX. Then $(z_n(q))_{n\geq 0}$ is q-LCX (see [12]) and q-SLCX (see [13]). Suppose that $(x_k)_{k\geq 0}$ is SM. Then $(z_n(q))_{n\geq 0}$ is SM for any fixed positive number q (see [11]). It is possible that $(z_n(q))_{n\geq 0}$ is q-SM.

Further, consider the linear transformation

$$z_n(q) = \sum_{k=0}^{n} \binom{n+k}{n-k} x_k(q), \quad n = 0, 1, 2, \ldots. \tag{5.1}$$

Liu and Wang [12] showed that (5.1) preserves the q-LCX property. Zhu and Sun [13] showed that (5.1) preserves the q-SLCX property. It is possible that (5.1) preserves the q-SM property.

On the contrary, the sequence $(B_n(q))_{n\geq 0}$ is not q-LCX since $B_{n-1}(q)B_{n+1}(q) - B_n^2(q) = -1$. This sequence possesses the so-called q-log-concavity and enjoys many interesting properties. We omit the details for brevity.

Acknowledgements The authors thank the anonymous referee for his/her helpful comments.

References

[1] M. N. S. SWAMY. *Properties of the polynomials defined by Morgan-Voyce.* Fibnoacci Quart., 1966, **4**: 73–81.

[2] M. N.S. SWAMY. *Further properties of Morgan-Voyce polynomials.* Fibnoacci Quart., 1968, **6**: 167–175.

[3] T. KOSHY. *Fibonacci and Lucas Numbers with Applications.* Wiley, New York, 2001.

[4] S. BERAHA, J. KAHANE, N. WEISS. *Limits of Zeros of Recursively Defined Families of Polynomials.* in G. Rota (Ed.), Studies in Foundations and Combinatorics. Academic Press, New York, 1978, 213–232.

[5] E. A. BENDER. *Central and local limit theorems applied to asymptotic enumeration.* J. Combinatorial Theory Ser. A, 1973, **15**: 91–111.

[6] E. R. CANFIELD. *Asymptotic normality in enumeration*, in *Handbook of Enumerative Combinatorics.* in Discrete Math. Appl. (Boca Raton), CRC Press, Boca Raton, FL, 2015, 255–280.

[7] S. KARLIN. *Total Positivity, Volume 1.* Stanford University Press, Stanford, 1968.

[8] L. W. SHAPIRO, S. GETU, W. -J. WOAN, et al. *The Riordan group.* Discrete Appl. Math., 1991, **34**: 229–239.

[9] Xi CHEN, Yi WANG. *Notes on the total positivity of Riordan arrays.* Linear Algebra Appl., 2019, **568**: 156–161.

[10] Huyile LIANG, Lili MU, Yi WANG. *Catalan-like numbers and Stieltjes moment sequences.* Discrete Math., 2016, **339**(2): 484–488.

[11] Yi WANG, Baoxuan ZHU. *Log-convex and Stieltjes moment sequences.* Adv. in Appl. Math., 2016, **81**: 115–127.

[12] L. L. LIU, Yi WANG. *On the log-convexity of combinatorial sequences.* Adv. in Appl. Math., 2007, **39**(4): 453–476.

[13] Baoxuan ZHU, Hua SUN. *Linear transformations preserving the strong q-log-convexity of polynomials.* Electron. J. Combin., 2015, **22**(3): Paper 3.26, 11 pp.