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Edge Partition of Graphs Embeddable in the Projective Plane and the Klein Bottle

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Dedicated to the Memory of Professor L. C. HSU on the Occasion of His 100th Birthday

Abstract In a previous paper by the author joint with Baogang XU published in *Discrete Math* in 2018, we show that every non-planar toroidal graph can be edge partitioned into a planar graph and an outerplanar graph. This edge partition then implies some results in thickness and outerthickness of toroidal graphs. In particular, if each planar graph has outerthickness at most 2 (conjectured by Chartrand, Geller and Hedetniemi in 1971 and the confirmation of the conjecture was announced by Gonçalves in 2005), then the outerthickness of toroidal graphs is at most 3 which is the best possible due to K_7.

In this paper we continue to study the edge partition for projective planar graphs and Klein bottle embeddable graphs. We show that (1) every non-planar but projective planar graph can be edge partitioned into a planar graph and a union of caterpillar trees; and (2) every non-planar Klein bottle embeddable graph can be edge partitioned into a planar graph and a subgraph of two vertex amalgamation of a caterpillar tree with a cycle with pendant edges. As consequences, the thinkness of projective planar graphs and Klein bottle embeddabe graphs are at most 2, which are the best possible, and the outerthickness of these graphs are at most 3.

Keywords surface; planar graph; edge partition; thickness; outerthickness; caterpillar tree; projective plane; Klein bottle

MR(2010) Subject Classification 05C10; 05C35

1. Introduction

Much work has been done in partitioning the edges of graphs such that each subset induces a subgraph of a certain type. A well-known result by Nash-Williams [1] gives a necessary and sufficient condition for a graph to admit an edge-partition into a fixed number of forests. An outerplanar graph is a planar graph that can be embedded in the plane without crossing edges, in such a way that all the vertices are incident with the same face (the infinite face). The thickness of a graph G, denoted by $\theta(G)$ (first defined by Tutte [2]), is the minimum number of planar subgraphs whose union is G. Similarly, the outerthickness $\theta_o(G)$ is obtained when "planar subgraphs" is replaced by "outerplanar subgraphs" in the previous definition. If Σ is a surface, define $\theta(\Sigma) = \max\{\theta(G) : G \text{ is embeddable in } \Sigma\}$, where the maximum is taken over all graphs embeddable in Σ. Define $\theta_o(\Sigma)$ analogously. Nash-Williams' result in fact shows that

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any planar graph can be edge-partitioned into three forests, and hence the outerthickness of any planar graph is at most 3.

The thickness of some special classes of graphs has been determined, including the complete graphs K_n (see [3, 4]), the complete bipartite graphs $K_{m,n}$ (see [5]) (except possibly if m and n are both odd, or $m \leq n$ and n takes some special values), and the hypercube Q_n (see [6]). See the survey paper [7] for more results on the thickness of graphs. Guy and Nowakowski [8, 9] determined the outerthickness of complete graphs, the hypercube and some complete bipartite graphs. It is known that the thickness problem is $\mathcal{NP}$-hard [10]. For many other classes of graphs, attention has been focused on finding upper bounds of thickness and outerthickness. Jünger et al. [11] have shown that a graph has thickness at most 2 if it contains no K_5-minor. Asano [12] proved that, if a graph G is triangle free and has orientable genus γ, then $\theta(G) \leq \gamma(G) + 1$. He also showed that all toroidal graphs have thickness at most 2. Dean and Hutchinson [13] strengthened Asano's result by proving that $\theta(G) \leq 6 + \sqrt{2\gamma(G) - 1}$. Xu and Zha [14] slightly improved Dean and Hutchinson's result to $\theta(G) \leq 3 + \sqrt{2\gamma(G) - 1}$ by introducing the technique of removing a maximal spanning disk from the embedding. Xu and Zha also improved Asano's result by dropping the triangle free condition, i.e., $\theta(G) \leq \gamma(G) + 1$ holds for all graphs with orientable genus γ. While this upper bound is in general weaker than Dean/ Hutchinson and Xu/Zha's upper bounds with the square root, it works better for graphs with small genus. Xu and Zha also obtained upper bounds for both thickness and outerthickness in terms of their nonorientable genus in [14].

In 1971, Chartrand, Geller and Hedetniemi [15] conjectured that every planar graph has an edge partition into two outerplanar graphs. Ding, Oporowski, Sanders and Vertigan [16] proved that every planar graph has an edge partition into two outerplanar graphs and a vee-forest, where a vee-forest is the disjoint union of a number of K_2's and $K_{1,2}$'s. They also showed that every graph with nonnegative Euler characteristic has an edge partition into two graphs of tree-width at most three. Kedlaya [17] showed that some planar graphs cannot be edge-partitioned into two outerplanar subgraphs such that one of them is outerplanarly embedded. In 2005 Gonçalves [18] announced that he had solved the Chartrand, Geller and Hedetniemi's conjecture.

In this paper, we first provide some technical results in Section 2. We then study the edge partition, thickness and outerthickness problems for graphs embedded in the projective plane and the Klein bottle in Section 3 and Section 4, respectively. We show that (i) Every non-planar but projective planar graph can be edge partitioned into a planar graph and a union of caterpillar trees; (ii) Every graph embedded in the Klein bottle can be edge partitioned into a planar graph and a subgraph of two vertex amalgamation of a caterpillar tree and a cycle with pendant edges; (iii) The thickness of all non-planar graphs embeddable in the projective plane and the Klein bottle is 2 (iv) If each planar graph has outerthickness at most 2 (announced by Goncalves [18]), then all projective plane or Klein bottle embeddable graphs have outerthickness at most 3.

2. Notation and technical results

We state and prove some technical results in this section. The following is obvious.

Lemma 2.1 *If G is a subgraph of H, then*

(i) Each edge partition of H induces an edge partition of G;

(ii) $\theta(G) \leq \theta(H)$ and $\theta_o(G) \leq \theta_o(H)$.

Lemma 2.1 may not be true if the subgraph relation is replaced by the minor relation. Example 2.2 shows that edge partition of G can be much more complicated than H when G is a minor of H. Example 2.2 also provides cases that $\theta(G) > \theta(H)$ and $\theta_o(G) > \theta_o(H)$. This may add difficulty to problems of finding edge partition problem, as well as the thickness and outerthickness problem, since many techniques and results on the minor relation may not be applied here.

Example 2.2 Let G be the complete graph K_n and H be the graph obtained by subdividing each edge of G with two degree 2 vertices. Let H_1 be the subgraph of H consisting of all matching edges each with two degree 2 vertices of H as endvertices, and H_2 be the subgraph of H with remaining edges, i.e., H_2 consists of all edges each with one of original vertex of K_n as one endvertex and the other endvertex is a degree 2 vertex. H_2 consists of n copies of disjoint stars and H_1 consists of $\binom{n}{2}$ matching edges of H. Hence H can be edge partitioned into two forests (one is a disjoint union of stars, and the other is a matching).

In Example 2.2, G is a minor of H. While H can be edge partitioned into two forests, G cannot be edge partitioned into less than $\lfloor \frac{n+7}{6} \rfloor$ planar graphs since $\theta(K_n) = \lfloor \frac{n+7}{6} \rfloor$ (see [3, 4]).

For many edge partition problems, such as partitioning into simpler subgraphs as discussed in this paper, adding multiple edges to a graph G does not increase the complexity of edge partition of G. Also the edge partition for a graph with a cut vertex can be reduced to the edge partition of its components of separation obtained by the cut vertex. It is clear that the thickness/outerthickness of a graph is equal to the maximum thickness/outerthickness of its blocks. Therefore in this paper we may assume that all graphs considered are simple and 2-connected unless specified for some technical reasons.

Let G be a graph and $\Psi(G)$ be an embedding of G in a surface Σ. Suppose C is a cycle of G, and x and y are two vertices on C. We assign a direction to C and define (xCy) to be the open path from x to y in this direction (x and y are not included), and $[xCy]$ for the path from x to y in this direction with end vertices included. Therefore, $[xCy] \cup [yCx] = C$. A subembedding Ψ^s is *spanning* if it contains all vertices of G. A spanning subembedding is *contractible* if it does not contain any noncontractible cycle of $\Psi(G)$. In particular a contractible spanning subembedding is a spanning disk if it is homeomorphic to a closed disk, in which case the boundary of this spanning subembedding is a contractible cycle of $\Psi(G)$. For any embedding, a spanning tree is always a contractible spanning subembedding. However, an embedding may not contain a spanning disk. An example is the unique embedding of the Heawood graph in the torus which is the dual embedding of K_7. It contains no spanning disk even though the embedding is 3-representative (or equivalently, a polyhedral embedding, or a wheel-neighborhood embedding). An edge e is *essential*, with respect to a contractible spanning subembedding Ψ^s if $e \cup \Psi^s$

contains a noncontractible cycle. Note that if e is an essential edge, then e is contained in every noncontractible cycle of $e \cup \Psi^s$, and all noncontratible cycles contained in $e \cup \Psi^s$ are homotopic since Ψ^s is contractible. An essential edge becomes a noncontractible loop if Ψ^s is contracted to a single point. We define homotopy classes of essential edges according to the corresponding homotopy classes of loops of these essential edges obtained by contracting Ψ^s to a single point.

In order to study edge partition and thickness/outerthickness of graphs embedded in surfaces, it is sometimes more convenient to add more edges to the graphs in the given embeddings. We apply Lemma 2.1 by adding edges to the embedding of G to obtain an embedded supergraph H of G with nice spanning subgraphs, then study the edge partition and thickness/outerthickness of H. In this way we obtain a better structure of embeddings. The following lemma in [14] will play an important role in our approach in this paper.

Lemma 2.3 *Let G be a simple graph and $\Psi(G)$ be an orientable genus embedding or a minimal surface embedding (i.e., with maximum Euler characteristic) of G in Σ. Then either $\Psi(G)$ contains a spanning disk, or there is a supergraph H with embedding $\Psi(H)$ in Σ such that H is simple, $V(H) = V(G)$, $\Psi(G)$ is a subembedding of $\Psi(H)$, and $\Psi(H)$ contains a spanning disk.*

The conclusion of Lemma 2.3 may not be true if the embedding is not an orientable genus embedding or a minimal surface embedding. For example, if G is a complete graph embedded in its orientable maximum surface Σ, then there does not exist an H embedded in Σ with G being a spanning subgraph of H such that the embedding of H contains a spanning disk. Lemma 2.3 includes the minimal surface embeddings as part of the assumption because, in the proof, we may need to cut a nonorientable surface open and cap off with disk(s) to obtain a surface with larger Euler characteristic. We cannot determine whether the resulting surface is orientable or nonorientable.

Let $\Psi(G)$ be an embedding of a graph G in a surface Σ. We now allow G to have multiple edges. If so then any two multiple edges form a noncontractible cycle (this is to prevent two multiple edges from forming a face of size 2). Suppose $\Psi(G)$ has a maximal spanning disk D, where maximal means it contains as many faces as possible. Denote the subgraph embedded in D by $D(G)$ (including all edges on the boundary of D). Then $\Psi(G) \setminus D(G)$ consists of essential edges only, which is called the subembedding of essential edges, and is denoted by G^e. We also use G^e to represent the subgraph consisting of all essential edges.

3. Edge partitions of projective planar graphs

In this section we study the edge partitions of graphs embedded in the Möbius band and the projective plane whose Euler characteristic is 1.

We first study embeddings in the Möbius band, which can be considered as the projective plane with the interior of a disk removed. Let Ψ be an embedding of a graph G in the Möbius band and e be an edge of Ψ. The edge e is called noncontractible if the resulting surface obtained by contracting e to a point is not homeomorphic to Möbius band any more. The edge e is nonseparating if e does not separate the Möbius band into two connected components. We have the following theorem

Theorem 3.1 *Let Ψ be an embedding of a graph G in the Möbius band M. Let the boundary of M be the simple closed curve B. If every edge of Ψ is noncontractible and nonseparating, then*

(i) All vertices of G are on the boundary B;

(ii) For any pair of edges $e_1 = xy$ and $e_2 = uv$ with x, y, u and v all being distinct, x and y are separated by u and v on B;

(iii) G is either a single cycle with pendant edges on some vertices, or a disjoint union of caterpillar trees.

Proof Let $e = xy$ be an edge. If one of x and y is an interior point of the Möbius band M, then clearly, after contracting e, the resulting surface is still homeomormhic to M. This implies that e is contractible, a contradiction. Therefore, all vertices are on B, the boundary of M, and (i) is true.

If we cap off the Möbius band with a disk D along the boundary B, the resulting surface is the projective plane. For each edge $e = xy$ we add an artificial edge e' which is a straight line segment contained in D joining x and y. Then $e \cup e'$ is a nonseparating loop in the projective plane. All nonseparating loops in the projective plane are noncontractible and they are all homotopic. Each pair of these loops intersect once (homotopically can be viewed as in the center of D) which implies that, homotopically, x and y are antipodal points of D. Hence (ii) is true.

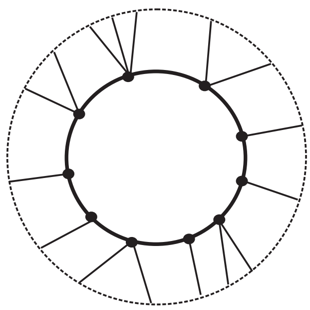

Figure 1 Graph embedded in the Möbius band

By (i), all vertices of G are embedded on the boundary B. We assume that $v_1, v_2, \ldots, v_n$ are vertices on B in this clockwise order. All edges of G can be viewed as edges in the projective plane as an essential edges with respect to the spanning disk D. There is only one homotopic class of noncontractible simple closed curve in the projective plane. This implies that, if a vertex v_i has degree ≥ 2, then all vertices of G incident to v_i are consecutively on a section b of B. Assume the first vertex on b incident to v_i is v_{i_1} and the last vertex on b incident to v_i is v_{i_k}. Then $v_iv_{i_1}, v_iv_{i_k}$ together with b, form a contractible triangle region. Therefore all vertices of G on b between v_{i_1} and v_{i_k} are of degree 1, i.e., each of these vertices is the end vertex of a pendant edge (see Figure 1).

Delete all pendant edges of G (do not iterate this procedure, just once) to obtain a subgraph G'. The vertices in G' have degree at most 2. If all vertices have degree 2, then G is a cycle, otherwise G is a path or disjoint union of paths. Adding back all pendant edges, we obtain (iii). This completes the proof. $\Box$

The following is the main theorem on edge partition for graphs embedded in the projective plane.

Theorem 3.2 *Every non-planar but projective planar graph can be edge partitioned into a planar graph and a union of caterpillar trees.*

Proof The fundamental group of the projective plane is $\mathcal{Z}_2$, and there is only one homotopic class of non-contractile simple closed curves, i.e., all non-contractible simple closed curves in the projective plane are homotopic. All embeddings of non-planar graphs in the projective plane are minimal surface embeddings. Let $\Psi(G)$ be an embedding of a non-planar simple graph G embedded in the projective plane. By Lemma 2.3, either $\Psi(G)$ contains a spanning disk, or there is a supergraph H with embedding $\Psi(H)$ in the projective plane such that H is simple, $V(H) = V(G)$, $\Psi(G)$ is a subembedding of $\Psi(H)$, and $\Psi(H)$ contains a spanning disk. Let D be such a spanning disk with maximal number of faces. Let D_G be the graph induced by $V(G)$ with all edges contained in D. Then there does not exist any edge e with both endvertices on the boundary of D such that $D_G \cup e$ are contractible. Therefore, all edges not contained in D_G are essential edges. Let G^e be the subembedding of essential edges. Choose any vertex u of degree at least 2 (u is not a degree 1 vertex as the endvertex of a pendant edge). Let $\mathrm{St}(u)$ be the subgraph of G^e consisting of all essential edges incident to u, and let $G_1 = D \cup \mathrm{St}(u)$. Then G_1 is planar because, if necessary, we can connect all essential edges incident to u to the boundary of D by permuting the clockwise order of essential edges incident to u.

By Theorem 3.1, G^e is either a single cycle with pendant edges on some vertices, or a disjoint union of caterpillar trees with some pendant edges on some vertices. Let $G_2 = G^e \backslash \mathrm{St}(u)$. Then G_2 is a union of caterpillar trees. This completes the proof. $\Box$

Corollary 3.3 *(i) The thickness of all non-planar but projective planar graphs is* 2*;*

(ii) If each planar graph has outerthickness at most 2 (announced by Gonçalves [18]), then all projective planar graphs have outerthickness at most 3.

4. Edge partitions for graphs embedded in the Klein bottle

In this section we study the edge partitions of graphs embedded in the Klein bottle whose Euler characteristic is 0. We will prove an edge partition result for embeddings in the Klein bottle similar to that we obtained for the embeddings in the torus in [14].

There are four types of noncontractible simple closed curves in the Klein bottle [19]. We can view the Klein bottle as the direct sum of two projective planes, or as twisted version of torus (cutting the torus open and then identify the two boundary circles reversely). These four types of curves are illustrated in Figures 2 and 3, respectively. Figure 2 is the direct sum model and Figure 3 is the twisted torus model. The twisted torus model is widely mentioned in the textbooks, and the directed sum model provides more combinatorial view which may help to understand the nature of the problem discussed in this paper. Each projective plane can be viewed as a crosscap, which is obtained by cutting off a disk and then identifying the antipodal points.

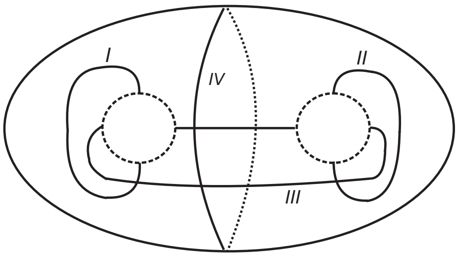

Figure 2 Four nonhomotopic simple closed curves in the Klein bottle - two cross-caps model

The four types of noncontractible simple closed curves are listed as follows with the directed sum model.

Type I: noncontractible and orientation reversing (called 1-sided curve) that crosses the first crosscap;

Type II: noncontractible and orientation reversing that crosses the second crosscap;

Type III: noncontractble, orientation preserving (called 2-sided curve), and nonseparating which crosses both crosscaps exactly once;

Type IV: noncontractible, orientation preserving, and separating which separates the Klein bottle into two M¨bius band (this curve becomes the boundary for the Möbius band, or the boundary of the hole of the projective plane which is used for the disk sum).

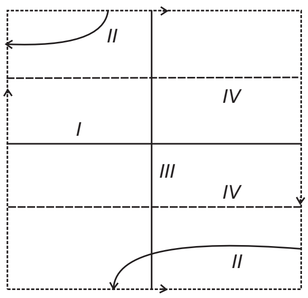

Figure 3 Four nonhomotopic simple closed curves in the Klein bottle - twisted torus model

Now suppose the graph G is embedded in the Klein bottle. By Lemma 2.3 we may assume that the embedding contains a (maximal) spanning disk D. Each essential edge corresponds to a loop obtained by contracting the spanning disk into a point. Two loops are either homotopically disjoint or cross transversely exactly once. Each essential edge has two endvertices on the boundary of the spanning disk D. Let e_i and e_j be two essential edges with $e_i = u_iv_i$ and $e_j = u_jv_j$, and u_i, v_i, u_j, v_j are all distinct. The two edges e_i and e_j are called crossing if the endvertices of these two essential edges are in this clockwise order u_i, u_j, v_i, v_j on the boundary of D. If e_i and e_j have common endvertex, then we may view these two edges as disjoint by homotopically moving e_i (or e_j) to make the endvertices distinct. If essential edges e_i and e_j are crossing, then both $G_D \cup e_i$ and $G_D \cup e_j$ are planar, but $G_D \cup e_i \cup e_j$ is not planar. If e_i and e_j are not crossing or e_i and e_j have a common endvertex, then $G_D \cup e_i \cup e_j$ is still planar. The following lemma explains whether two essential edges are crossing or not.

Lemma 4.1 *(i) Two Type I essential edges are crossing, and similarly two Type II essential edges are crossing;*

(ii) Two Type III or two Type IV essential edges are not crossing, respectively. Each pair bounds a cylinder;

(iii) Type I essential edges and Type II essential edges are not crossing;

(iv) Both Type I and Type II essential edges are crossing with Type III essential edges;

(v) Both Type I and Type II essential edges are not crossing with Type IV essential edges;

(vi) For a given embedding in the Klein bottle with a spanning disk D, Type III and Type IV essential edges do not exist simultaneously. In fact, this is true for embeddings in any surfaces with a spanning disk.

Proof Two homotopic orientation preserving simple closed curves are topologically disjoint and they bound a cylinder. On the other hand, two homotopic orientation reversing simple closed curves are crossing transversely and is the boundary of a disk with a self-attached pinch point. This can be viewed locally as two homotopic simple closed curves on a projective plane. Therefore, (i) and (ii) are true. The Klein bottle is the direct sum of two projective planes and Type I and Type II essential edges belong to different projective planes and therefore (iii) is true. A Type III simple closed curve passes each of two crosscaps exactly once, hence it crosses each of Type I loops and Type II loops exactly once. This implies (iv). Type IV loops separates two crosscaps and therefore it is homotopically disjoint from Type I and Type II curves, and hence (v) is true. Type IV loops are separating curves and therefore if any loop crosses a Type IV loop then it must cross with an even number times. But any two loops obtained by contracting the disk D are either homotopically disjoint or intersecting transversely exactly once. Therefore, (vi) is true. □

We now have our main theorem for embeddings in the Klein bottle.

Theorem 4.2 *Every graph embedded in the Klein bottle can be edge partitioned into a planar graph and a subgraph of two vertex amalgamation of a caterpillar tree and a cycle with pendant edges.*

Proof By Lemma 4.1 (iv), Type III and Type IV essential edges do not exist simultaneously. We divide our discussion into two cases according to whether Type III or Type IV essential edges exist. If none of Type III or Type IV essential edges exist, the proof will be the same as the case with Type IV essential edges since we may artificially add a Type IV essential edge to the embedding.

Case I. There exist Type IV essential edges (noncontractible, $e \cup G_D$ separates the Klein bottle with any Type IV edge e).

Let $e = xy$ be a Type IV essential edge. Since a Type IV curve separates the Klein bottle into two Möbius bands, say the left Möbius band and the right Möbius band, $D \cup xy$ separates G^e into two subgraphs, G^e_l and G^e_r, which are the essential edges on the left Möbius band and the essential edges on the right Möbius band, respectively. Let $C = \partial D$ be the boundary of D. Then $C = [xCy] \cup [yCx]$. All Type IV essential edges are homotopic to each other and their endvertices are contained consecutively in two disjoint sections of C, one contains x and the other contains y. All Type I essential edges are contained in G^e_l, and their endvertices are contained consecutively in $[yCx]$, and all Type II essential edges are contained in G^e_r, and their endvertices are contained consecutively in $[xCy]$. The only possible common endvertices of essential edges in G^e_l and G^e_r are x and y (in the case that x and/or y is also the endvertex of Type I and Type II essential edges).

Let $A_{IV} = St(x) \cup \{\text{all Type IV essential edges}\}$. If x is incident to both Type I and Type II essential edges, let $G_1 = G_D \cup A_{IV}$; If x is also incident to a Type I essential edge but not any of Type II essential edges, choose a vertex u_2 in (xCy) such that u_2 is an endvertex of a Type II essential edge and let $G_1 = G_D \cup A_{IV} \cup \mathrm{St}(u_2)$; If x is also incident to a Type II essential edge but not any of Type I essential edges, choose a vertex u_1 in (yCx) such that u_1 is an endvertex of an Type I essential edge and let $G_1 = G_D \cup A_{IV} \cup \mathrm{St}(u_1)$; If x is not incident to any of Type I and Type II essential edges, choose a vertex u_1 in (yCx) such that u_1 is an endvertex of a Type I essential edge and choose a vertex u_2 in (xCy) such that u_2 is an endvertex of a Type II essential edge, and let $G_1 = G_D \cup A_{IV} \cup \mathrm{St}(u_1) \cup \mathrm{St}(u_2)$;

By Lemma 4.1 (iii) and (v), G_1 is planar (see Figure 4 for the last case).

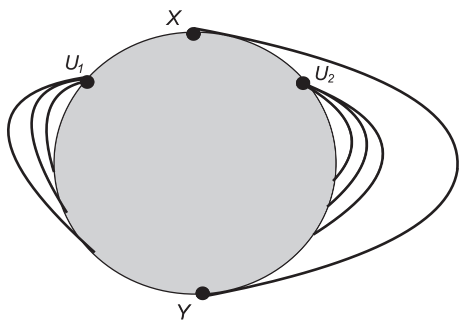

Figure 4 Embedding with Type IV essential edges

Let $G_2 = G \backslash G_1, G_{2l} = G^e_l \backslash G_1$, and $G_{2r} = G^e_r \backslash G_1$. Then $G_2 = G_{2l} \cup G_{2r}$, and G_{2l} and G_{2r} can only possibly have y as a common vertex (because if G^e_l and G^e_r also have x as a common vertex then $St(x)$ is contained in G_1). In all four sub-cases above, G_{2l} contains only Type I edges and G_{2r} contains only Type II edges. Both G_{2l} and G_{2r}, together with the regions they are embedded in, are homeomorphic to a Möbius band with a vertex removed. By Lemma 3.1 (iii), G_{2l} and G_{2r}

each is a union of caterpillar tree (since a vertex is removed from the boundary of each Möbius band, respectively). Therefore, G_2 $(= G_{2l} \cup G_{2_r})$ is a union of caterpillar tree because G_{2l} and G_{2r} can only possibly be attached at y. The theorem is true in this case when Type IV essential edges exist

Case II. There exist Type III essential edges (noncontractible, nonseparating, and orientation preserving).

Let $e = xy$ be a Type III essential edge. Therefore, by Lemma 4.1 (vi) there does not exist any Type IV essential edge. We may assume both Type I and Type II essential edges exist, or the edge partition problem will be simpler. The vertices x and y separate C into two sections: $[xCy]$ and $[yCx]$. By Lemma 4.1 (iii)–(v), all endvertices of Type I essential edges and all endvertices of Type II essential edges are contained separately on two different sections of C, with possible two common vertices. For each Type I essential edge $e = uv$, one of u and v is contained in $[xCy]$ and the other is contained in $[yCx]$, i.e., two endvertices of each Type I essential edge are separated by x and y on C. Similarly, two endvertices of each Type II essential edge are also separated by x and y on C. Half endvertices (counting with multiplicity) of Type I essential edges and half endvertices of Type II essential edges are contained in $[xCy]$, on two different sections. The other half endvertices of Type I essential edges and the other half endvertices of Type II essential edges are contained in $[yCx]$, on two different sections. The relative positions of these endvertices are illustrated in Figure 5.

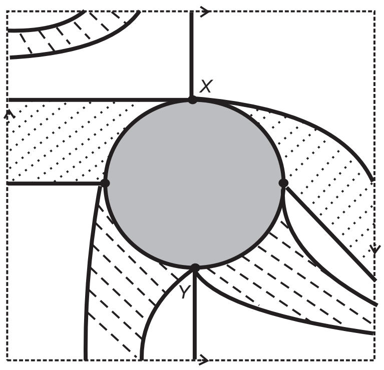

Figure 5 Embedding with Type III essential edges

If there are two disjoint Type III edges, say x_1y_1 and x_2y_2, then x_1y_1 and x_2y_2, together with G_D, form a cylinder H. We may assume x_1y_1 and x_2y_2 are Type III edges so that outside of the cylinder contains no other Type III edges. Let G_1 be the subgraph of G contained in the cylinder H (including boundary edges). Then G_1 is planar. Let $G_2 = G \backslash G_1, G_{2l} = G_l^e \backslash G_1$, and $G_{2r} = G_r^e \backslash G_1$. G_{2l} is the subgraph of G^e induced by all Type I essential edges, and G_{2r} is the subgraph of G^e induced by all Type II essential edges. The subgraph G_{2l} and G_{2r} can have at most two common vertices, one in the middle of section (x_2Cy_2) and the other in the middle of section (y_1Cx_1). G_{2l} is a graph embedded in the Möbius band with the section on the boundary from x_1 to x_2 being removed, and G_{2r} is a graph embedded in the other Möbius band with the section on the boundary from y_2 to y_1 being removed. By Theorem 3.1 both G_{2l} and G_{2r} are caterpillar trees or disjoint union of caterpillar trees. Since G_{2l} and G_{2r} have at most two common vertices, G_2 is isomorphic to a subgraph of two vertex amalgamation of two caterpillar trees. The theorem is true in this case.

If there are more than one Type III edges all with a common vertex, say x, let $G_1 = G_D \cup St(x)$. Then G_1 is planar. Let $G_2 = G \backslash G_1, G_{2l} = G_l^e \backslash G_1$, and $G_{2r} = G_r^e \backslash G_1$. Then G_{2l} is a graph embedded in the Möbius band with the vertex x being removed, and G_{2r} is a graph embedded in the other Möbius band with the section on the boundary being removed. By Theorem 3.1 both G_{2l} and G_{2r} are caterpillar tree or disjoint union of caterpillar trees. Since G_{2l} and G_{2r} have at most two common vertices, G_2 is isomorphic to a subgraph of two vertex amalgamation of two caterpillar trees. The theorem is true in this case.

Now we assume that xy is the only Type III edge. let $G_1 = G_D \cup St(x)$. Then G_1 is planar. Let $G_2 = G \backslash G_1, G_{2l} = G_l^e \backslash G_1$, and $G_{2r} = G_r^e \backslash G_1$. Then G_{2l} is a graph embedded in the Möbius band with the vertex x being removed, and G_{2r} is a graph embedded in the other Möbius band. By Theorem 3.1, G_{2l} is a caterpillar tree or disjoint union of caterpillar trees, and G_{2r} is a a cycle with pendant edges, a caterpillar tree, or disjoint union of caterpillar trees. Hence G_2 is isomorphic to a subgraph of two vertex amalgamation of a caterpillar tree with a cycle with pendant edges. The theorem is also true in this case. This completes the proof. □

Corollary 4.3 *(i) The thickness of all graphs embeddable in the Klein bottle but not planar is* 2*;*

(ii) If each planar graph has outerthickness at most 2 (announced by Gonçalves [18]), then all Klein bottle embeddable graphs have outerthickness at most 3.

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