Journal of Mathematical Research with Applications
Nov., 2019, Vol. 39, No. 6, pp. 593–602
DOI:10.3770/j.issn:2095-2651.2019.06.006
Http://jmre.dlut.edu.cn

Some Properties of a Class of Refined Eulerian Polynomials

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Dedicated to the Memory of Professor L. C. HSU on the Occasion of His 100th Birthday

Abstract Recently, Sun defined a new kind of refined Eulerian polynomials, namely,

$$A_n(p,q)=\sum_{\pi\in\mathfrak{S}_n} p^{\mathrm{odes}(\pi)}q^{\mathrm{edes}(\pi)}$$

for $n\geq 1$, where $\mathfrak{S}_n$ is the set of all permutations on $\{1,2,\ldots,n\}$, $\mathrm{odes}(\pi)$ and $\mathrm{edes}(\pi)$ enumerate the number of descents of permutation π in odd and even positions, respectively. In this paper, we obtain an exponential generating function for $A_n(p,q)$ and give an explicit formula for $A_n(p,q)$ in terms of Eulerian polynomials $A_n(q)$ and $C(q)$, the generating function for Catalan numbers. In certain cases, we establish a connection between $A_n(p,q)$ and $A_n(p,0)$ or $A_n(0,q)$, and express the coefficients of $A_n(0,q)$ by Eulerian numbers $A_{n,k}$. Consequently, this connection discovers a new relation between Euler numbers E_n and Eulerian numbers $A_{n,k}$.

Keywords Eulerian polynomial; Eulerian number; Euler number; descent; alternating permutation; Catalan number

MR(2010) Subject Classification 05A05; 05A15; 05A10

1. Introduction

Let $\mathfrak{S}_n$ denote the set of all permutations on $[n]=\{1,2,\ldots,n\}$. For a permutation $\pi=a_1a_2\ldots a_n\in\mathfrak{S}_n$, an index $i\in[n-1]$ is called a descent of π if $a_i>a_{i+1}$, and $\mathrm{des}(\pi)$ denotes the number of descents of π. It is well known that $A_{n,k}$, the Eulerian number [1, A008292], counts the number of permutations $\pi\in\mathfrak{S}_n$ with k descents and obeys the following recurrence [2]

$$A_{n,k}=(n-k)A_{n-1,k-1}+(k+1)A_{n-1,k},\quad n>k\geq 0$$

with $A_{n,0}=1$ for $n\geq 0$ and $A_{n,k}=0$ for $1\leq n\leq k$ or $k<0$. The exponential generating function [2] for $A_{n,k}$ is

$$\mathcal{E}(q;t)=1+\sum_{n\geq 1}A_n(q)\frac{t^n}{n!}=1+\sum_{n\geq 1}\sum_{k=0}^{n-1}A_{n,k}q^k\frac{t^n}{n!}=\frac{1-q}{e^{t(q-1)}-q},\tag{1.1}$$

Received March 30, 2019; Accepted September 4, 2019
Supported by "Liaoning BaiQianWan Talents Program" and by the Fundamental Research Funds for the Central Universities (Grant No. 3132019323).
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where

$$A_n(q) = \sum_{\pi\in\mathfrak{S}_n} q^{\mathrm{des}(\pi)} = \sum_{k=0}^{n-1} A_{n,k}q^k,$$

is the classical Eulerian polynomial [3]. The Eulerian polynomials have a rich history and appear in a large number of contexts in combinatorics; see [4] for a detailed exposition.

Recently, Sun [5] introduced a new kind of refined Eulerian polynomials defined by

$$A_n(p,q) = \sum_{\pi\in\mathfrak{S}_n} p^{\mathrm{odes}(\pi)}q^{\mathrm{edes}(\pi)}$$

for $n \geq 1$, where $\mathrm{odes}(\pi)$ and $\mathrm{edes}(\pi)$ enumerate the number of descents of permutation π in odd and even positions, respectively. The polynomial $A_n(p,q)$ is a bivariate polynomial of degree $n-1$, and the monomial with degree $n-1$ is exactly $p^{\lfloor\frac{n}{2}\rfloor}q^{\lfloor\frac{n-1}{2}\rfloor}$, where $\lfloor x\rfloor$ denotes the largest integer $\leq x$. When $p=q$, $A_n(p,q)$ reduces to the Eulerian polynomial $A_n(q)$.

For convenience, define

$$\tilde{A}_n(p,q) = \begin{cases} A_n(p,q), & \text{if } n = 2m+1,\\ (1+q)A_n(p,q), & \text{if } n = 2m+2, \end{cases}$$

for $n \geq 1$ and $\tilde{A}_0(p,q) = A_0(p,q) = 1$. Sun [5] showed that the (modified) refined Eulerian polynomial $\tilde{A}_n(p,q)$ is palindromic (symmetric) of darga $\lfloor\frac{n}{2}\rfloor$. She also provided certain explicit formulas for special cases, namely,

$$A_n(p,1) = \frac{n!}{2^{\lfloor\frac{n}{2}\rfloor}}(1+p)^{\lfloor\frac{n}{2}\rfloor}, \tag{1.2}$$

$$A_n(1,q) = \frac{n!}{2^{\lfloor\frac{n-1}{2}\rfloor}}(1+q)^{\lfloor\frac{n-1}{2}\rfloor}. \tag{1.3}$$

Note that a permutation $\pi\in\mathfrak{S}_n$ such that $\mathrm{odes}(\pi) = \lfloor\frac{n}{2}\rfloor$ and $\mathrm{edes}(\pi) = 0$ (or $\mathrm{odes}(\pi) = 0$ and $\mathrm{edes}(\pi) = \lfloor\frac{n-1}{2}\rfloor$) is exactly an alternating (or reverse alternating) permutation. Recall that a permutation $\pi = a_1a_2\cdots a_n \in \mathfrak{S}_n$ is alternating (or reverse alternating) [6] if $a_1 > a_2 < a_3 > \cdots$ (or $a_1 < a_2 > a_3 < \cdots$). It is well known that the Euler number E_n (see [1, A000111]) counts the (reverse) alternating permutations in $\mathfrak{S}_n$, which has the remarkable generating function [6],

$$\begin{aligned}\sum_{n\geq 0} E_n\frac{t^n}{n!} &= \tan(t) + \sec(t)\\ &= 1 + t + \frac{t^2}{2!} + 2\frac{t^3}{3!} + 5\frac{t^4}{4!} + 16\frac{t^5}{5!} + 61\frac{t^6}{6!} + 272\frac{t^7}{7!} + 1385\frac{t^8}{8!} + \cdots.\end{aligned}$$

It produces that

$$\sum_{n\geq 0} E_{2n}\frac{t^{2n}}{(2n)!} = \sec(t), \quad \sum_{n\geq 0} E_{2n+1}\frac{t^{2n+1}}{(2n+1)!} = \tan(t).$$

For this reason E_{2n} is sometimes called a secant number and E_{2n+1} a tangent number. See [6] for a survey of alternating permutations. The following associated generating functions are useful,

$$\sum_{n\geq 0}(-1)^n E_{2n}\frac{t^{2n}}{(2n)!} = \frac{2}{e^t + e^{-t}} = \frac{2e^t}{e^{2t}+1} = \mathcal{E}(-1;t)e^{-t},$$

$$\sum_{n\geq 0}(-1)^n E_{2n+1}\frac{t^{2n+1}}{(2n+1)!} = \frac{e^t-e^{-t}}{e^t+e^{-t}} = \frac{e^{2t}-1}{e^{2t}+1} = \mathcal{E}(-1;t)-1.$$

which establishes a connection between Eulerian numbers and Euler numbers, that is,

$$E_{2n+1} = (-1)^n A_{2n+1}(-1), \tag{1.4}$$

$$E_{2n+3} = (-1)^n 2A'_{2n+2}(-1), \tag{1.5}$$

where $A'_n(q)$ is the derivative of $A_n(q)$ with respect to q.

The remainder of this paper is organized as follows. The next section will be devoted to building an exponential generating function for $A_n(p,q)$ and to establishing an explicit formula for $A_n(p,q)$ in terms of Eulerian polynomials $A_n(q)$ and $C(q)=\frac{1-\sqrt{1-4q}}{2q}$. The third section will set up a connection between $A_n(p,q)$ and $A_n(p,0)$ or $A_n(0,q)$, and express the coefficients of $A_n(0,q)$ by Eulerian numbers $A_{n,k}$.

2. The explicit formula for $A_n(p,q)$

In this section, we consider the bivariate polynomials $A_n(p,q)$ and find an explicit formula for $A_n(p,q)$. First, we need the following lemma.

Lemma 2.1 *For any integer $n \geq 1$, there holds*

$$A_{2n}(p,q) = (1+p)A_{2n-1}(p,q) + (p+q)\sum_{i=1}^{n-1}\binom{2n-1}{2i-1}A_{2i-1}(p,q)A_{2n-2i}(p,q), \tag{2.1}$$

$$A_{2n+1}(p,q) = A_{2n}(p,q) + p\sum_{i=0}^{n-1}\binom{2n}{2i}A_{2i}(p,q)A_{2n-2i}(q,p) + q\sum_{i=1}^{n}\binom{2n}{2i-1}A_{2i-1}(p,q)A_{2n-2i+1}(p,q). \tag{2.2}$$

Proof For any $\pi = a_1a_2\cdots a_{2n} \in \mathfrak{S}_{2n}$, if $a_k = 2n$, then π can be partitioned into $\pi = \pi_1(2n)\pi_2$ with $\pi_1 = a_1a_2\ldots a_{k-1}$ and $\pi_2 = a_{k+1}a_{k+2}\cdots a_{2n}$. Let $S = \{a_1, a_2, \ldots, a_{k-1}\}$. Then S is a $(k-1)$-subset of $[2n-1]$ and π_2 is a certain permutation of $[2n-1]-S$. If π_2 is empty, that is $a_{2n} = 2n$, then $\pi_1 \in \mathfrak{S}_{2n-1}$, which are totally counted by $A_{2n-1}(p,q)$ according to $\mathrm{odes}(\pi_1)$ and $\mathrm{edes}(\pi_1)$. If π_2 is not empty, that is $1 \leq k < 2n$, then

$$\mathrm{odes}(\pi) = \mathrm{odes}(\pi_1) + \mathrm{odes}(\pi_2),\ \ \mathrm{edes}(\pi) = \mathrm{edes}(\pi_1) + \mathrm{edes}(\pi_2) + 1,\ \text{ when } k \text{ even},$$

$$\mathrm{odes}(\pi) = \mathrm{odes}(\pi_1) + \mathrm{edes}(\pi_2) + 1,\ \ \mathrm{edes}(\pi) = \mathrm{edes}(\pi_1) + \mathrm{odes}(\pi_2),\ \text{ when } k \text{ odd}.$$

Therefore, there are $\binom{2n-1}{k-1}$ choices to choose $S \in [2n-1]$, all permutations π_1 of S are counted by $A_{k-1}(p,q)$ and all permutations π_2 of $[2n-1]-S$ are counted by $A_{2n-k}(p,q)$, so all permutations $\pi = \pi_1(2n)\pi_2$ are counted by $A_{2i-1}(p,q)qA_{2n-2i}(p,q)$ when $k = 2i$ for $1 \leq i < n$, and counted by $A_{2n-2i}(p,q)pA_{2i-1}(q,p)$ when $k = 2(n-i)+1$ for $1 \leq i \leq n$. Note that $A_k(p,q) = A_k(q,p)$ when k is odd. To summarize all these cases, we obtain (2.1).

Similarly, one can prove (2.2), the details are omitted. □

Let $A^{(e)}(p,q;t)$ and $A^{(o)}(p,q;t)$ be the exponential generating functions for $A_{2n}(p,q)$ and $A_{2n+1}(p,q)$, respectively, i.e.,

$$A^{(e)}(p,q;t)=\sum_{n\geq 1}A_{2n}(p,q)\frac{t^{2n}}{(2n)!},$$
$$A^{(o)}(p,q;t)=\sum_{n\geq 0}A_{2n+1}(p,q)\frac{t^{2n+1}}{(2n+1)!}.$$

Then Lemma 2.1 suggests that

$$\frac{\partial A^{(e)}(p,q;t)}{\partial t}=A^{(o)}(p,q;t)\big((1+p)+(p+q)A^{(e)}(p,q;t)\big), \tag{2.3}$$
$$\frac{\partial A^{(o)}(p,q;t)}{\partial t}=1+A^{(e)}(p,q;t)+p(A^{(e)}(p,q;t)+1)A^{(e)}(q,p;t)+qA^{(o)}(p,q;t)^2. \tag{2.4}$$

Noting that $(1+q)A_{2n}(p,q)=(1+p)A_{2n}(q,p)$ and $A_{2n-1}(p,q)=A_{2n-1}(q,p)$ for $n\geq 1$, one has $A^{(e)}(q,p;t)=\frac{1+q}{1+p}A^{(e)}(p,q;t)$ and $A^{(o)}(q,p;t)=A^{(o)}(p,q;t)$. Then after simplification, (2.4) produces

$$\begin{aligned}\frac{\partial A^{(o)}(p,q;t)}{\partial t}&=\frac{1}{2}\frac{\partial(A^{(o)}(p,q;t)+A^{(o)}(q,p;t))}{\partial t}\\&=1+(1+q)A^{(e)}(p,q;t)+\frac{(1+q)(p+q)}{2(1+p)}A^{(e)}(p,q;t)^2+\frac{p+q}{2}A^{(o)}(p,q;t)^2.\end{aligned} \tag{2.5}$$

Let $y=\frac{p+q}{2(1+pq)}$ and $x=yC(y^2)$, where $C(y)=\frac{1-\sqrt{1-4y}}{2y}$ is the generating function of the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$ for $n\geq 0$. By the relation $C(y)=1+yC(y)^2$, we have $y=\frac{x}{1+x^2}$, $\frac{2x}{1+x^2}=\frac{p+q}{1+pq}$, $\frac{2x}{(1+x)^2}=\frac{p+q}{(1+p)(1+q)}$ and $\frac{(1+x)^2}{1+x^2}=\frac{(1+p)(1+q)}{1+pq}$. Define

$$B^{(e)}(p,q;t)=\frac{1+x}{1+q}\Big\{\frac{\mathcal{E}(x;\sqrt{\frac{1+pq}{1+x^2}}t)+\mathcal{E}(x;-\sqrt{\frac{1+pq}{1+x^2}}t)}{2}-1\Big\},$$
$$B^{(o)}(p,q;t)=\frac{\mathcal{E}(x;\sqrt{\frac{1+pq}{1+x^2}}t)-\mathcal{E}(x;-\sqrt{\frac{1+pq}{1+x^2}}t)}{2\sqrt{\frac{1+pq}{1+x^2}}}.$$

Now taking partial derivative for $B^{(e)}(p,q;t)$ and $B^{(o)}(p,q;t)$ with respect to t, we obtain

$$\begin{aligned}\frac{\partial B^{(e)}(p,q;t)}{\partial t}&=\frac{(1+x)}{(1+q)}\frac{(1+pq)}{(1+x^2)}B^{(o)}(p,q;t)\Big((1+x)+\frac{2x(1+q)}{1+x}B^{(e)}(p,q;t)\Big),\\&=B^{(o)}(p,q;t)\big((1+p)+(p+q)B^{(e)}(p,q;t)\big),\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}\frac{\partial B^{(o)}(p,q;t)}{\partial t}&=1+(1+q)B^{(e)}(p,q;t)+\\&\quad\frac{x(1+q)^2}{(1+x)^2}B^{(e)}(p,q;t)^2+\frac{x(1+pq)}{1+x^2}B^{(o)}(p,q;t)^2,\\&=1+(1+q)B^{(e)}(p,q;t)+\frac{(1+q)(p+q)}{2(1+p)}B^{(e)}(p,q;t)^2+\frac{p+q}{2}B^{(o)}(p,q;t)^2.\end{aligned} \tag{2.7}$$

By (2.3), (2.4), (2.6) and (2.7), one can see that $A^{(e)}(p,q;t)$, $A^{(o)}(p,q;t)$ and $B^{(e)}(p,q;t)$, $B^{(o)}(p,q;t)$ satisfy the same differential equations of order one. On the other hand, it is routine to verify that for $1 \le n \le 3$ the coefficients of t^{2n} in $A^{(e)}(p,q;t)$ and $B^{(e)}(p,q;t)$ coincide, as well as the ones of t^{2n-1} in $A^{(o)}(p,q;t)$ and $B^{(o)}(p,q;t)$. Hence we obtain the exponential generating functions of $A_{2n}(p,q)$ and $A_{2n-1}(p,q)$ for $n \ge 1$ as follows.

Theorem 2.2 *There hold*

$$A^{(e)}(p,q;t) = \frac{1+x}{1+q}\Big\{\frac{\mathcal{E}(x;\sqrt{\frac{1+pq}{1+x^2}}t)+\mathcal{E}(x;-\sqrt{\frac{1+pq}{1+x^2}}t)}{2} - 1\Big\}, \tag{2.8}$$

$$A^{(o)}(p,q;t) = \frac{\mathcal{E}(x;\sqrt{\frac{1+pq}{1+x^2}}t)-\mathcal{E}(x;-\sqrt{\frac{1+pq}{1+x^2}}t)}{2\sqrt{\frac{1+pq}{1+x^2}}}, \tag{2.9}$$

where $x = yC(y^2)$, $y = \frac{p+q}{2(1+pq)}$ *and* $C(y) = \frac{1-\sqrt{1-4y}}{2y}$.

Comparing the coefficient of $\frac{t^n}{n!}$ on both sides of (2.8) and (2.9), we have the following explicit formula for $A_n(p,q)$.

Corollary 2.3 *For any integer* $n \ge 0$, *there hold*

$$A_{2n+1}(p,q) = A_{2n+1}\big(\frac{p+q}{2(1+pq)}C\big(\frac{(p+q)^2}{4(1+pq)^2}\big)\big)\big(\frac{1+pq}{C\big(\frac{(p+q)^2}{4(1+pq)^2}\big)}\big)^n, \tag{2.10}$$

$$\begin{aligned} A_{2n+2}(p,q) = &\frac{1+\frac{p+q}{2(1+pq)}C\big(\frac{(p+q)^2}{4(1+pq)^2}\big)}{1+q} A_{2n+2}\big(\frac{p+q}{2(1+pq)}C\big(\frac{(p+q)^2}{4(1+pq)^2}\big)\big) \\ &\big(\frac{1+pq}{C\big(\frac{(p+q)^2}{4(1+pq)^2}\big)}\big)^{n+1}, \end{aligned} \tag{2.11}$$

where $C(y) = \frac{1-\sqrt{1-4y}}{2y}$.

In the case $q = 1$ in (2.10) and (2.11), by $C(\frac{1}{4}) = 2$ and $A_n(1) = n!$, one has

$$A_{2n+1}(p,1) = \frac{(2n+1)!}{2^n}(1+p)^n,$$

$$A_{2n+2}(p,1) = \frac{(2n+2)!}{2^{n+1}}(1+p)^{n+1},$$

which is equivalent to (1.2).

Similarly, the case $p = 1$ in (2.10) and (2.11) also generates an equivalent form to (1.3).

3. The special case $p = 0$ or $q = 0$

In this section, we concentrate on the special case $p = 0$ or $q = 0$.

By the symmetry of $\tilde{A}_n(p,q)$ for $n \ge 1$, there is

$$A_n(p,0) = \begin{cases} A_n(0,p), & \text{if } n = 2m+1, \\ (1+p)A_n(0,p), & \text{if } n = 2m+2. \end{cases} \tag{3.1}$$

In fact, one can represent $\tilde{A}_n(p,q)$ in terms of $A_n(p,0)$ or $A_n(0,q)$.

Theorem 3.1 *For any integer $n \geq 1$, there holds*

$$\tilde{A}_n(p,q) = (1+pq)^{\lfloor \frac{n}{2} \rfloor} A_n\big(\frac{p+q}{1+pq}, 0\big). \tag{3.2}$$

Equivalently, for $n \geq 0$ there are

$$A_{2n+1}(p,q) = (1+pq)^n A_{2n+1}\big(0, \frac{p+q}{1+pq}\big), \tag{3.3}$$

$$A_{2n+2}(p,q) = (1+p)(1+pq)^n A_{2n+2}\big(0, \frac{p+q}{1+pq}\big). \tag{3.4}$$

Proof The case $q = 0$ in Corollary 2.3 gives rise to

$$A_{2n+1}(p,0) = A_{2n+1}\big(\frac{p}{2}C(\frac{p^2}{4})\big)C\big(\frac{p^2}{4}\big)^{-n}, \tag{3.5}$$

$$A_{2n+2}(p,0) = \big(1+\frac{p}{2}C(\frac{p^2}{4})\big)A_{2n+2}\big(\frac{p}{2}C(\frac{p^2}{4})\big)C\big(\frac{p^2}{4}\big)^{-n-1}. \tag{3.6}$$

Then resetting $p := \frac{p+q}{1+pq}$, by Corollary 2.3, we get

$$A_{2n+1}\big(\frac{p+q}{1+pq}, 0\big) = A_{2n+1}(p,q)(1+pq)^{-n} = \tilde{A}_{2n+1}(p,q)(1+pq)^{-n},$$

$$A_{2n+2}\big(\frac{p+q}{1+pq}, 0\big) = (1+q)A_{2n+2}(p,q)(1+pq)^{-n-1} = \tilde{A}_{2n+2}(p,q)(1+pq)^{-n-1},$$

which is equivalent to (3.2), or by (3.1), equivalent to (3.3) and (3.4). □

Remark 3.2 As stated by Sun [5], the refined Eulerian polynomials $\tilde{A}_n(p,q)$ can be expanded in terms of gamma basis, that is,

$$\tilde{A}_n(p,q) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,j}(p+q)^j(1+pq)^{\lfloor \frac{n}{2} \rfloor - j},$$

and she conjectured that for any $n \geq 1$, all $c_{n,j}$ are positive integers. From Theorem 3.1,

$$A_n(p,0) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} c_{n,j} p^j,$$

it is obvious that $c_{n,j}$ are positive because $c_{n,j}$ counts the number of permutations $\pi \in \mathfrak{S}_n$ with j odd descents and with no even descents, and one can easily construct such a permutation $\pi = \pi_1(2j+1)(2j+2)\cdots n$, where π_1 is an alternating permutation on $[2j]$.

Let $a_{n,k}$ be the number of permutations $\pi \in \mathfrak{S}_n$ with k even descents and with no odd descents. Then

$$A_n(0,q) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} a_{n,k} q^k, \quad n \geq 1.$$

By (3.1), it is clear that $c_{n,k} = a_{n,k}$ when n is odd and $c_{n,k} = a_{n,k} + a_{n,k-1}$ when n is even. Now we can establish several connections between Eulerian numbers $A_{n,k}$ and $a_{n,k}$.

Corollary 3.3 *For any integers $n \geq k \geq 0$, there hold*

$$A_{2n+1,k} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{n-k+2i}{i} 2^{k-2i} a_{2n+1,k-2i}, \tag{3.7}$$

$$A_{2n+2,k} = \sum_{\substack{2i+j=k \text{ or } k-1 \\ i,j\geq 0}} \binom{n-j}{i} 2^j a_{2n+2,j}. \tag{3.8}$$

Equivalently,

$$a_{2n+1,k} = \frac{1}{2^k} \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} (-1)^i \frac{n-k+2i}{n-k+i} \binom{n-k+i}{i} A_{2n+1,k-2i}, \tag{3.9}$$

$$a_{2n+2,k} = \frac{1}{2^k} \sum_{\substack{2i+j+r=k \\ i,j,r\geq 0}} (-1)^{r+i} \frac{n-k+2i}{n-k+i} \binom{n-k+i}{i} A_{2n+2,j}. \tag{3.10}$$

Specially, Euler numbers can be represented by Eulerian numbers as follows

$$E_{2n+1} = a_{2n+1,n} = \frac{1}{2^n}\Big(- A_{2n+1,n} + 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^i A_{2n+1,n-2i}\Big), \tag{3.11}$$

$$E_{2n+2} = a_{2n+2,n} = \frac{1}{2^n}\Big(- \sum_{j=0}^{n} (-1)^{n-j} A_{2n+2,j} + 2 \sum_{\substack{0\leq 2i+j\leq n \\ i,j\geq 0}} (-1)^{n-i-j} A_{2n+2,j}\Big). \tag{3.12}$$

Proof By (3.3) and (3.4), the case $p = q$ produces

$$A_{2n+1}(q) = (1+q^2)^n A_{2n+1}\big(0, \frac{2q}{1+q^2}\big) = \sum_{j=0}^{n} a_{2n+1,j} (2q)^j (1+q^2)^{n-j}, \tag{3.13}$$

$$A_{2n+2}(q) = (1+q)(1+q^2)^n A_{2n+2}\big(0, \frac{2q}{1+q^2}\big) = (1+q)\sum_{j=0}^{n} a_{2n+2,j} (2q)^j (1+q^2)^{n-j}. \tag{3.14}$$

Comparing the coefficients of q^k on both sides of (3.13) and (3.14), we get (3.7) and (3.8).

By (3.1) and (3.6), using the series expansion [7],

$$C(t)^\alpha = \sum_{i\geq 0} \frac{\alpha}{2i+\alpha} \binom{2i+\alpha}{i} t^i,$$

we have

$$\begin{aligned} A_{2n+1}(0,q) &= A_{2n+1}(q,0) = A_{2n+1}\Big(\frac{q}{2} C\Big(\frac{q^2}{4}\Big)\Big) C\Big(\frac{q^2}{4}\Big)^{-n} \\ &= \sum_{j=0}^{2n} A_{2n+1,j} \Big(\frac{q}{2}\Big)^j C\Big(\frac{q^2}{4}\Big)^{j-n} \\ &= \sum_{j=0}^{2n} A_{2n+1,j} \sum_{i\geq 0} \frac{j-n}{2i+j-n} \binom{2i+j-n}{i} \Big(\frac{q}{2}\Big)^{2i+j} \\ &= \sum_{j=0}^{2n} \sum_{i\geq 0} A_{2n+1,j} (-1)^i \frac{n-j}{n-i-j} \binom{n-i-j}{i} \Big(\frac{q}{2}\Big)^{2i+j}, \end{aligned} \tag{3.15}$$

and

$$A_{2n+2}(0,q) = \frac{A_{2n+2}(q,0)}{1+q} = \frac{1+\frac{q}{2}C\Big(\frac{q^2}{4}\Big)}{1+q} A_{2n+2}\Big(\frac{q}{2} C\Big(\frac{q^2}{4}\Big)\Big) C\Big(\frac{q^2}{4}\Big)^{-n-1}$$

$$
\begin{aligned}
&=\frac{1}{1+\frac{q}{2}C\Big(\frac{q^2}{4}\Big)}A_{2n+2}\Big(\frac{q}{2}C\Big(\frac{q^2}{4}\Big)\Big)C\Big(\frac{q^2}{4}\Big)^{-n}\\
&=\sum_{j=0}^{2n+1}A_{2n+2,j}\sum_{r\geq 0}(-1)^r\Big(\frac{q}{2}\Big)^{j+r}C\Big(\frac{q^2}{4}\Big)^{r+j-n}\\
&=\sum_{j=0}^{2n+1}A_{2n+2,j}\sum_{r\geq 0}(-1)^r\sum_{i\geq 0}\frac{r+j-n}{2i+j+r-n}\binom{2i+r+j-n}{i}\Big(\frac{q}{2}\Big)^{2i+j+r}\\
&=\sum_{j=0}^{2n+1}\sum_{r,i\geq 0}A_{2n+2,j}(-1)^{r+i}\frac{n-j-r}{n-i-j-r}\binom{n-i-j-r}{i}\Big(\frac{q}{2}\Big)^{2i+j+r}. \qquad (3.16)
\end{aligned}
$$

Then taking the coefficients of t^k on both sides of (3.15) and (3.16), we get (3.9) and (3.10).

Note that $a_{n,\lfloor\frac{n-1}{2}\rfloor}$ is the number of permutations $\pi\in\mathfrak{S}_n$ with $\lfloor\frac{n-1}{2}\rfloor$ even descents and with no odd descents, such π are exactly the reverse alternating permutations and vice versa. Clearly, $a_{n,\lfloor\frac{n-1}{2}\rfloor}=E_n$ for $n\geq 1$. Setting $k=n$ in (3.9) and (3.10), we obtain (3.11) and (3.12). □

Lemma 3.4 *For any integer $n\geq 1$, there hold*

$$
A_n(0,-1)=\begin{cases}(-1)^m\frac{E_n}{2^m}, & \text{if } n=2m+1,\\ (-1)^m\frac{E_{n+1}}{2^{m+1}}, & \text{if } n=2m+2.\end{cases} \qquad (3.17)
$$

Proof Setting $p=q=-1$ in (3.3) and (3.4), by (1.4) and (1.5) we have

$$
\begin{aligned}
A_{2n+1}(0,-1)=&2^{-n}A_{2n+1}(-1)=(-1)^n\frac{E_{2n+1}}{2^n},\\
A_{2n+2}(0,-1)=&2^{-n}\lim_{q\to-1}\frac{A_{2n+2}(q)}{1+q}=2^{-n}A'_{2n+2}(-1)=(-1)^n\frac{E_{2n+3}}{2^{n+1}},
\end{aligned}
$$

which is equivalent to (3.17). □

The case $q=-1$ in (3.3) and (3.4), together with Lemma 3.4, leads to

Corollary 3.5 *For any integer $n\geq 0$, there are*

$$
\begin{aligned}
A_{2n+1}(p,-1)=&\Big(\frac{p-1}{2}\Big)^nE_{2n+1},\\
A_{2n+2}(p,-1)=&\frac{p+1}{2}\Big(\frac{p-1}{2}\Big)^nE_{2n+3}.
\end{aligned}
$$

The case $p=3$ in Corollary 3.5 produces two settings counted by tangent numbers E_{2n+1}. See [6] for further information on various combinatorial interpretations of Euler numbers E_n.

Let $\pi=a_1a_2\cdots a_{n-1}a_n\in\mathfrak{S}_n$, define the reversal of π to be $\pi^r=a_na_{n-1}\cdots a_2a_1$, the complement of π to be $\pi^c=(n+1-a_1)(n+1-a_2)\cdots(n+1-a_{n-1})(n+1-a_n)$ and the reversal-complement of π to be $\pi^{rc}:=(\pi^r)^c=(\pi^c)^r$. For any $\sigma\in\mathfrak{S}_{2n+1}$, note that i is a descent of σ if and only if $2n+1-i$ is a descent of σ^{rc}. Specially, i is an odd (even) descent of σ if and only if $2n+1-i$ is an even (odd) descent of σ^{rc}. Now we can return to consider the recurrence relations for $a_{n,k}$.

Theorem 3.6 *For any integers $n \geq k \geq 1$, there hold*

$$a_{2n,k} = (n-k)a_{2n-1,k-1} + (k+1)a_{2n-1,k}, \tag{3.18}$$

$$a_{2n+1,k} = (n-k+1)a_{2n,k-2} + (n+1)a_{2n,k-1} + (k+1)a_{2n,k} \tag{3.19}$$

with $a_{n,0} = 1$ and $a_{n,\lfloor \frac{n-1}{2} \rfloor} = E_n$.

Proof Let $\alpha_{n,k}$ be the set of permutations $\pi \in \mathfrak{S}_n$ with k even descents and with no odd descents, so $|\alpha_{n,k}| = a_{n,k}$. In the $k = 0$ case, $\alpha_{n,0}$ only contains the natural permutation $123 \cdots (2n-1)(2n)$, and in the $k = \lfloor \frac{n-1}{2} \rfloor$ case $\alpha_{n,\lfloor \frac{n-1}{2} \rfloor}$ is exactly the set of reverse alternating permutations in $\mathfrak{S}_n$. This implies $a_{n,0} = 1$ and $a_{n,\lfloor \frac{n-1}{2} \rfloor} = E_n$.

Set $X = \{2i | 1 \leq i \leq n-1\}$ and denote by $\mathrm{Des}(\pi)$ the descent set of $\pi \in \mathfrak{S}_n$. For any $\pi \in \alpha_{2n,k}$, $\mathrm{Des}(\pi)$ is a k-subset of X. In order to prove (3.18), there are exactly three cases to consider, i.e.,

Case 1. Given $\pi = a_1 a_2 \cdots a_{2n-2} a_{2n-1} \in \alpha_{2n-1,k}$, let $\pi_1^* = \pi a_{2n}$ with $a_{2n} = 2n$, one can easily check that $\pi_1^* \in \alpha_{2n,k}$ and π can be recovered by deleting $a_{2n} = 2n$ in π_1^*. So the total number of such π_1^* in the set $\alpha_{2n,k}$ is $a_{2n-1,k}$.

Case 2. Given $\pi = a_1 a_2 \cdots a_{2j} a_{2j+1} \cdots a_{2n-2} a_{2n-1} \in \alpha_{2n-1,k}$ with $2j \in \mathrm{Des}(\pi)$, define $\pi_2^* = a_{2j+1} \cdots a_{2n-2} a_{2n-1} (2n) a_1 a_2 \cdots a_{2j}$. Clearly, subject to $a_{2j} > a_{2j+1}$, we obtain $\pi_2^* \in \alpha_{2n,k}$ and vice versa. In this case, there are totally $k a_{2n-1,k}$ contributions to the set $\alpha_{2n,k}$.

Case 3. Given $\pi = a_1 a_2 \cdots a_{2j} a_{2j+1} \cdots a_{2n-2} a_{2n-1} \in \alpha_{2n-1,k-1}$ with $2j \in X - \mathrm{Des}(\pi)$, define $\pi_3^* = a_{2j+1} \cdots a_{2n-2} a_{2n-1} (2n) a_1 a_2 \cdots a_{2j}$. Similarly, subject to $a_{2j} < a_{2j+1}$, we have $\pi_3^* \in \alpha_{2n,k}$ and vice versa. In this case, there are totally $(n-k) a_{2n-1,k-1}$ contributions to the set $\alpha_{2n,k}$.

Hence, summarizing the above three cases generates (3.18) immediately.

In order to prove (3.19), by the relation $c_{n,k} = a_{n,k} + a_{n,k-1}$ when n is even, we need take the equivalent form into account,

$$a_{2n+1,k} = (n-k+1)c_{2n,k-1} + k c_{2n,k} + a_{2n,k}. \tag{3.20}$$

Let $\beta_{n,k}$ be the set of permutations $\theta \in \mathfrak{S}_n$ with k odd descents and with no even descents. So $|\beta_{n,k}| = c_{n,k}$. Set $Y = \{2i-1 | 1 \leq i \leq n\}$. For any $\theta \in \beta_{2n,k}$, $\mathrm{Des}(\theta)$ is a k-subset of Y. Similarly, there are precisely three cases to consider.

Case I. Given $\theta = b_1 b_2 \cdots b_{2n} \in \alpha_{2n,k}$, let $\theta_1^* = \theta b_{2n+1}$ with $b_{2n+1} = 2n+1$. Obviously $\theta_1^* \in \alpha_{2n+1,k}$ and θ can be easily obtained by removing $b_{2n+1} = 2n+1$ in θ_1^*. Then there are exactly $a_{2n,k}$ such θ_1^*'s in the set $\alpha_{2n+1,k}$.

Case II. Given $\theta = b_1 b_2 \cdots b_{2j-1} b_{2j} b_{2j+1} \cdots b_{2n} \in \beta_{2n,k}$ with $2j-1 \in \mathrm{Des}(\theta)$, define $\theta_2^* = (b_1 b_2 \cdots b_{2j-1})^{rc} (2n+1) b'_{2j} b'_{2j+1} \cdots b'_{2n} = b'_1 b'_2 \cdots b'_{2j-1} (2n+1) b'_{2j} b'_{2j+1} \cdots b'_{2n}$, where $b'_i = 2n + 1 - b_{2j-i}$ for $1 \leq i \leq 2j-1$ and $b'_{2j} b'_{2j+1} \cdots b'_{2n}$ is a permutation on $[2n] - \{b'_1, b'_2, \ldots, b'_{2j-1}\}$ which has the same relative order as $b_{2j} b_{2j+1} \cdots b_{2n}$. This procedure is naturally invertible due to $b_{2j-1} > b_{2j}$. Clearly, $2\ell - 1$ with $1 \leq \ell < j$ is an odd descent of θ if and only if $2j - 2\ell$ is an even descent of θ_2^*, and $2\ell - 1$ with $j \leq \ell \leq n$ is an odd descent of θ if and only if 2ℓ is an even descent of θ_2^*. Then we get $\theta_2^* \in \alpha_{2n+1,k}$. In this case, there are totally $k c_{2n,k}$ contributions to the set $\alpha_{2n+1,k}$. For example, let $\theta = 4\ 1\ 8\ 3\ 5\ 7\ 10\ 2\ 6\ 9 \in \beta_{10,3}$ and $\mathrm{Des}(\theta) = \{1, 3, 7\}$, we have three $\theta_2^* \in \alpha_{11,3}$, namely,

$$7\ 11\ 1\ 8\ 3\ 4\ 6\ 10\ 2\ 5\ 9, \quad 3\ 10\ 7\ 11\ 2\ 4\ 6\ 9\ 1\ 5\ 8, \quad 1\ 4\ 6\ 8\ 3\ 10\ 7\ 11\ 2\ 5\ 9.$$

Case III. Given $\theta = b_1b_2\cdots b_{2j-1}b_{2j}b_{2j+1}\cdots b_{2n} \in \beta_{2n,k-1}$ with $2j-1 \in Y - \mathrm{Des}(\theta)$, define $\theta_3^* = (b_1b_2\cdots b_{2j-1})^{rc}(2n+1)b'_{2j}b'_{2j+1}\cdots b'_{2n} = b'_1b'_2\cdots b'_{2j-1}(2n+1)b'_{2j}b'_{2j+1}\cdots b'_{2n}$, where $b'_i = 2n+1-b_{2j-i}$ for $1 \le i \le 2j-1$ and $b'_{2j}b'_{2j+1}\cdots b'_{2n}$ is a permutation on $[2n] - \{b'_1, b'_2, \ldots, b'_{2j-1}\}$ which has the same relative order as $b_{2j}b_{2j+1}\cdots b_{2n}$. This procedure is also invertible subject to $b_{2j-1} < b_{2j}$. Similar to Case II, $\theta_3^* \in \alpha_{2n+1,k}$. In this case, there are totally $(n-k+1)c_{2n,k-1}$ contributions to the set $\alpha_{2n+1,k}$.

Hence, summing over all the three cases yields (3.20) immediately. □

Table 1 illustrates this triangle for n up to 10 and k up to 4.

n/k	0	1	2	3	4
1	1				
2	1				
3	1	2			
4	1	5			
5	1	13	16		
6	1	28	61		
7	1	60	297	272	
8	1	123	1011	1385	
9	1	251	3651	10841	7936
10	1	506	11706	50666	50521

Table 1 The first values of $a_{n,k}$

Acknowledgements We are grateful to the referees for the helpful suggestions and comments.

References

[1] N. J. A. SLOANE. *The On-Line Encyclopedia of Integer Sequences.* http://oeis.org/, Maintained by The OEIS Foundation Inc.

[2] L. COMTET. *Advanced Combinatorics.* D. Reidel Publishing Company, Dordrecht-Holland, 1974.

[3] D. FOATA, M.-P. SCHÜTZENBERGER. *Théorie Géométrique des Polynômes Eulériens.* Lecture Notes in Mathematics, Vol. 138, Springer-Verlag, 1970.

[4] T. K. PETERSEN. *Eulerian Numbers.* Birkhauser, 2015.

[5] Hua SUN. *A new class of refined Eulerian polynomials.* J. Integer Seq., 2018, **21**(5): 1–9.

[6] R. P. STANLEY. *A Survey of Alternating Permutations.* Amer. Math. Soc., Providence, RI, 2010.

[7] R.P. STANLEY. *Catalan Numbers.* Cambridge University Press, Cambridge, 2015.