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Powers of the Catalan Generating Function and Lagrange's 1770 Trinomial Equation Series

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Dedicated to the Memory of Professor L. C. HSU on the Occasion of His 100th Birthday

Abstract The Catalan numbers $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$ are given by $C(n) = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$. They are named for Eugene Catalan who studied them as early as 1838. They were also found by Leonhard Euler (1758), Nicholas von Fuss (1795), and Andreas von Segner (1758). The Catalan numbers have the binomial generating function

$$\mathbf{C}(z) = \sum_{n=0}^{\infty} C(n) z^n = \frac{1 - \sqrt{1-4z}}{2z}$$

It is known that powers of the generating function $\mathbf{C}(z)$ are given by

$$\mathbf{C}^a(z) = \sum_{n=0}^{\infty} \frac{a}{a+2n}\binom{a+2n}{n} z^n.$$

The above formula is not as widely known as it should be. We observe that it is an immediate, simple consequence of expansions first studied by J. L. Lagrange. Such series were used later by Heinrich August Rothe in 1793 to find remarkable generalizations of the Vandermonde convolution. For the equation $x^3 - 3x + 1 = 0$, the numbers $\frac{1}{2k+1}\binom{3k}{k}$ analogous to Catalan numbers occur of course. Here we discuss the history of these expansions. and formulas due to L. C. Hsu and the author.

Keywords Catalan numbers; Vandermonde convolution; Lagrange and Bürmann series; Rothe's formula (or general identity of Rothe-Hagen)

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The Catalan numbers $1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, \ldots$ are given by the formula $C(n) = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$. They arise in the study of the number of parenthesizations of a product and also in the number of ways to place non-intersecting diagonals in an n-sided regular polygon. Also $C(n)$ counts the number of planar trees with n edges. They are named for Eugene Catalan who studied them as early as 1838 and 1839. They were also found by Leonhard Euler (1758), Nicholas von Fuss (1795), and Andreas von Segner (1758). See Gould [1] for a detailed history and bibliography of 470 references. Today one might add a thousand new references.

The Catalan numbers have the binomial generating function

$$\mathbf{C}(z) = \sum_{n=0}^{\infty} C(n) z^n = \frac{1 - \sqrt{1-4z}}{2z} \tag{1}$$

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which satisfies the quadratic equation

$$\mathbf{C}(z) = 1 + z\mathbf{C}^2(z). \tag{2}$$

Incidentally, iteration of this relation yields

$$\mathbf{C}(z) = \sum_{k=0}^{n} z^k \mathbf{C}^k(z) + z^{n+1}\mathbf{C}^{n+2}(z). \tag{3}$$

It is known that powers of the generating function $\mathbf{C}(z)$ are given by

$$\mathbf{C}^a(z) = \sum_{n=0}^{\infty} \frac{a}{a+2n}\binom{a+2n}{n} z^n. \tag{4}$$

This is given as part of an exercise by Riordan [2, p.153]. It is stated and used by Shapiro [3] in a study of the number of terminal points at a given height in a planar (or ordered) tree.

Formula (4) is not as widely known as it should be. We observe that it is an immediate, simple consequence of expansions first studied by Lagrange [4] in his monumental 1770 paper, presented to the Berlin Academy, in which he found infinite series expansions for roots of trinomial equations such as

$$x^n + Ax + B = 0, \;\; x^n + Ax^m + C = 0, \quad \text{etc.} \tag{5}$$

Lagrange was motivated to find such series expansions by his futile effort to solve the general n-th degree algebraic equation in closed form using a finite number of root extractions as was known for $n = 2, 3,$ and 4. Out of this work came the Lagrange and Bürmann series for solving the equation $x = w + a\phi(x)$ for x.

Such series were used later by Heinrich August Rothe [5] to find remarkable generalizations of the Vandermonde convolution

$$\sum_{k=0}^{n} \binom{a}{k}\binom{c}{n-k} = \binom{a+c}{n}, \tag{6}$$

where a and c are any complex numbers, about which I have written a number of papers starting in 1956. An account of my work to date can be found in [6]. Some of Rothe's formulas were listed in the famous Synopsis of Johann Georg Hagen [7]. It was from Hagen's Synopsis that I first learned of Rothe's work and I dubbed the main binomial identities as Rothe-Hagen convolutions. In 1955 I learned from the late Raymond Claire Archibald (mathematics librarian at Brown University) that there were only two known copies of Rothe's 1793 thesis in existence, one at Pulkovo Observatory in Russia and the other in the library of the Royal Astronomical Society at Edinburgh, Scotland. But unfortunately the copy in Russia had been destroyed by German bombing in the Second World War. Happily, I was able through the good services of the Astronomer Royal of Scotland to obtain a microfilm of the copy at Edinburgh. I had this printed out and a Xerox copy prepared for study.

In my papers [8], [9] and book [10], emphasis is laid on the following general identity of Rothe-Hagen

$$\sum_{k=0}^{n} (p+qk) A_k(a,b) A_{n-k}(c,b) = \frac{p(a+c)+qna}{a+c} A_n(a+c,b), \tag{7}$$

where

$$A_k(a,b) = \frac{a}{a+bk}\binom{a+bk}{k}. \tag{8}$$

For $p = 1$, $q = 0$ this gives

$$\sum_{k=0}^{n} A_k(a,b)A_{n-k}(c,b) = A_n(a+c,b), \tag{9}$$

that is

$$\sum_{k=0}^{n} \frac{a}{a+bk}\binom{a+bk}{k}\frac{c}{c+(n-k)b}\binom{c+(n-k)b}{n-k} = \frac{a+c}{a+c+bn}\binom{a+c+bn}{n}. \tag{10}$$

Another valuable formula is

$$\sum_{k=0}^{n} \frac{a}{a+bk}\binom{a+bk}{k}\binom{c+(n-k)b}{n-k} = \binom{a+c+bn}{n}, \tag{11}$$

which I have always preferred to write as,

$$\sum_{k=0}^{n} A_k(a,b)C_{n-k}(c,b) = C_n(a+c,b), \tag{12}$$

where

$$C_k(a,b) = \binom{a+bk}{k}. \tag{13}$$

I have also called attention to the identity

$$\sum_{k=0}^{n}\binom{a+bk}{k}\binom{c+(n-k)b}{n-k} = \sum_{k=0}^{n}\binom{a+d+bk}{k}\binom{c-d+(n-k)b}{n-k} \tag{14}$$

All of these identities follow from the formal power series

$$\sum_{n=0}^{\infty} A_n(a,b)z^n = x^a, \tag{15}$$

and

$$\sum_{n=0}^{\infty} C_n(a,b)z^n = \frac{x^{a+1}}{(1-b)x+b}, \tag{16}$$

where in both expressions

$$zx^b = x - 1. \tag{17}$$

As for the powers of the Catalan generating function, we have only to set $b = 2$, solve (17) (now the simple quadratic $zx^2 - x + 1 = 0$), pick the proper sign, and we get expansion (4) at once.

Because of the similarity of (15) and (16) we have a companion to (4),

$$\mathbf{H}(x) = \sum_{n=0}^{\infty}\binom{a+2n}{n}z^n = \frac{x^{a+1}}{3-2x} = \frac{\mathbf{C}^{a+1}(z)}{3-2\mathbf{C}(z)}, \tag{18}$$

in terms of the generating function for the Catalan numbers. We may also write this in the form

$$(3-2\mathbf{C}(z))\sum_{n=0}^{\infty}\binom{a+2n}{n}z^n = \mathbf{C}(z)\sum_{n=0}^{\infty}\frac{a}{a+2n}\binom{a+2n}{n}z^n. \tag{19}$$

Solving this for $\mathbf{C}(z)$ gives us the following curious expression for the binomial Catalan generating function:

$$\mathbf{C}(z) = \frac{3A}{2A+B}, \tag{20}$$

where

$$A = \sum_{n=0}^{\infty} \binom{a+2n}{n} z^n \quad \text{and} \quad B = \sum_{n=0}^{\infty} \frac{a}{a+2n} \binom{a+2n}{n} z^n. \tag{21}$$

The more general formulas of Rothe and their history are given in [9], where I also showed how Rothe's formula (7) also yields the generalized binomial theorem of N. H. Abel

$$\sum_{k=0}^{n} (p+qk) B_k(a,b) B_{n-k}(c,b) = \frac{p(a+c)+qna}{a+c} B_n(a+c,b) \tag{22}$$

where

$$B_k(a,b) = \frac{a}{a+bk} \frac{(a+bk)^k}{k!}. \tag{23}$$

The derivation of (22) from (7) merely uses the limit

$$\lim_{h\to 0} \binom{\frac{a+bk}{h}}{k} \frac{h^k}{k!} = \frac{(a+bk)^k}{k!}. \tag{24}$$

Relations (22) and (7) have also been discussed in detail in [11], with analogous formulas for powers of Dirichlet series.

A useful account of series expansions of all the roots of the trinomial equation with numerical examples was given by J. Sutherland Frame [12]. For the equation $x^3 - 3x + 1 = 0$, the numbers $\frac{1}{2k+1}\binom{3k}{k}$ analogous to Catalan numbers occur of course.

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