Fekete-Szegő Functional Problems for Certain Subclasses of Bi-Univalent Functions Involving the Hohlov Operator

Pinhong LONG$^{1,*}$, Huo TANG$^{2}$, Wenshuai WANG$^{1}$

1. School of Mathematics and Statistics, Ningxia University, Ningxia 750021, P. R. China;
2. School of Mathematics and Statistics, Chifeng University, Inner Mongolia 024000, P. R. China

Abstract In the paper the new subclasses $\mathcal{N}^{a,b,c}_\mu(\lambda; \phi)$ and $\mathcal{M}^{a,b,c}_\lambda(\phi)$ of the function class $\sum$ of bi-univalent functions involving the Hohlov operator are introduced and investigated. Then, the corresponding Fekete-Szegő functional inequalities as well as the bound estimates of the coefficients $a_2$ and $a_3$ are obtained. Furthermore, several consequences and connections to some of the earlier known results also are given.

Keywords Fekete-Szegő problem; analytic function; bi-univalent function; Gaussian hypergeometric function; Hohlov operator

MR(2010) Subject Classification 30C45; 30C50; 30C55

1. Introduction

For the set $\mathbb{C}$ of complex numbers, let $\mathcal{A}$ be the class of normalized analytic function $f(z)$ by

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

in the open unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.

Let $\mathcal{S}$ denote the subclass of $\mathcal{A}$ consisting of all functions which are univalent in $\Delta$. Due to the Koebe one quarter theorem [1], the inverse $f^{-1}$ of $f \in \mathcal{S}$ satisfies

$$f^{-1}(f(z)) = z, \quad z \in \Delta$$

and

$$f(f^{-1}(w)) = w, \quad w \in \Delta_\rho,$$

where $\rho \in [\frac{1}{4}, 1]$ denotes the radius of the image $f(\Delta)$ and $\Delta_\rho = \{z \in \mathbb{C} : |z| < \rho\}$. It is well known that

$$f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \quad (1.2)$$
If the function \( f \in \mathcal{A} \) and its inverse \( f^{-1} \) are univalent in \( \Delta \), then it is bi-univalent. Denote by \( \Sigma \) the class of all bi-univalent functions \( f \in \mathcal{A} \) in \( \Delta \).

Given two analytic functions \( f \) and \( g \), if there exists an analytic \( w \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) for \( z \in \Delta \) so that \( f(z) = g(w(z)) \), then \( f \) is subordinate to \( g \), i.e., \( f \prec g \).

For given \( f, g \in \mathcal{A} \), define the Hadamard product or convolution \( f \ast g \) by

\[
(f \ast g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \quad (z \in \Delta),
\]

where \( f(z) \) is given by Eq. (1.1) and \( g(z) = z + \sum_{k=2}^{\infty} b_k z^k \), and the Gaussian hypergeometric function \( \sum_{2F1(a,b,c,z)} \) by

\[
2F1(a,b,c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} = 1 + \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1}} \frac{z^{n-1}}{(n-1)!}, \quad z \in \Delta
\]

for the complex parameters \( a, b \) and \( c \) with \( c \neq 0, -1, -2, -3, \ldots \), where \((\ell)_n\) denotes the Pochhammer symbol or shifted factorial by

\[
(\ell)_n = \frac{\Gamma(\ell+n)}{\Gamma(\ell)} = \begin{cases} 
1, & \text{if } n = 0, \ell \in \mathbb{C} \setminus \{0\} \\
\ell(\ell+1)(\ell+2) \cdots (\ell+n-1), & \text{if } n \in \mathbb{N} \{1, 2, 3, \ldots\}.
\end{cases}
\]

Hohlov [2,3] ever considered the convolution operator \( T_c^{a,b} \) later named by himself as follows:

\[
T_c^{a,b} f(z) = z \sum_{2F1} (a,b,c;z) \ast f(z) = z + \sum_{n=2}^{\infty} p_n a_n z^n, \quad z \in \Delta,
\]

where

\[
p_n = \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (n-1)!}.
\]

Here, we also note that there exist some reduced versions of Hohlov operator \( T_c^{a,b} \) for suitable parameters \( a, b \) and \( c \), for example, Carlson-Shaffer operator \( L(a,c) = T_c^{a,1} \) (see [4]), Ruscheweyh derivative operator \( D^\delta = \sum_{1+\delta} (\delta < 1) \) (see [5]), Owa-Srivastava fractional differential operator \( \Omega_{\mu} = \sum_{2+1} (0 \leq \mu < 1) \) (see [6,7]), Choi-Saigo-Srivastava operator \( \sum_{3+1} (-1 < \lambda, 0 \leq \mu) \) (see [8]), Noor integral operator \( \sum_{n+1} = \sum_{2+1} \) (see [9]).

In 1967, Lewin [10] introduced the analytic and bi-univalent function and proved that \( |a_2| < 1.51 \). Moreover, Brannan and Clunie [11] conjectured that \( |a_2| \leq \sqrt{2} \), and Netanyahu [12] obtained that \( \max f \sum |a_2| = \frac{3}{4} \). Later, Styer and Wright [13] showed that there exists function \( f(z) \) so that \( |a_2| > \frac{4}{5} \). However, so far the upper bound estimate \( |a_2| < 1.485 \) of coefficient for functions in \( \sum \) by Tan [14] is best. Unfortunately, as for the coefficient estimate problem for every Taylor-Maclaurin coefficient \( |a_n| (n \in \mathbb{N} \setminus \{1, 2\}) \) it is probably still an open problem.

For the work of Brannan and Taha [15] and Srivastava et al. [16], a great deal of subclasses of analytic and bi-univalent functions class \( \sum \) were introduced and investigated, and the non-sharp estimates of first two Taylor-Maclaurin coefficients \( |a_2| \) and \( |a_3| \) were given; refer to Deniz [17], Frasin and Aouf [18], Hayami and Owa [19], Li and Wang [20], Ma and Minda [21], Magesh and Yamini [22], Patil and Naik [23,24], Srivastava et al. [25,26], Tang et al. [27,28] and Xu et al. [29,30] for more detailed information. Recently, Srivastava et al. [31,32] gave some new
subclasses of the function class $\sum$ of analytic and bi-univalent functions to unify the work of Deniz [17], Frasin [33], Srivastava et al. [34], Srivastava et al. [35], Keerthi and Raja [36] and Xu et al. [29], etc.

Since Fekete-Szegö [37] considered the determination of the sharp upper bounds for the subclass of $S$, Fekete-Szegö functional problem was studied in many classes of functions; refer to Orhan and Răducanu [38] for class of starlike functions, Abdel-Gawad [39] for class of quasi-convex functions, Magesh and Balaji [40] for class of convex and starlike functions, Koepf [41] for class of close-to-convex functions, Tang et al. [28] for classes of $m$-mold symmetric bi-univalent functions, Panigrahi and Raina [42] for class of quasi-subordination functions.

Besides, Murugusundaramoorthy et al. [35,43,44] and Patil and Naik [45] ever introduced and investigated several new subclasses of the function class $\sum$ of analytic and bi-univalent functions associated with the Hohlov operator. Stimulated by the statements above, in the paper we will introduce and investigate the new subclasses of the function class $\sum$ of analytic and bi-univalent functions involving the Hohlov operator, and consider the corresponding bound estimates of the coefficients $a_2$ and $a_3$ and Fekete-Szegö functional inequalities. Moreover, several consequences and connections to some of the earlier known results also will be given.

Now we will introduce the following general subclasses of bi-univalent functions.

**Definition 1.1** A function $f(z) \in \sum$ given by (1.1), belongs to the class $\mathcal{N}_{\sum}^{a,b,c}(\mu, \lambda; \phi)$ if the following subordinations are satisfied:

\[
(1 - \lambda) \left( \frac{T_{a,b} f(z)}{z} \right)^\mu + \lambda \left( \frac{T_{a,b} f(z)}{z} \right)^{\mu - 1} < \phi(z)
\]

and

\[
(1 - \lambda) \left( \frac{T_{a,b} g(w)}{w} \right)^\mu + \lambda \left( \frac{T_{a,b} g(w)}{w} \right)^{\mu - 1} < \phi(w)
\]

for $z, w \in \Delta$, where $\mu, \lambda \in [0, \infty)$ satisfy $\mu^2 + \lambda^2 > 0$ and the function $g$ is the inverse of $f$ given by (1.2).

**Definition 1.2** A function $f(z) \in \sum$ given by (1.1), belongs to the class $\mathcal{M}_{\sum}^{a,b,c}(\lambda; \phi)$ if the following subordinations are satisfied:

\[
(1 - \lambda) \left( \frac{T_{a,b} f(z)}{I_{a,b} f(z)} \right)^\mu + \lambda \left( \frac{T_{a,b} f(z)}{I_{a,b} f(z)} \right)^{\mu - 1} < \phi(z)
\]

and

\[
(1 - \lambda) \left( \frac{T_{a,b} g(w)}{I_{a,b} g(w)} \right)^\mu + \lambda \left( \frac{T_{a,b} g(w)}{I_{a,b} g(w)} \right)^{\mu - 1} < \phi(w)
\]

for $z, w \in \Delta$, where $0 \leq \lambda \leq 1$ and the function $g$ is the inverse of $f$ given by (1.2).

**Remark 1.3** Let

\[
\phi(z) = \left( \frac{1 + z}{1 - z} \right)^\alpha \text{ for } 0 < \alpha \leq 1
\]
or
\[
\phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad \text{for } 0 \leq \beta < 1
\]  

(1.12)

in Definitions 1.1 and 1.2. The class \( N^{a,b,c}_\lambda(\mu, \phi) \) (resp., \( M^{a,b,c}_\lambda(\mu, \phi) \)) reduces to \( \hat{N}^{a,b,c}_\lambda(\mu, \lambda; \alpha) \) (resp., \( \hat{M}^{a,b,c}_\lambda(\lambda; \alpha) \)). Further, if \( a = c \) and \( b = 1 \), then the classes \( N^{a,b,c}_\lambda(\mu, \phi) \) and \( M^{a,b,c}_\lambda(\lambda; \phi) \) are just \( N^{a,1,a}_\lambda(\mu, \phi) = N^{0}_\lambda(\mu, \phi) \) and \( M^{1,a}_\lambda(\lambda; \phi) = M^{\Delta}_\lambda(\lambda; \phi) \), respectively; refer to Tang et al. [27] and Ali et al. [46].

**Lemma 1.4** ([1, 47]) Let \( P \) be the class of all analytic functions \( h(z) \) of the following form

\[
h(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \Delta
\]

satisfying \( \Re h(z) > 0 \) and \( h(0) = 1 \). Then the sharp estimates \( |c_n| \leq 2 \) (\( n \in \mathbb{N} \)). Particularly, the equality holds for all \( n \) for the next function

\[
h(z) = \frac{1 + z}{1 - z} = 1 + \sum_{n=1}^{\infty} 2z^n.
\]

**2. Coefficient estimates for the class \( N^{a,b,c}_\lambda(\mu, \phi) \)**

Define the functions \( s \) and \( t \) in \( P \) by

\[
s(z) = \frac{1 + u(z)}{1 - u(z)} = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad t(w) = \frac{1 + v(w)}{1 - v(w)} = 1 + \sum_{n=1}^{\infty} d_n w^n, \quad z, w \in \Delta.
\]  

(2.1)

Therefore, from (2.1) we infer that

\[
u(z) = \frac{s(z) - 1}{s(z) + 1} = \frac{c_1}{1} z + \frac{1}{1} \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \cdots, \quad z \in \Delta
\]  

(2.2)

and

\[
v(w) = \frac{t(w) - 1}{t(w) + 1} = \frac{d_1}{1} w + \frac{1}{1} \left( d_2 - \frac{d_1^2}{2} \right) w^2 + \cdots, \quad w \in \Delta.
\]  

(2.3)

Let \( \phi \in P \) with \( \phi'(0) > 0 \) satisfying \( \phi(\Delta) \) being symmetric with respect to the real axis. Assume that the series expansion form of \( \phi \) is denoted by

\[
\phi(z) = 1 + \sum_{n=1}^{\infty} E_n z^n, \quad E_1 > 0, \quad z \in \Delta.
\]  

(2.4)

By (2.2–2.4), it follows that

\[
\phi(u(z)) = 1 + \frac{1}{2} E_1 c_1 z + [\frac{1}{2} E_1 (c_2 - \frac{c_1^2}{2}) + \frac{1}{4} E_2 c_1^2] z^2 + \cdots, \quad z \in \Delta
\]  

(2.5)

and

\[
\phi(v(w)) = 1 + \frac{1}{2} E_1 d_1 w + [\frac{1}{2} E_1 (d_2 - \frac{d_1^2}{2}) + \frac{1}{4} E_2 d_1^2] w^2 + \cdots, \quad w \in \Delta.
\]  

(2.6)

Now we consider the coefficient estimates for the class \( N^{a,b,c}_\lambda(\mu, \phi) \) and establish the next theorem.
Functions

Obviously, from (2.5), (2.6) and (2.10)–(2.13), we obtain that

$$|a_2| \leq \min \left\{ \frac{E_1}{(\lambda + \mu)p_2}, \sqrt{\frac{E_1}{2(\lambda + \mu)(\mu - 1)p_2^2 + p_3}}, \frac{E_1^{3/2}}{\sqrt{|\Phi|}} \right\} \quad (2.7)$$

and

$$|a_3| \leq \frac{E_1}{(\lambda + \mu)p_3} + \min \left\{ \frac{E_1^2}{(\lambda + \mu)^2p_2^2}, \frac{2(|E_1| + |E_2 - E_1|)}{(2\lambda + \mu)(\mu - 1)p_2^2 + 2p_3) \right\}, \quad (2.8)$$

where

$$\Phi = \Phi(\lambda, \mu, E_1, E_2, p_2, p_3) = (2\lambda + \mu)\frac{1}{2}(\mu - 1)p_2^2 + p_3|E_1^2 + (E_1 - E_2)(\lambda + \mu)^2p_2^2. \quad (2.9)$$

**Proof** Assume that \( f(z) \in \Lambda^{a,b,c}(\mu, \lambda; \phi) \). Hence, by Definition 1.1 there exist two analytic functions \( u(z), v(z) : \Delta \to \Delta \) with \( u(0) = 0 \) and \( v(0) = 0 \) so that

$$1 - \lambda \left( \frac{T_{a,b}f(z)}{z} \right)^\mu + \lambda(T_{a,b}f)'(z)\left( \frac{T_{a,b}f(z)}{z} \right)^{\mu-1} = \phi(u(z)) \quad (2.10)$$

and

$$1 - \lambda \left( \frac{T_{a,b}g(w)}{w} \right)^\mu + \lambda(T_{a,b}g)'(w)\left( \frac{T_{a,b}g(w)}{w} \right)^{\mu-1} = \phi(v(w)). \quad (2.11)$$

Expanding the left half parts of (2.10) and (2.11), we have that

$$1 - \lambda \left( \frac{T_{a,b}f(z)}{z} \right)^\mu + \lambda(T_{a,b}f)'(z)\left( \frac{T_{a,b}f(z)}{z} \right)^{\mu-1}$$

$$= 1 + (\lambda + \mu)p_2a_2z + (2\lambda + \mu)\left( \frac{1}{2}(\mu - 1)p_2^2a_2^2 + p_3a_3 \right)z^2 + \cdots \quad (2.12)$$

and

$$1 - \lambda \left( \frac{T_{a,b}g(w)}{w} \right)^\mu + \lambda(T_{a,b}g)'(w)\left( \frac{T_{a,b}g(w)}{w} \right)^{\mu-1}$$

$$= 1 - (\lambda + \mu)p_2a_2w + (2\lambda + \mu)\left( \frac{1}{2}[(\mu - 1)p_2^2 + 4p_3)a_2^2 - p_3a_3 \right)w^2 + \cdots \quad (2.13)$$

Obviously, from (2.5), (2.6) and (2.10)–(2.13), we obtain that

$$\lambda + \mu)p_2a_2 = \frac{E_1c_1}{2}, \quad (2.14)$$

$$(2\lambda + \mu)\left[ \frac{1}{2}(\mu - 1)p_2^2a_2^2 + p_3a_3 \right] = \frac{1}{2}(c_2 - \frac{c_1^2}{2})E_1 + \frac{1}{4}c_1^2E_2, \quad (2.15)$$

$$-(\lambda + \mu)p_2a_2 = \frac{E_1d_1}{2} \quad (2.16)$$

and

$$2(\lambda + \mu)\left[ \frac{1}{2}[(\mu - 1)p_2^2 + 4p_3)a_2^2 - p_3a_3 \right] = \frac{1}{2}(d_2 - \frac{d_1^2}{2})E_1 + \frac{1}{4}d_1^2E_2. \quad (2.17)$$

From (2.14) and (2.16), we know that

$$a_2 = \frac{E_1c_1}{2p_2(\lambda + \mu)} = -\frac{E_1d_1}{2p_2(\lambda + \mu)} = \frac{E_1d_1}{2p_2(\lambda + \mu)}. \quad (2.18)$$
which derives
\[ c_1 = -d_1 \]  
(2.19)
and
\[ E_1^2(c_1^2 + d_1^2) = 8(\lambda + \mu)^2p_2^2a_2^2. \]  
(2.20)
By (2.15) and (2.17), we have that
\[ c_1^2(E_2 - E_1) + E_1(c_2 + d_2) = 2(2\lambda + \mu)[(\mu - 1)p_2^2 + 2p_3]a_1^2. \]  
(2.21)
Therefore, from (2.19)-(2.21) we obtain
\[ a_2^2 = \frac{(c_2 + d_2)E_1^3}{2(2\lambda + \mu)[(\mu - 1)p_2^2 + 2p_3](E_1^2 + 4(E_1 - E_2)(\lambda + \mu)^2p_2^2)}. \]  
(2.22)
Hence, by Lemma 1.4 we may remark that
\[ |a_2| \leq \frac{E_1^{3/2}}{\sqrt{|\Phi|}}. \]
In addition, from (2.20) and (2.21) we get that
\[ |a_2| \leq \frac{E_1}{(\lambda + \mu)|p_2|} \]
and
\[ |a_2| \leq \sqrt{\frac{E_1 + |E_2 - E_1|}{(2\lambda + \mu)(\mu - 1)p_2^2 + p_3}}, \]
which yield the desired results on $|a_2|$ in (2.7).
Similarly, (2.15) and (2.17) imply that
\[ E_1(c_2 - d_2) = 4(2\lambda + \mu)p_3(a_3 - a_2^2). \]  
(2.23)
Then, by (2.19), (2.20) and (2.23), it follows that
\[ a_3 = \frac{E_1(c_2 - d_2)}{4(2\lambda + \mu)p_3} + \frac{E_1^2(c_1^2 + d_1^2)}{8(\lambda + \mu)^2p_2^2}. \]
So, we obtain from Lemma 1.4 that
\[ |a_3| \leq \frac{E_1}{(2\lambda + \mu)|p_3|} + \frac{E_1^2}{(\lambda + \mu)^2p_2^2}. \]
On the other hand, by (2.21) and (2.23) we infer that
\[ a_3 = \frac{2[c_1^2(E_2 - E_1) + E_1(c_2 + d_2)|p_3 + E_1(c_2 - d_2)[(\mu - 1)p_2^2 + 2p_3]}{4(2\lambda + \mu)|p_3|[(\mu - 1)p_2^2 + 2p_3}|. \]
Thus, from Lemma 1.4 we see that
\[ |a_3| \leq \frac{E_1}{(2\lambda + \mu)|p_3|} + \frac{2(E_1 + |E_2 - E_1|)}{(2\lambda + \mu)(\mu - 1)p_2^2 + 2p_3} \]
\[ \Box \]
When $\mu = 1$, $\mathcal{A}_{\Sigma}^{a,b,c}((1, \lambda; \phi)) = \mathcal{A}_{\Sigma}^{a,b,c}(\lambda; \phi)$. Hence, by Theorem 2.1 we immediately get the next corollary.
If $f(z)$ given by (1.1) belongs to the class $N^{a,b,c}_\Sigma(\lambda; \phi)$, then

$$|a_2| \leq \min\left\{ \frac{E_1}{(1+\lambda)|p_2|}, \sqrt{\frac{E_1 + |E_2 - E_1|}{(1+2\lambda)|p_3|}}, \frac{E_1^{3/2}}{|\Phi|} \right\}$$

and

$$|a_3| \leq \frac{E_1}{(1+2\lambda)|p_3|} + \min\left\{ \frac{E_1^2}{(1+\lambda)^2|p_2|^2}, \frac{(E_1 + |E_2 - E_1|)}{(1+2\lambda)|p_3|} \right\},$$

where

$$\Phi = \Phi(\lambda, E_1, E_2, p_2, p_3) = (1+2\lambda)E_1^2p_3 + (E_1 - E_2)(1+\lambda)^2p_2^2.$$ 

**Remark 2.3** Moreover, under the conditions of the parameters $a = c$ and $b = 1$ and Remark 1.3, if we choose some suitable parameters $\mu$ and $\lambda$ as well as $\phi$, we also provide the following reduced versions for $N^{a,b,c}_\Sigma(\mu, \lambda; \phi)$ in Theorem 2.1:

(i) $N^{a,1,a}_\Sigma(\mu, \lambda; \alpha) = H(a, \theta, \lambda, \alpha) = H(a, \lambda, \beta)$, refer to Çağlar et al. [48];

(ii) $N^{a,1,a}_\Sigma(1, \lambda; \phi) = H(a, \lambda, \phi), N^{a,1,a}_\Sigma(\mu, 1; \phi) = H\Sigma(\phi)$, refer to Kumar et al. [49];

(iii) $N^{a,1,a}_\Sigma(1, 1; \phi) = H(a, \lambda, \beta)$, refer to Ali et al. [46];

(iv) $N^{a,1,a}_\Sigma(1, \lambda; \alpha) = B(\alpha, \lambda, \beta), N^{a,1,a}_\Sigma(1, \lambda; \beta) = B(\beta, \lambda)$, refer to Frasin and Aouf [18];

(v) $N^{a,1,a}_\Sigma(1, 1; \alpha) = B(\alpha, \lambda, \beta), N^{a,1,a}_\Sigma(1, 1; \beta) = B(\beta, \lambda)$, refer to Srivastava et al. [16].

Next, we will consider Fekete-Szegö functional problems for the class $N^{a,b,c}_\Sigma(\mu, \lambda; \phi)$.

**Corollary 2.4** If $f(z)$ given by (1.1) belongs to the class $N^{a,b,c}_\Sigma(\mu, \lambda; \phi)$ and $\rho \in \mathbb{R}$, then

$$|a_3 - \rho a_2^2| \leq \begin{cases} \frac{E_1}{(1+\rho)p_3}, & \text{if } (2\lambda + \mu)(1-\rho)p_3|E_1^2| \leq |\Phi| \\ \frac{E_1}{|\Phi|}, & \text{if } (2\lambda + \mu)(1-\rho)p_3|E_1^2| \geq |\Phi|, \end{cases}$$

where $\Phi = \Phi(\lambda, \mu, E_1, E_2, p_2, p_3)$ is the same as in Theorem 2.1.

**Proof** From (2.23), it follows that

$$a_3 - a_2^2 = \frac{E_1(c_2 - d_2)}{4(2\lambda + \mu)p_3}.$$ 

By (2.22) we easily obtain that

$$a_3 - \rho a_2^2 = \frac{E_1\{(1-\rho)(2\lambda + \mu)p_3E_1^2 + \Phi\}c_2 + [(1-\rho)(2\lambda + \mu)p_3E_1^2 - \Phi]d_2}{4(2\lambda + \mu)p_3\Phi}.$$ 

Hence, from Lemma 1.4 it follows

$$|a_3 - \rho a_2^2| \leq \begin{cases} \frac{E_1}{(1+\rho)p_3}, & \text{if } (2\lambda + \mu)(1-\rho)p_3|E_1^2| \leq |\Phi|; \\ \frac{E_1}{|\Phi|}, & \text{if } (2\lambda + \mu)(1-\rho)p_3|E_1^2| \geq |\Phi|. \end{cases}$$

**Corollary 2.5** If $f(z)$ given by (1.1) belongs to the class $N^{a,b,c}_\Sigma(\mu, \lambda; \phi)$, then

$$|a_3 - a_2^2| \leq \frac{E_1}{(2\lambda + \mu)p_3},$$ 

**Corollary 2.6** If $f(z)$ given by (1.1) belongs to the class $N^{a,b,c}_\Sigma(\mu, \lambda; \phi)$, then

$$|a_3| \leq \begin{cases} \frac{E_1}{(2\lambda + \mu)p_3}, & \text{if } (2\lambda + \mu)|p_3|E_1^2 \leq |\Phi|; \\ \frac{E_1}{|\Phi|}, & \text{if } (2\lambda + \mu)|p_3|E_1^2 \geq |\Phi|. \end{cases}$$
where $\Phi = \Phi(\lambda, \mu, E_1, E_2, p_2, p_3)$ is the same as in Theorem 2.1.

**Remark 2.7** Without Hohlav operator, we may refer to the subclass $B_{\Sigma,m}(\lambda; \phi)$ of $m$-fold symmetric bi-univalent functions (see Tang et al. [28] for Remark 2.7).

Hence, with (2.5), (2.6) and (3.4)–(3.7), we deduce that

3. Coefficient estimates for the class $M_{\Sigma}^{a,b,c}(\lambda; \phi)$

Now we study the coefficient estimates for the class $M_{\Sigma}^{a,b,c}(\lambda; \phi)$ and give the next theorem.

**Theorem 3.1** If $f(z)$ given by (1.1) belongs to the class $M_{\Sigma}^{a,b,c}(\lambda; \phi)$, then

$$|a_2| \leq \min \left\{ \frac{E_1}{(1 + \lambda)|p_2|}, \sqrt{\frac{E_1 + |E_2 - E_1|}{2(1 + 2\lambda)p_3 - (1 + 3\lambda)p_2^2}}, \frac{E_1^3/2}{\sqrt{\Theta}} \right\}$$

and

$$|a_3| \leq \frac{E_1}{2(1 + 2\lambda)|p_3|} + \min \left\{ \frac{E_1^2}{(1 + \lambda)^2p_2^2}, \frac{E_1 + |E_2 - E_1|}{2(1 + 2\lambda)p_3 - (1 + 3\lambda)p_2^2} \right\},$$

where

$$\Theta = \Theta(\lambda, E_1, E_2, p_2, p_3) = |2(1 + 2\lambda)p_3 - (1 + 3\lambda)p_2^2|E_1^2 + (E_1 - E_2)(1 + \lambda)^2p_2^2.$$

**Proof** Assume that $f(z) \in M_{\Sigma}^{a,b,c}(\lambda; \phi)$. Then, by Definition 1.2 there exist two analytic functions $u(z), v(z) : \Delta \rightarrow \Delta$ with $u(0) = 0$ and $v(0) = 0$ so that

$$(1 - \lambda)\frac{z(T_c^a b f)'(z)}{T_c^a b f(z)} + \lambda(1 + \frac{z(T_c^a b f)''(z)}{T_c^a b f(z)}) = \phi(u(z))$$

and

$$(1 - \lambda)\frac{w(T_c^a b g)'(w)}{T_c^a b g(w)} + \lambda(1 + \frac{w(T_c^a b g)''(w)}{(T_c^a b g)'(w)}) = \phi(v(w)).$$

Expanding the left half parts of (3.4) and (3.5), we obtain that

$$= 1 + (1 + \lambda)p_2a_2 + [2(1 + 2\lambda)p_3a_3 - (1 + 3\lambda)p_2^2a_2^2]z^2 + \cdots$$

and

$$= 1 - (1 + \lambda)p_2a_2 + \{2(1 + 2\lambda)p_3(2a_2^3 - a_3) - (1 + 3\lambda)p_2^2a_2^2\}w^2 + \cdots.$$

Hence, with (2.5), (2.6) and (3.4)–(3.7), we deduce that

$$E_1c_1 = \frac{E_1c_1}{2},$$

$$2(1 + 2\lambda)p_3a_3 - (1 + 3\lambda)p_2^2a_2^2 = \frac{1}{2}(c_2 - \frac{c_1^2}{2})E_1 + \frac{1}{4}c_1^2E_2.$$
\[-(1 + \lambda)p_2 a_2 = \frac{E_1 d_1}{2}\] (3.10)

and

\[2(1 + 2\lambda)p_3(2a_3^2 - a_3) - (1 + 3\lambda)p_2 a_2^2 = \frac{1}{2} (d_2 - \frac{d_1^2}{2}) E_1 + \frac{1}{4} d_1^2 E_2.\] (3.11)

From (3.8) and (3.10), we know that

\[a_2 = \frac{E_1 c_1}{2p_2(1 + \lambda)} = - \frac{E_1 d_1}{2p_2(1 + \lambda)},\] (3.12)

which implies

\[c_1 = -d_1\] (3.13)

and

\[E_1^2 (c_1^2 + d_1^2) = 8(1 + \lambda)^2 p_2^2 a_2^2.\] (3.14)

By (3.9), (3.11) and (3.13), we have that

\[c_1^2(E_2 - E_1) + E_1(c_2 + d_2) = 8(1 + 2\lambda)p_3 a_2^2 - 4(1 + 3\lambda)p_2 a_2^2.\] (3.15)

Therefore, from (3.12)–(3.15) we obtain

\[a_2 = \frac{(c_2 + d_2)E_1^3}{8(1 + 2\lambda)p_3 - 4(1 + 3\lambda)p_2^2 |E_1^2 + (E_1 - E_2)(1 + \lambda)^2 p_2^2|}.\] (3.16)

Hence, by Lemma 1.4 we derive

\[|a_2| \leq \frac{E_1^3/2}{\sqrt{|2(1 + 2\lambda)p_3 - (1 + 3\lambda)p_2^2 |E_1^2 + (E_1 - E_2)(1 + \lambda)^2 p_2^2|}}.\]

In addition, from (3.14) and (3.15) we get that

\[|a_2| \leq \frac{E_1}{(1 + \lambda)|p_2|}\]

and

\[|a_2| \leq \sqrt{\frac{E_1 + |E_2 - E_1|}{2(1 + 2\lambda)p_3 - (1 + 3\lambda)p_2^2}},\]

which yield the desired results on \(|a_2|\) in (3.1).

Similarly, from (3.9) and (3.11), it follows

\[E_1(c_2 - d_2) = 8(1 + 2\lambda)p_3 (a_3 - a_2^2).\] (3.17)

Then, by (3.13), (3.14) and (3.17), one gets

\[a_3 = \frac{E_1(c_2 - d_2)}{8(1 + 2\lambda)p_3} + \frac{E_1^2 (c_1^2 + d_1^2)}{8(1 + \lambda)^2 p_2^2}\]

So, we obtain from Lemma 1.4 that

\[|a_3| \leq \frac{E_1}{2(1 + 2\lambda)|p_3|} + \frac{E_1^2}{(1 + \lambda)^2 p_2^2}.\]
On the other hand, by (3.15) and (3.17) we infer that

\[
a_3 = \frac{E_1(c_2 - d_2)}{8(1 + 2\lambda)p_3} + \frac{c_1^2(E_2 - E_1) + E_1(c_2 + d_2)}{8(1 + 2\lambda)^2p_3 - 4(1 + 3\lambda)p_2^2}.
\]

Thus, from Lemma 1.4 we see that

\[
|a_3| \leq \frac{E_1}{2(1 + 2\lambda)|p_3|} + \frac{E_1 + |E_2 - E_1|}{2(1 + 2\lambda)p_3 - (1 + 3\lambda)p_2^2}. \quad \Box
\]

**Remark 3.2** Clearly, under the conditions of the parameters \(a = c\) and \(b = 1\) and Remark 1.3, if we take some suitable parameter \(\lambda\) and \(\phi\), we also provide the following reduced versions for \(\mathcal{M}^{a,b,c}_\Sigma(\mu;\lambda;\phi)\) in Theorem 3.1:

(i) \(\mathcal{M}^{a,1,a}_\Sigma(\phi) = \mathcal{M}^\Sigma(\phi)\), refer to Ali et al. [46];

(ii) \(\mathcal{M}^{a,1,a}_\Sigma(1;\alpha), \mathcal{M}^{a,1,a}_\Sigma(1;\beta)\), or \(\mathcal{M}^{a,1,a}_\Sigma(0;\alpha)\) and \(\mathcal{M}^{a,1,a}_\Sigma(0;\beta)\), refer to Brannan and Taha [15].

**Theorem 3.3** If \(f(z)\) given by (1.1) belongs to the class \(\mathcal{M}^{a,b,c}_\Sigma(\lambda;\phi)\) and \(\rho \in \mathbb{R}\), then

\[
|a_3 - \rho a_2^2| \leq \frac{E_1}{4(1 + 2\lambda)|p_3|}, \quad \text{if} \quad 2(1 + 2\lambda)(1 - \rho)p_3|E_2^2| \leq |\Theta|;
\]

\[
\left\{ \frac{E_1}{4(1 + 2\lambda)|p_3|}, \quad \text{if} \quad 2(1 + 2\lambda)(1 - \rho)p_3|E_2^2| > |\Theta|, \right. \quad \text{if} \quad 2(1 + 2\lambda)(1 - \rho)p_3|E_2^2| \geq |\Theta|\right\}.
\]

where \(\Theta = \Theta(\lambda, E_1, E_2, p_2, p_3)\) is the same as in Theorem 3.1.

**Proof** From (3.17), it follows that

\[
a_3 - a_2^2 = \frac{E_1(c_2 - d_2)}{8(1 + 2\lambda)p_3}.
\]

By (3.16) we easily obtain that

\[
a_3 - \rho a_2^2 = \frac{E_1([2(1 - \rho)(1 + 2\lambda)p_3|E_2^2| + \Theta|c_2 + |2(1 - \rho)(1 + 2\lambda)p_3|E_2^2 - \Theta|d_2]}}{8(1 + 2\lambda)p_3}. \quad (3.19)
\]

Then, from Lemma 1.4 we show that

\[
|a_3 - \rho a_2^2| \leq \left\{ \frac{E_1}{4(1 + 2\lambda)|p_3|}, \quad \text{if} \quad 2(1 + 2\lambda)(1 - \rho)p_3|E_2^2| \leq |\Theta|; \right. \quad \text{if} \quad 2(1 + 2\lambda)(1 - \rho)p_3|E_2^2| \geq |\Theta|, \quad \Box
\]

**Corollary 3.4** If \(f(z)\) given by (1.1) belongs to the class \(\mathcal{M}^{a,b,c}_\Sigma(\lambda;\phi)\), then

\[
|a_3 - a_2^2| \leq \frac{E_1}{4(1 + 2\lambda)|p_3|}.
\]

**Corollary 3.5** If \(f(z)\) given by (1.1) belongs to the class \(\mathcal{M}^{a,b,c}_\Sigma(\lambda;\phi)\), then

\[
|a_3| \leq \left\{ \frac{E_1}{4(1 + 2\lambda)|p_3|}, \quad \text{if} \quad 2(1 + 2\lambda)|p_3|E_2^2| \leq |\Theta|; \right. \quad \text{if} \quad 2(1 + 2\lambda)|p_3|E_2^2| \geq |\Theta|\right\}
\]

where \(\Theta = \Theta(\lambda, E_1, E_2, p_2, p_3)\) is the same as in Theorem 3.3.

**Remark 3.6** Similarly, without Hohlov operator we may refer to the subclass \(\mathcal{M}^{a,b,c}_\Sigma(\lambda;\phi)\) of \(m\)-fold symmetric bi-univalent functions (see Tang et al. [28] for \(m = 1\)) for Fekete-Szegö functional problems about \(\mathcal{M}^{a,b,c}_\Sigma(\lambda;\phi)\).
Acknowledgements

We thank the referees for their time and comments so that this article is greatly improved.

References


