Journal of Mathematical Research with Applications
Jan., 2020, Vol. 40, No. 1, pp. 13–25
DOI:10.3770/j.issn:2095-2651.2020.01.002
Http://jmre.dlut.edu.cn

Explicit Iterative Sequences of Positive Solutions for a Class of Fractional Differential Equations on an Infinite Interval

Haiyan ZHANG[1,2,*], **Yaohong LI**[1]
1. *School of Mathematics and Statistics, Suzhou University, Anhui* 234000*, P. R. China;*
2. *School of Mathematics Sciences, Anhui University, Anhui* 230601*, P. R. China*

Abstract By applying the monotone iterative method, this study develops two explicit monotone iterative sequences for approximating the minimal and maximal positive solutions. At the same time, by applying the Banach fixed-point theory, an explicit iterative sequence and error estimate for approximating the unique positive solution is obtained. Some examples are given to illustrate the application of the results.

Keywords monotone iterative method; iterative sequences; fractional differential equations; infinite interval

MR(2010) Subject Classification 34A08; 34B18

1. Introduction

Fractional calculus has gained wide attention from both theoretical and applied perspectives over the last few decades. A detailed description of the subject can be found in [1–5]. We note that most of the available literature on the solvability of fractional differential equations are focused on the finite interval. Few papers discuss fractional differential equations on the infinite intervals [6–12]. Zhao and Ge [6] applied the Schauder's fixed-point theorem to study the existence of positive solutions for the following nonlocal fractional boundary value problem:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \quad 1 < \alpha \le 2, \\ u(0) = 0, \lim_{t\to+\infty} D_{0^+}^{\alpha-1} u(t) = \beta u(\xi), \end{cases} \tag{1.1}$$

where $t \in J = [0, +\infty)$, $f \in C(J \times \mathbb{R}, J)$, $0 < \xi < +\infty$, $D_{0^+}^{\alpha}$ is the standard Riemann-Liouville fractional derivative of order α. Su and Zhang [7] applied the same method as [6] to obtain the existence result of solutions for a boundary value problem of fractional order:

$$\begin{cases} D_{0^+}^{\alpha} u(t) + f(t, u(t), D_{0^+}^{\alpha-1} u(t)) = 0, \quad 1 < \alpha \le 2, \\ u(0) = 0, \lim_{t\to+\infty} D_{0^+}^{\alpha-1} u(t) = u_{\infty}, \quad u_{\infty} \in \mathbb{R}, \end{cases} \tag{1.2}$$

where $t \in J = [0, +\infty)$, $f \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

Received October 17, 2018; Accepted April 11, 2019
Supported by the University Natural Science Foundation of Anhui Provincial Education Department (Grant Nos. KJ2017A442; KJ2018A0452; KJ2019A0666) and the Foundation of Suzhou University (Grant Nos. 2016XJGG13; 2019XJZY02; 2019XJSN03).
* Corresponding author
E-mail address: liz.zhang@yeah.net (Haiyan ZHANG); liz.zhanghy@163.com (Yaohong LI)

By using the Schauder's fixed-point theorem and the Banach's contraction mapping principle, the authors in [8] found sufficient conditions for the existence and uniqueness of solutions for the boundary value problem:

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \quad 2 < \alpha \le 3, \\ u(0) = u'(0) = 0, D^{\alpha-1}_{0+}u(\infty) = \xi I^{\beta}_{0+}u(\eta), \quad \beta > 0, \end{cases} \tag{1.3}$$

where $t \in J = [0, +\infty)$, $f \in C(J \times \mathbb{R}, \mathbb{R})$, $\xi \in \mathbb{R}$, $\eta \in J$, D^{α}_{0+} is the Riemann-Liouville fractional derivatives of order α, and I^{β}_{0+} is the Riemann-Liouville fractional integral of order β.

Liang and Zhang [9] considered the following nonlinear fractional differential equations with multipoint fractional boundary conditions on an unbounded domain:

$$\begin{cases} D^{\alpha}_{0+}u(t) + a(t)f(t, u(t)) = 0, \quad 2 < \alpha \le 3, \\ u(0) = u'(0) = 0, D^{\alpha-1}_{0+}u(+\infty) = \sum_{i=1}^{m-2} \beta_i u(\xi_i), \end{cases} \tag{1.4}$$

where $t \in J = [0, +\infty)$, D^{α}_{0+}, $D^{\alpha-1}_{0+}$ are the Caputo fractional derivatives, and $0 < \xi_1 < \xi_2 < \ldots < \xi_{m-2} < +\infty$ and $\beta_i > 0$, $i = 1, 2, \ldots, m-2$, satisfy $0 < \sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha-1} < \Gamma(\alpha)$. By using the fixed-point index theory, the authors provided sufficient conditions for the existence of multiple positive solutions to the above multi-point fractional boundary value problem.

Motivated by the aforementioned papers, an interesting and important question is if we know the existence of the solution, how can we seek it? This question motivates the study of iterative sequences of positive solutions for the following fractional boundary value problem (FBVP):

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t, u(t), D^{\alpha-1}_{0+}u(t)) = 0, \quad n-1 < \alpha \le n, \\ u^{(j)}(0) = 0 (j = 0, 1, 2, \ldots, n-2), D^{\alpha-1}_{0+}u(+\infty) = \int_0^{+\infty} h(t)u(t)\mathrm{d}t, \end{cases} \tag{1.5}$$

where $t \in J = [0, +\infty)$, $f \in C(J \times \mathbb{R} \times \mathbb{R}, J)$, $h(t) \in L[0, +\infty)$, D^{α}_{0+} is the standard Riemann-Liouville fractional derivative of order α. Here, we emphasize that the nonlinearity term depends on the unknown function's lower-order fractional derivative and the boundary value depends on the unknown function's integral.

By applying the monotone iterative method in this study, we first develop two computable explicit monotone iterative sequences for approximating the minimal and maximal positive solutions. We also obtain an explicit iterative sequence for approximating the unique positive solution and provide an error estimate for approximation, which is more interesting and valuable for devising the auxiliary routine that establishes the existence of solutions for the problem at hand and provides method for obtaining solutions. For the application and details of the method, refer to [13–17] and the references therein. Furthermore, the nonlinearity term of our research depends on the unknown function's lower-order fractional derivative, which is different from [6, 8, 9]. Finally, the main results extend the fractional derivative from the low-order to the high-order fractional derivatives.

The remainder of this paper is organized as follows: Section 2 presents basic definitions and related lemmas that will be used. In Section 3, the main results and proof are presented. In Section 4, two examples are provided to illustrate the main results.

2. Preliminaries and lemmas

First, we introduce the assumptions that will play an important role in our main results.

(H_1) $h(t) \in L[0,+\infty)$ and $\int_0^{+\infty} h(t)t^{\alpha-1}\mathrm{d}t := \Delta < \Gamma(\alpha), f(t,0,0) \not\equiv 0, \forall t \in J.$

(H_2) There exist nonnegative integrable functions $a(t), b(t), c(t)$ defined on $[0,+\infty)$ and constants $p, q \geq 0$, such that

$$|f(t,u,v)| \leq a(t) + b(t)|u|^p + c(t)|v|^q,$$

and

$$\int_0^{+\infty} a(t)\mathrm{d}t = a^* < +\infty, \quad \int_0^{+\infty} b(t)(1+t^{\alpha-1})^p\mathrm{d}t = b^* < +\infty, \quad \int_0^{+\infty} c(t)\mathrm{d}t = c^* < +\infty.$$

(H_3) f is nondecreasing with respect to the second and last variables.

(H_4) There exist nonnegative integrable functions $d(t), e(t)$ defined on $[0,+\infty)$, such that

$$|f(t,u,v) - f(t,u',v')| \leq d(t)|u-u'| + e(t)|v-v'|, \quad t \in J, \ u,u',v,v' \in R,$$

and

$$\int_0^{+\infty} d(t)(1+t^{\alpha-1})\mathrm{d}t = d^* < +\infty, \int_0^{+\infty} e(t)\mathrm{d}t = e^* < +\infty, \quad \int_0^{+\infty} |f(t,0,0)|\mathrm{d}t = \lambda < +\infty.$$

Secondly, we present definitions and lemmas that are useful to the proof of main results.

Definition 2.1 ([2,3]) *The Riemann-Liouville fractional derivative of order α for a continuous function f is given by*

$$D_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)}(\frac{\mathrm{d}}{\mathrm{d}x})^n \int_0^x (x-t)^{n-\alpha-1} f(t)\mathrm{d}t, \quad \alpha > 0, \ n = [\alpha]+1,$$

provided that the right-hand side is pointwise defined on $(0,+\infty)$.

Definition 2.2 ([2,3]) *The Riemann-Liouville fractional integral of order α for a function f is given by*

$$I_{0+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\mathrm{d}t, \quad \alpha > 0,$$

provided that the integral exists.

Lemma 2.3 *Let $y \in C[0,+\infty)$ with $\int_0^{+\infty} h(t)t^{\alpha-1}\mathrm{d}t \neq \Gamma(\alpha)$. For $n-1 < \alpha \leq n$, then the fractional boundary value problem*

$$\begin{cases} D_{0+}^{\alpha} u(t) + y(t) = 0, 0 < t < +\infty, \\ u^{(j)}(0) = 0 (j = 0,1,2,\ldots,n-2), D_{0+}^{\alpha-1} u(+\infty) = \int_0^{+\infty} h(t)u(t)\mathrm{d}t, \end{cases} \tag{2.1}$$

has a unique solution

$$u(t) = \int_0^{\infty} G(t,s)y(s)\mathrm{d}s, \tag{2.2}$$

where

$$G(t,s) = G_1(t,s) + G_2(t,s), \tag{2.3}$$

with

$$G_1(t,s)=\frac{1}{\Gamma(\alpha)}\begin{cases} t^{\alpha-1}-(t-s)^{\alpha-1}, 0\le s\le t\le +\infty,\\ t^{\alpha-1}, 0\le t\le s\le +\infty,\end{cases}$$

$$G_2(t,s)=\frac{t^{\alpha-1}}{\Gamma(\alpha)-\Delta}\int_0^{+\infty}h(t)G_1(t,s)\mathrm{d}t. \tag{2.4}$$

Proof This proof is similar to [18, Lemma 2.3], so we omit it. □

Remark 2.4 From (2.2)–(2.4), we have

$$D_{0^+}^{\alpha-1}u(t)=\int_0^{\infty}G^*(t,s)y(s)\mathrm{d}s,$$

where

$$G^*(t,s)=G_1^*(t,s)+G_2^*(t,s), \tag{2.5}$$

with

$$G_1^*(t,s)=\begin{cases} 0, 0\le s\le t\le +\infty,\\ 1, 0\le t\le s\le +\infty,\end{cases}$$

$$G_2^*(t,s)=\frac{\Gamma(\alpha)}{\Gamma(\alpha)-\Delta}\int_0^{+\infty}h(t)G_1(t,s)\mathrm{d}t.$$

Lemma 2.5 *For $(s,t)\in J\times J$, if condition (H_1) satisfies, then*

$$0\le G(t,s)\le \frac{t^{\alpha-1}}{\Gamma(\alpha)-\Delta},\quad 0\le \frac{G(t,s)}{1+t^{\alpha-1}}\le \frac{1}{\Gamma(\alpha)-\Delta},$$

and

$$0\le G^*(t,s)\le \frac{\Gamma(\alpha)}{\Gamma(\alpha)-\Delta}.$$

Proof From (2.3), we can easily determine that

$$0\le G_1(t,s)\le \frac{t^{\alpha-1}}{\Gamma(\alpha)},\quad \forall (t,s)\in J\times J,$$

and

$$0\le G_2(t,s)\le \frac{t^{\alpha-1}}{\Gamma(\alpha)-\Delta}\int_0^{+\infty}\frac{h(t)t^{\alpha-1}}{\Gamma(\alpha)}\mathrm{d}t=\frac{\Delta t^{\alpha-1}}{\Gamma(\alpha)(\Gamma(\alpha)-\Delta)},\quad \forall (t,s)\in J\times J,$$

then

$$0\le G(t,s)=G_1(t,s)+G_2(t,s)\le \frac{t^{\alpha-1}}{\Gamma(\alpha)-\Delta},\quad \forall (t,s)\in J\times J.$$

So

$$0\le \frac{G(t,s)}{1+t^{\alpha-1}}\le \frac{1}{\Gamma(\alpha)-\Delta},\quad \forall (t,s)\in J\times J.$$

By direct calculation from (2.5), we can easily obtain

$$0\le G^*(t,s)=G_1^*(t,s)+G_2^*(t,s)\le 1+\frac{\Delta}{\Gamma(\alpha)-\Delta}=\frac{\Gamma(\alpha)}{\Gamma(\alpha)-\Delta},\quad \forall (t,s)\in J\times J.$$

Therefore, the proof is completed. □

Define two Banach spaces

$$X=\{u\in C(J,\mathbb{R}):\sup_{t\in J}\frac{|u(t)|}{1+t^{\alpha}}<+\infty\},$$

$$Y = \{u \in X : D_{0^+}^{\alpha-1}u(t) \in C(J, \mathbb{R}), \sup_{t \in J} |D_{0^+}^{\alpha-1}u(t)| < +\infty\}$$

equipped with norms $\|u\|_X = \sup_{t\in J} \frac{|u(t)|}{1+t^\alpha}$ and $\|u\|_Y = \max\{\|u\|_X, \sup_{t\in J} |D_{0^+}^{\alpha-1}u(t)|\}$.

Lemma 2.6 *If condition* (H_2) *is satisfied, then*

$$\int_0^{+\infty} |f(s, u(s), D_{0^+}^{\alpha-1}u(s))|\mathrm{d}s \le a^* + b^*\|u\|_Y^p + c^*\|u\|_Y^q, \quad \forall u \in Y.$$

Proof For $\forall u \in Y$, by condition (H_2), we have

$$\begin{aligned}
&\int_0^{+\infty} |f(s, u(s), D_{0^+}^{\alpha-1}u(s))|\mathrm{d}s \le \int_0^{+\infty} [a(s) + b(s)|u(s)|^p + c(s)|D_{0^+}^{\alpha-1}u(s)|^q]\mathrm{d}s \\
&\le a^* + \int_0^{+\infty} b(s)(1+s^{\alpha-1})^p \frac{|u(s)|^p}{(1+s^{\alpha-1})^p}\mathrm{d}s + \int_0^{+\infty} c(s)|D_{0^+}^{\alpha-1}u(s)|^q\mathrm{d}s \\
&\le a^* + b^*\|u\|_Y^p + c^*\|u\|_Y^q. \quad \square
\end{aligned}$$

Lemma 2.7 *If condition* (H_4) *is satisfied, then*

$$\int_0^{+\infty} |f(s, u(s), D_{0^+}^{\alpha-1}u(s))|\mathrm{d}s \le (d^* + e^*)\|u\|_Y + \lambda, \quad \forall u \in Y.$$

Proof For $\forall u \in Y$, by condition (H_4), we have

$$\begin{aligned}
&\int_0^{+\infty} |f(s, u(s), D_{0^+}^{\alpha-1}u(s))|\mathrm{d}s = \int_0^{+\infty} |f(s, u(s), D_{0^+}^{\alpha-1}u(s)) - f(s,0,0) + f(s,0,0)|\mathrm{d}s \\
&\le \int_0^{+\infty} |f(s, u(s), D_{0^+}^{\alpha-1}u(s)) - f(s,0,0)|\mathrm{d}s + \int_0^{+\infty} |f(s,0,0)|\mathrm{d}s \\
&\le \int_0^{+\infty} d(s)(1+t^{\alpha-1})\frac{|u(s)|}{1+t^{\alpha-1}}\mathrm{d}s + \int_0^{+\infty} e(s)|D_{0^+}^{\alpha-1}u(s)|\mathrm{d}s + \int_0^{+\infty} |f(s,0,0)|\mathrm{d}s \\
&\le (d^* + e^*)\|u\|_Y + \lambda. \quad \square
\end{aligned}$$

Lemma 2.8 ([8]) *Let $U \subset X$ be a bounded set. Then U is a relatively compact in X if the following conditions hold:*

(i) For any $u(t) \in U$, $\frac{u(t)}{1+t^{\alpha-1}}$ and $D_{0^+}^{\alpha-1}u(t)$ are equicontinuous on any compact interval of J;

(ii) For any $\varepsilon > 0$, there is a constant $T = T(\varepsilon) > 0$ such that $|\frac{u(t_1)}{1+t_1^{\alpha-1}} - \frac{u(t_2)}{1+t_2^{\alpha-1}}| < \varepsilon$ and $|D_{0^+}^{\alpha-1}u(t_1) - D_{0^+}^{\alpha-1}u(t_2)| < \varepsilon$ for any $t_1, t_2 \ge T$ and $u \in U$.

Define an operator T associated with FBVP (1.5) by

$$Tu(t) = \int_0^{\infty} G(t,s) f(t, u(t), D_{0^+}^{\alpha-1}u(t))\mathrm{d}s. \tag{2.6}$$

Applying Lemma 2.3, it is easy to prove that FBVP (1.5) has a solution if and only if the operator equation $u = Tu$ has a fixed point, where T is given by (2.6).

Lemma 2.9 *Assume that the conditions* (H_1) *and* (H_2) *are satisfied. Then the operator T : $Y \to Y$ is completely continuous.*

Proof We first show that the operator $T : Y \to Y$ is relatively compact.

(i) Let $\Omega=\{u|u\in Y,\|u\|_Y\le M\}$. For $\forall u\in\Omega$, by Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned}\|Tu\|_X&=\sup_{t\in J}\Big|\int_0^{+\infty}\frac{G(t,s)}{1+t^{\alpha-1}}f(s,u(s),D_{0^+}^{\alpha-1}u(s))\mathrm{d}s\Big|\\&\le\frac{1}{\Gamma(\alpha)-\Delta}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|\mathrm{d}s\\&\le\frac{1}{\Gamma(\alpha)-\Delta}[a^*+b^*M^p+c^*M^q]\end{aligned}\tag{2.7}$$

and

$$\begin{aligned}\sup_{t\in J}|D_{0^+}^{\alpha-1}Tu(t)|&=\sup_{t\in J}\Big|\int_0^{\infty}G^*(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))\mathrm{d}s\Big|\\&\le\frac{\Gamma(\alpha)}{\Gamma(\alpha)-\Delta}\int_0^{+\infty}|f(s,u(s),D_{0^+}^{\alpha-1}u(s))|\mathrm{d}s\\&\le\frac{\Gamma(\alpha)}{\Gamma(\alpha)-\Delta}[a^*+b^*M^p+c^*M^q].\end{aligned}\tag{2.8}$$

So

$$\|Tu\|_Y=\max\{\|Tu\|_X,\sup_{t\in J}|D^{\alpha-1}Tu(t)|\}\le\frac{\max\{1,\Gamma(\alpha)\}}{\Gamma(\alpha)-\Delta}[a^*+b^*M^p+c^*M^q],$$

which implies that $T\Omega$ is uniformly bounded.

(ii) Let $I\subset J$ be any interval. Then, for all $t_1,t_2\in I,t_2>t_1$ and $u\in\Omega$, we obtain

$$\begin{aligned}\Big|\frac{Tu(t_2)}{1+t_2^{\alpha-1}}-\frac{Tu(t_1)}{1+t_1^{\alpha-1}}\Big|&\le\Big|\int_0^{+\infty}\Big(\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\Big)f(s,u(s),D_{0^+}^{\alpha-1}u(s))\mathrm{d}s\Big|\\&\le\int_0^{+\infty}\Big|\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\Big||f(s,u(s),D_{0^+}^{\alpha-1}u(s))|\mathrm{d}s.\end{aligned}\tag{2.9}$$

Since $G(t,s)\in C(J\times J)$, for any compact set $I\times I, G(t,s)/(1+t^{\alpha-1})$ is uniformly continuous. Furthermore, the function $G(t,s)/(1+t^{\alpha-1})$ only depends on t for $s\ge t$, which implies that $G(t,s)/(1+t^{\alpha-1})$ is uniformly continuous on $I\times(J\setminus I)$. That is, for all $s\in J$ and $t_1,t_2\in I$, we have

$$\forall\epsilon>0,\exists\delta(\epsilon)\text{ such that if }|t_1-t_2|<\delta,\text{ then }\Big|\frac{G(t_2,s)}{1+t_2^{\alpha-1}}-\frac{G(t_1,s)}{1+t_1^{\alpha-1}}\Big|<\epsilon.\tag{2.10}$$

By Lemma 2.6, for all $u\in\Omega$, we find that

$$\int_0^{+\infty}|f(t,u(t),D_{0^+}^{\alpha-1}u(t))|\mathrm{d}s\le a^*+b^*M^p+c^*M^q<\infty,$$

together with (2.9) and (2.10), which implies that $Tu(t)/(1+t^{\alpha-1})$ is equicontinuous on I.

Note that

$$D_{0^+}^{\alpha-1}Tu(t)=\int_0^{\infty}G^*(t,s)f(s,u(s),D_{0^+}^{\alpha-1}u(s))\mathrm{d}s$$

and the function $G^*(t,s)\in C(J\times J)$ does not depend on t, which implies that $D_{0^+}^{\alpha-1}Tu(t)$ is equicontinuous on I. Hence, we prove that the operator $T:Y\to Y$ is equicontinuous on any compact interval I of J.

(iii) We show the operator T is equiconvergent at ∞. Since

$$\lim_{t\to\infty}\frac{G(t,s)}{1+t^{\alpha-1}}=\frac{1}{\Gamma(\alpha)-\Delta}\int_0^{+\infty}h(t)G_1(t,s)\mathrm{d}t,$$

by knowledge of limit theory, it is easy to infer that for any $\epsilon > 0$, there exists a constant $C = C(\epsilon) > 0$, for any $t_1, t_2 \geq C$ and $s \in J$, such that

$$|\frac{G(t_2,s)}{1+t_2^{\alpha-1}} - \frac{G(t_1,s)}{1+t_1^{\alpha-1}}| < \epsilon.$$

Therefore, by Lemma 2.6 and (2.9), we conclude that $Tu(t)/1+t^{\alpha-1}$ is equiconvergent at ∞. As the function $G^*(t,s)$ does not depend on t, we can easily determine that $D_{0^+}^{\alpha-1}Tu(t)$ is equiconvergent at ∞.

From the above three steps, by Lemma 2.8, it follows that the operator $T : Y \to Y$ is relatively compact.

Next, we show that the operator $T : Y \to Y$ is continuous. Let $u_n, u \in Y$, such that $u_n \to u(n \to \infty)$. Then $\|u_n\|_Y < \infty, \|u\|_Y < \infty$. By Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned}
\|Tu_n\|_X =& \sup_{t\in J}\Big|\int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))\mathrm{d}s\Big| \\
\leq& \frac{1}{\Gamma(\alpha)-\Delta}\int_0^{+\infty} |f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))|\mathrm{d}s \\
\leq& \frac{1}{\Gamma(\alpha)-\Delta}[a^* + b^*\|u_n\|_Y^p + c^*\|u_n\|_Y^q], \\
\sup_{t\in J}|D_{0^+}^{\alpha-1}u_n(t)| =& \sup_{t\in J}\Big|\int_0^{\infty} G^*(t,s) f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))\mathrm{d}s\Big| \\
\leq& \frac{\Gamma(\alpha)}{\Gamma(\alpha)-\Delta}\int_0^{+\infty} |f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))|\mathrm{d}s \\
\leq& \frac{\Gamma(\alpha)}{\Gamma(\alpha)-\Delta}[a^* + b^*\|u_n\|_Y^p + c^*\|u_n\|_Y^q],
\end{aligned}$$

and by the Lebesgue dominated convergence theorem and continuity of f, we obtain

$$\lim_{n\to\infty}\int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))\mathrm{d}s = \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} f(s,u(s),D_{0^+}^{\alpha-1}u(s))\mathrm{d}s,$$

$$\lim_{n\to\infty}\int_0^{\infty} G^*(t,s) f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s))\mathrm{d}s = \int_0^{\infty} G^*(t,s) f(s,u(s),D_{0^+}^{\alpha-1}u(s))\mathrm{d}s.$$

Then, as $n \to \infty$,

$$\|Tu_n - Tu\|_X \leq \sup_{t\in J}\int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} |f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s)) - f(s,u(s),D_{0^+}^{\alpha-1}u(s))|\mathrm{d}s \to 0,$$

and, as $n \to \infty$,

$$\sup_{t\in J}|D_{0^+}^{\alpha-1}u_n(t) - D_{0^+}^{\alpha-1}u(t)| \leq \sup_{t\in J}\int_0^{\infty} G^*(t,s)|f(s,u_n(s),D_{0^+}^{\alpha-1}u_n(s)) - f(s,u(s),D_{0^+}^{\alpha-1}u(s))|\mathrm{d}s \to 0.$$

This implies that the operator T is continuous.

In view of the above arguments, we infer that the operator $T : Y \to Y$ is completely continuous. Therefore, the proof is completed. □

3. Main results

In this section, we provide the main results for discussing iterative sequences of positive solutions of FBVP (1.5).

For convenience, we set $L = \frac{1}{\Gamma(\alpha)-\Delta}\max\{1, \Gamma(\alpha)\}$. Define a cone $P \subset Y$ by $P = \{u \in Y : u(t) \geq 0, t \in J\}$.

Theorem 3.1 *Assume the conditions* (H_1), (H_2) *and* (H_3) *are satisfied. There exists a positive constant R such that FBVP (1.5) has a minimal and maximal positive solution v^*, u^*, respectively, in $(0, Rt^{\alpha-1}]$, which can be given by means of the following two explicit monotone iterative sequences:*

$$v_{n+1}(t) = \int_0^{\infty} G(t,s)f(t, v_n(t), D_{0^+}^{\alpha-1}v_n(t))\mathrm{d}s, \quad \textit{with } v_0(t) = 0, \tag{3.1}$$

$$u_{n+1}(t) = \int_0^{\infty} G(t,s)f(t, u_n(t), D_{0^+}^{\alpha-1}u_n(t))\mathrm{d}s, \quad \textit{with } u_0(t) = Rt^{\alpha-1}. \tag{3.2}$$

Moreover,

$$v_0 \leq v_1 \leq \cdots \leq v_n \leq \cdots \leq v^* \leq \cdots \leq u^* \leq \cdots \leq u_n \leq \cdots \leq u_1 \leq u_0.$$

Proof First, Lemma 2.5 leads to the fact that $(Tu)(t) \geq 0$ for any $u \in P, t \in J$. Therefore, $T(P) \subset P$.

Next, for $0 \leq p, q < 1$, choose

$$R \geq \max\{3La^*, (3Lb^*)^{1/(1-p)}, (3Lc^*)^{1/(1-q)}\}$$

and define $B = \{u \in P, \|u\|_Y \leq R\}$. For any $u \in B$, by Lemmas 2.5 and 2.6 [similar to (2.7) and (2.8)], we obtain

$$\|Tu\|_X = \sup_{t\in J}\Big|\int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} f(s, u(s), D_{0^+}^{\alpha-1}u(s))\mathrm{d}s\Big| \leq L[a^* + b^*R^p + c^*R^q] \leq R$$

and

$$\sup_{t\in J}|D_{0^+}^{\alpha-1}u(t)| = \sup_{t\in J}\Big|\int_0^{\infty} G^*(t,s)f(s, u(s), D_{0^+}^{\alpha-1}u(s))\mathrm{d}s\Big| \leq L[a^* + b^*R^p + c^*R^q] \leq R.$$

This implies that $\|Tu\|_Y \leq R$ for all $u \in B$. Hence, $T(B) \subset B$.

Let $v_0(t) = 0, v_1(t) = Tv_0(t), v_2(t) = T^2v_0(t) = Tv_1(t)$, for all $t \in J$. Since $v_0(t) = 0 \in B$ and $T : B \to B$, we obtain $v_1 \in T(B) \subset B$ and $v_2 \in T(B) \subset B$. The condition (H_3) and definition of operator T imply that the operator T is nondecreasing. Thus,

$$v_1(t) = Tv_0(t) \geq 0 = v_0(t), \quad \forall t \in J.$$

By the nondecreasing property of the operator T, we obtain

$$v_2(t) = Tv_1(t) \geq Tv_0(t) = v_1(t), \quad \forall t \in J.$$

Define a sequence $v_{n+1}(t) = Tv_n(t)$, $n = 0, 1, 2, \ldots$. By induction, the sequence $\{v_n\}|_n^{\infty} \subset T(B) \subset B$ and satisfies

$$v_{n+1}(t) \geq v_n(t), \quad \forall t \in J,\ n = 0, 1, 2, \ldots. \tag{3.3}$$

By the complete continuity of the operator T, it is easy to determine that $\{v_n\}|_n^\infty$ has a convergent subsequence $\{v_{n_k}\}|_{k=1}^\infty$ and there exists a $v^* \in B$ such that $v_{n_k} \to v^*$ as $k \to \infty$. This, together with (3.3), means that $\lim_{n\to\infty} v_n = v^*$.

Since T is continuous and $v_{n+1} = Tv_n$, we can obtain $Tv^* = v^*$, that is, v^* is a fixed point of the operator T.

Define $u_0(t) = Rt^{\alpha-1}$, $u_1(t) = Tu_0(t)$, $u_2(t) = T^2u_0(t) = Tu_1(t)$, $t \in J$. Since $u_0(t) \in B$ and $T : B \to B$, we obtain $u_1 \in T(B) \subset B$ and $u_2 \in T(B) \subset B$. Using Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned} u_1(t) &= \int_0^{+\infty} G(t,s)f(s,u_0(s),D_{0^+}^{\alpha-1}u_0(s))\mathrm{d}s \le Lt^{\alpha-1}\int_0^{+\infty} f(s,u_0(s),D_{0^+}^{\alpha-1}u_0(s))\mathrm{d}s \\ &\le Lt^{\alpha-1}[a^* + b^*\|u_0\|_Y^p + c^*\|u_0\|_Y^q] \le Lt^{\alpha-1}[a^* + b^*R^p + c^*R^q] \\ &\le Rt^{\alpha-1} = u_0(t), \quad \forall t \in J. \end{aligned}$$

Since T is nondecreasing, we obtain

$$u_2(t) = Tu_1(t) \le Tu_0(t) = u_1(t), \quad \forall t \in J.$$

Using the above argument, we define $u_{n+1} = Tu_n$, $n = 0,1,2,\ldots$. By induction, the sequence $\{u_n\}|_n^\infty \subset T(B) \subset B$ and satisfies

$$u_{n+1}(t) \ge u_n(t), \quad \forall t \in J, \ n = 0,1,2,\ldots. \tag{3.4}$$

Similar to earlier arguments, it can be proven that there exists a u^* such that $\lim\limits_{n\to\infty} u_n = u^*$.

Since T is continuous and $u_{n+1} = Tu_n$, we obtain $Tu^* = u^*$, that is, u^* is a fixed point of the operator T.

Finally, we infer that u^* and v^* are the maximal and minimal positive solutions of FBVP (1.5), respectively, in $(0, Rt^{\alpha-1}]$. We first assume that $w \in (0, Rt^{\alpha-1}]$ is any positive solution of FBVP (1.5). Then $v_0(t) = 0 \le w(t) \le Rt^{\alpha-1} = u_0(t)$ and $Tw = w$. Applying the monotone property of T, we obtain that $v_1(t) = Tv_0(t) \le w(t) \le Tu_0(t) = u_1(t)$, for all $t \in J$.

Repeating the above step several times, we obtain

$$v_n(t) \le w(t) \le u_n(t), \quad t \in J, \ n = 0,1,2,\ldots. \tag{3.5}$$

Since $u^* = \lim_{n\to\infty} u_n$ and $v^* = \lim_{n\to\infty} v_n$, it follows from (3.4) and (3.5) that

$$v_0 \le v_1 \le \cdots \le v_n \le \cdots \le v^* \le w \le u^* \le \cdots \le u_n \le \cdots \le u_1 \le u_0. \tag{3.6}$$

Again $f(t,0,0) \ne 0$ for all $t \in J$, it follows that zero is not a solution of FBVP (1.5). From (3.6), we know that v^* and u^* are the minimal and maximal positive solutions of FBVP (1.5), respectively, in $(0, Rt^{\alpha-1}]$, which can be established by means of two explicit monotone iterative sequences in (3.1) and (3.2).

With regard to the difference range of p, q, the method is similar, so we omit the details, thereby completing the proof. □

Theorem 3.2 *Assume the conditions* (H_1) *and* (H_4) *are satisfied. If*

$$m = L(d^* + e^*) < 1, \tag{3.7}$$

then FBVP (1.5) has a unique positive solution $\overline{u}(t)$ in P. Moreover, there exists a monotone iterative $u_n(t)$, such that $u_n(t) \to \overline{u}(t)$ as $n \to \infty$ uniformly on any finite interval of J, where

$$u_n(t) = \int_0^{\infty} G(t,s) f(t, u_{n-1}(t), D_{0^+}^{\alpha-1} u_{n-1}(t)) \mathrm{d}s. \tag{3.8}$$

In addition, there exists an error estimate for the approximation sequence

$$\|u_n - \overline{u}\|_Y = \frac{m^n}{1-m} \|u_1 - u_0\|_Y, \quad n = 1, 2, \ldots. \tag{3.9}$$

Proof Choose $r \geq L\lambda/(1-m)$, where m is defined by (3.7) and $\lambda = \int_0^{+\infty} |f(t,0,0)| \mathrm{d}t < +\infty$ is defined in condition (H_4).

First, we show that $TB_r \subset B_r$, where $B_r = \{u \in P, \|u\|_Y \leq r\}$. For any $u \in B_r$, by Lemmas 2.5 and 2.7, we obtain

$$\|Tu\|_X \leq \sup_{t \in J} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} |f(s, u(s), D_{0^+}^{\alpha-1} u(s))| \mathrm{d}s \leq L[(d^* + e^*)\|u\|_Y + \lambda] = m\|u\|_Y + L\lambda$$

and

$$\begin{aligned} \sup_{t \in J} |D_{0^+}^{\alpha-1} u(t)| &= \sup_{t \in J} \Big| \int_0^{\infty} G^*(t,s) f(s, u(s), D_{0^+}^{\alpha-1} u(s)) \mathrm{d}s \Big| \leq L[(d^* + e^*)\|u\|_Y + \lambda] \\ &= m\|u\|_Y + L\lambda, \end{aligned}$$

which implies

$$\|Tu\|_Y \leq m\|u\|_Y + L\lambda \leq r, \quad \forall u \in B_r.$$

We now show that T is a contraction. For any $u_1, u_2 \in B_r$, by condition (H_4), we obtain

$$\begin{aligned} \|Tu_1 - Tu_2\|_X \leq& \sup_{t \in J} \int_0^{+\infty} \frac{G(t,s)}{1+t^{\alpha-1}} |f(s, u_1(s), D_{0^+}^{\alpha-1} u_1(s)) - f(s, u_2(s), D_{0^+}^{\alpha-1} u_2(s))| \mathrm{d}s \\ \leq& L \int_0^{+\infty} [d(s)(1+s^{\alpha-1}) \frac{|u_1(s) - u_2(s)|}{1+s^{\alpha-1}} + e(s)|D_{0^+}^{\alpha-1} u_1(s) - D_{0^+}^{\alpha-1} u_2(s)|] \mathrm{d}s \\ =& m\|u_1 - u_2\|_Y \end{aligned}$$

and

$$\begin{aligned} &\sup_{t \in J} |D_{0^+}^{\alpha-1} u_1(t) - D_{0^+}^{\alpha-1} u_2(t)| \\ &\quad \leq \sup_{t \in J} \int_0^{\infty} G^*(t,s) |f(s, u_1(s), D_{0^+}^{\alpha-1} u_1(s)) - f(s, u_2(s), D_{0^+}^{\alpha-1} u_2(s))| \mathrm{d}s \leq m\|u_1 - u_2\|_Y, \end{aligned}$$

which implies

$$\|Tu_1 - Tu_2\|_Y \leq m\|u_1 - u_2\|_Y, \forall u_1, u_2 \in B_r. \tag{3.10}$$

As $m < 1$, then T is a contraction. Hence, the Banach fixed-point theorem ensures that T has a unique fixed point $\overline{u}$ in P. That is, FBVP (1.5) has a unique positive solution $\overline{u}$ in P.

Furthermore, for any $u_0 \in P, \|u_n - \overline{u}\|_Y \to 0$ as $n \to \infty$, where $u_n = Tu_{n-1}$ $(n = 1, 2, \ldots)$. From (3.10), we obtain

$$\|u_n - u_{n-1}\|_Y \leq m^{n-1} \|u_1 - u_0\|_Y,$$

and

$$\begin{aligned}\|u_n - u_j\|_Y &\le \|u_n - u_{n-1}\|_Y + \|u_{n-1} - u_{n-2}\|_Y + \cdots + \|u_{j+1} - u_j\|_Y \\ &\le \frac{m^n(1 - m^{j-n})}{1 - m}\|u_1 - u_0\|_Y. \end{aligned} \tag{3.11}$$

Allowing $j \to \infty$ on both sides of (3.11), we obtain

$$\|u_n - \overline{u}\|_Y \le \frac{m^n}{1 - m}\|u_1 - u_0\|_Y.$$

Hence, the result (3.9) is satisfied, and the proof is completed. □

4. Example

In this section, we provide the following two examples to illustrate our results.

Example 4.1 Consider fractional differential equations on an infinite interval

$$\begin{cases} D_{0^+}^{2.5}u(t) + \frac{2}{(10+t)^2} + \frac{e^{-2t}|u(t)|^p}{(1+\sqrt{t^3})^p} + \frac{2t|D_{0^+}^{1.5}u(t)|^q}{(3+t^2)^2} = 0, \quad t \in [0, +\infty), \\ u(0) = u'(0) = 0, \ D_{0^+}^{1.5}u(+\infty) = \int_0^{+\infty} t^{-1.5}e^{-t}u(t)\mathrm{d}t, \end{cases} \tag{4.1}$$

where $\alpha = 2.5$, $h(t) = t^{-1.5}e^{-t}$ and

$$f(t, u, v) = \frac{2}{(10+t)^2} + \frac{e^{-2t}|u|^p}{(1+\sqrt{t^3})^p} + \frac{2t|v|^q}{(3+t^2)^2}, \quad 0 \le p, q \le 1.$$

It is clear that $\Gamma(2.5) = 1.32934 > \Delta = \int_0^{+\infty} h(t)t^{1.5}\mathrm{d}t = 1$, $f(t, 0, 0) \not\equiv 0$. Hence, the condition (H_1) is satisfied.

We observe that

$$\begin{aligned}|f(t, u, v)| &\le \frac{2}{(10+t)^2} + \frac{e^{-2t}|u|^p}{(1+\sqrt{t^3})^p} + \frac{2t|v|^q}{(3+t^2)^2} \\ &\triangleq a(t) + b(t)|u|^p + c(t)|v|^q,\end{aligned}$$

$$a(t) = \frac{2}{(10+t)^2}, \ b(t) = \frac{e^{-2t}}{(1+\sqrt{t^3})^p}, \ c(t) = \frac{2t}{(3+t^2)^2},$$

and

$$a^* = \int_0^{+\infty} a(t)\mathrm{d}t = \int_0^{+\infty} \frac{2}{(10+t)^2}\mathrm{d}t = \frac{1}{5} < +\infty,$$

$$b^* = \int_0^{+\infty} b(t)(1 + t^{\alpha-1})^p\mathrm{d}t = \int_0^{+\infty} \frac{e^{-2t}}{(1+\sqrt{t^3})^p}(1+\sqrt{t^3})^p\mathrm{d}t = \frac{1}{2} < +\infty,$$

$$c^* = \int_0^{+\infty} c(t)\mathrm{d}t = \int_0^{+\infty} \frac{2t}{(3+t^2)^2}\mathrm{d}t = \frac{1}{3} < +\infty,$$

which implies that the condition (H_2) is satisfied.

From the expression for function f, we can easily determine that f is nondecreasing with respect to the second and last variables. Therefore, the condition (H_3) is satisfied.

Hence by Theorem 3.1, it follows that there exists a positive constant R such that FBVP (4.1) has minimal and maximal positive solutions v^*, u^*, respectively, in $(0, Rt^{\alpha-1}]$, which can be given by means of two explicit monotone iterative sequences in (3.1) and (3.2).

Example 4.2 Consider fractional differential equations on an infinite interval

$$\begin{cases} D_{0^+}^{2.5}u(t)+\frac{2}{(10+t)^2}+\frac{e^{-10t}u(t)}{(1+\sqrt{t^3})^p}+\frac{2tD_{0^+}^{1.5}u(t)}{(10+t^2)^2}=0, \quad t\in[0,+\infty), \\ u(0)=u'(0)=0,\ D_{0^+}^{1.5}u(+\infty)=\int_0^{+\infty}t^{-1.5}e^{-t}u(t)\mathrm{d}t, \end{cases} \tag{4.2}$$

where $\alpha=2.5$, $h(t)=t^{-1.5}e^{-t}$ and

$$f(t,u,v)=\frac{2}{(10+t)^2}+\frac{e^{-10t}u}{(1+\sqrt{t^3})^p}+\frac{2tv}{(10+t^2)^2},\quad p>0.$$

It is clear that $\Gamma(2.5)=1.32934>\Delta=\int_0^{+\infty}h(t)t^{1.5}\mathrm{d}t=1$, $f(t,0,0)\not\equiv 0$. Hence, the condition (H_1) is satisfied.

Observing that

$$\begin{aligned} |f(t,u,v)-f(t,u',v')| &\le \frac{e^{-10t}}{(1+\sqrt{t^3})^p}|u-u'|+\frac{2t}{(10+t^2)^2}|v-v'| \\ &\triangleq d(t)|u-u'|+e(t)|v-v'|, \end{aligned}$$

$$d(t)=\frac{e^{-2t}}{(1+\sqrt{t})^p},\ e(t)=\frac{2t}{(10+t^2)^2},$$

and

$$d^*=\int_0^{+\infty}d(t)(1+t^{\alpha-1})^p\mathrm{d}t=\int_0^{+\infty}\frac{e^{-10t}}{(1+\sqrt{t^3})^p}(1+\sqrt{t^3})^p\mathrm{d}t=\frac{1}{10}<+\infty,$$

$$e^*=\int_0^{+\infty}e(t)\mathrm{d}t=\int_0^{+\infty}\frac{2t}{(10+t^2)^2}\mathrm{d}t=\frac{1}{10}<+\infty,$$

$$\lambda=\int_0^{+\infty}f(t,0,0)\mathrm{d}t=\int_0^{+\infty}\frac{2}{(10+t)^2}\mathrm{d}t=\frac{1}{5}<+\infty,$$

implies that the condition (H_4) is satisfied. By direct computation, we obtain

$$m=L(d^*+e^*)=4.03638\times(\frac{1}{10}+\frac{1}{10})=0.80728<1.$$

Therefore, all conditions of Theorem 3.2 are satisfied. Hence, Theorem 3.2 ensures that FBVP (4.2) has a unique positive solution, which can be obtained by the limits from the iterative sequences in (3.8).

Acknowledgements We thank the referees for their time and comments.

References

[1] K.S. MILLER, B. ROSS. *An Introduction to the Fractional Calculus and Fractional Differential Equations.* Wiley, New York, 1993.

[2] I. PODLUBNY. *Fractional Differential Equations: an Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications.* Academic Press, New York, 1998.

[3] A. A. KILBAS, H. M. SRIVASTAVA, J. J. TRUJILLO. *Theory and Applications of Fractional Differential Equations.* Elsevier, Asterdam, 2006.

[4] K. DIETHELM. *The Analysis of Fractional Differential Equations: An Application-Oriented Exposition using Differential Operators of Caputo Type*. Springer-Verlag, Berlin, 2010.

[5] Yong ZHOU, Jinrong WANG, Lu ZHANG. *Basic Theory of Fractional Differential Equations*. World Scientific, Singapore, 2014.

[6] Xiangkui ZHAO, Weigao GE. *Unbounded solutions for a fractional boundary value problems on the infinite interval*. Acta Appl. Math., 2010, **109**(2): 495–505.

[7] Xinwei SU, Shuqin ZHANG. *Unbounded solutions to a boundary value problem of fractional order on the half-line*. Comput. Math. Appl., 2011, **61**(4): 1079–1087.

[8] Guotao WANG. *Explicit iteration and unbounded solutions for fractional integral boundary value problem on an infinite interval*. Appl. Math. Lett., 2015, **47**: 1–7.

[9] Sihua LIANG, Jihui ZHANG. *Existence of multiple positive solutions for m-point fractional boundary value problems on an infinite interval*. Math. Comput. Model., 2011, **54**(5-6): 1334–1346.

[10] A. GUEZANE-LAKOUD, A. KILICMAN. *Unbounded solution for a fractional boundary value problem*. Adv. Differ. Equ., 2014, **2014**(1): Article ID 154.

[11] S. A. HUSSAIN SHAH, M. REHMAN. *A note on terminal value problems for fractional differential equations on infinite interval*. Appl. Math. lett., 2016, **52**: 118–125.

[12] Changlong YU, Junfang WANG, Yanping GUO. *Solvability for integral boundary value problems of fractional differential equation on infinite intervals*. J. Nonlinear Sci. Appl., 2016, **9**(1): 160–170.

[13] Bingxian LI, Shurong SUN, Zhenlai HAN. *Successively iterative method for a class of high-order fractional differential equations with multi-point boundary value conditions on half-line*. Bound. Value Probl., 2016, **2016**(1): Article ID 5.

[14] Zhongli WEI, Dong WEI, Junling CHE. *Periodic boundary value problems for fractional differential equations involving a Riemann-Liouville fractional derivative*. Nonlinear Anal., 2010, **73**(10): 3232–3238.

[15] Guotao WANG, D. BALEANU, Lihong Zhang. *Monotone iterative method for a class of nonlinear fractional differential equations*. Fract. Calc. Appl. Anal., 2012, **15**(2): 244–252.

[16] Zhanbing BAI, Shuo ZHANG, Sujing SUN, et al. *Monotone iterative method for fractional differential equations*. Electron. J. Differential Equations, 2016, **2016**(6): 1–8.

[17] Lishan LIU, Xinqiu ZHANG, Juan JIANG, et al. *The unique solution of a class of sum mixed monotone operator equations and its application to fractional boundary value problems*. J. Nonlinear Sci. Appl., 2016, **9**(5): 2943–2958.

[18] Limin GUO, Xingqiu ZHANG. *Existence of positive solutions for a class of fractional differential equations with integral boundary value conditions on an infinite interval*. J. Sys. Sci. Math. Scis., 2014, **34**(6): 752–762. (in Chinese)