Directional Preimage Entropy for $\mathbb{Z}_+^k$-Actions

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Abstract In this paper, a new type of entropy, directional preimage entropy including topological and measure theoretic versions for $\mathbb{Z}_+^k$-actions, is introduced. Some of their properties including relationships and the invariance are obtained. Moreover, several systems including $\mathbb{Z}_+^k$-actions generated by the expanding maps, $\mathbb{Z}_+^k$-actions defined on finite graphs and some infinite graphs with zero directional preimage branch entropy are studied.

Keywords $\mathbb{Z}_+^k$-actions; directional preimage entropy; infinite graph

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1. Introduction

Entropy including measure-theoretic entropy and topological entropy, a measure of the complexity of a system, plays a very important role in the study of dynamical systems. Measure-theoretic entropy gives the maximum average information we can get from a system, and topological entropy measures the exponential growth rate of the number of different orbits. It is well known that they are related by a variational principle. Specifically, let f be a continuous map on a compact topological space X, $h_{\text{top}}(f) = \sup_{\mu \in \mathcal{M}_f(X)} h_\mu(f)$, where $\mathcal{M}_f(X)$ is the set of all f-invariant measures of X. For more details about Measure-theoretic entropy and topological entropy, one can refer to Walters's book [1].

In order to get a further understanding of a system, many types of entropies are introduced and studied from different points of view, most of which are for $\mathbb{Z}$-actions or $\mathbb{Z}_+$-actions. For important results on $\mathbb{Z}$-actions or $\mathbb{Z}_+$-actions, the reader can refer to [2–4]. However, it is necessary to study $\mathbb{Z}^k$-actions or $\mathbb{Z}_+^k$-actions due to the research need of the lattice statistical mechanics. For a high-dimensional system, we can study its complexity by considering the complexity of its subsystems. For instance, the complexity of a high-dimensional system can be described by directional entropy partially. In 1988, Milnor [5] proposed the concept of directional entropy based on the problem of cellular automata and presented the problem whether the directional entropy is continuous in directions. In 1985, Sinai [6] proved that the directional entropy is upper semi-continuous for a $\mathbb{Z}^2$-action generated by cellular automata mappings. But Thouvenot proved

that the directional entropy is not upper semi-continuous for more general systems. Following them, Park [7–9] also studied the questions on the continuity of the directional entropy. In 1999, Boyle and Lind [10] considered the concepts of directional entropy including measure-theoretic directional entropy and topological directional entropy for expansive subdynamics of $\mathbb{Z}^k$-actions. And the techniques of coding and shading were used to prove that the directional entropy is continuous on every expansive branch. For the classical entropies, the systems are considered with respect to the future behavior. While we can also study the system from the view of past time. Langevin, Hurley [11] and Nitecki [12] gave the definition of topological preimage entropy by studying the exponential growth rate of the number of the reverse orbits. Lately, Cheng and Newhouse [13] introduced a new version of topological preimage entropy, and the concept of measure-theoretic preimage entropy is also given. Then a variational principle relating them is obtained. In 2005, Zhang, Zhu and He [14] considered the preimage entropy for non-autonomous system. And in [15, 16], Zhu generalized them to random dynamical systems.

Our purpose is to formulate and study a new type of entropy, directional preimage entropy for $\mathbb{Z}_+^k$-actions, a concept combining the directional entropy with the preimage entropy, which will give us a further understanding of the complexity of a system.

This paper is organised as follows: In Section 2, some definitions and notations are given and the directional preimage entropy for $\mathbb{Z}_+^k$-actions is introduced using spanning sets and separated sets. In Section 3, the properties of directional preimage entropy for $\mathbb{Z}_+^k$-actions are investigated. Some relationships among these entropies and the invariance are obtained. In the last section, several systems with zero directional preimage branch entropy including $\mathbb{Z}_+^k$-actions generated by the expanding maps, $\mathbb{Z}_+^k$-actions defined on finite graphs and some infinite graphs are studied.

2. Directional preimage entropy for $\mathbb{Z}_+^k$-actions

Let (X, ρ) be a compact metric space. If $\alpha^{\vec{n}} := \alpha(\vec{n}, \cdot) \colon X \to X$, $\vec{n} \in \mathbb{Z}_+^k$ satisfy

(1) $\alpha^{\vec{0}} = \mathrm{id}$, where id is the identity on X;

(2) For any $\vec{m}, \vec{n} \in \mathbb{Z}_+^k$, $\alpha^{\vec{m}+\vec{n}} = \alpha^{\vec{m}} \circ \alpha^{\vec{n}}$;

(3) For any $\vec{n} \in \mathbb{Z}_+^k$, $\alpha^{\vec{n}}$ is a continuous map,

then α is called a continuous $\mathbb{Z}_+^k$-action on X. Let $\vec{e}_i = (0, \ldots, 1, \ldots, 0)$, where the i-th element is 1, and put $f_i := \alpha^{\vec{e}_i}$, $i = 1, 2, \ldots, k$. And f_i is said to be a generator of α. And the commutative law holds for all generators, that is, for any $i, j \in \{1, 2, \ldots, k\}$, $f_i \circ f_j = f_j \circ f_i$.

Let K be a compact subset of X. For any $\varepsilon > 0$, a subset $E \subset X$ is said to be an (E, ε, ρ)-spanning set of K, if for any $x \in K$, there exists $y \in E$ such that $\rho(x, y) < \varepsilon$. Let $r(E, \varepsilon, \rho, K)$ denote the smallest cardinality of any (E, ε, ρ)-spanning set of K. For any $\varepsilon > 0$, a subset $F \subset K$ is said to be an (F, ε, ρ)-separated set of K, if $x, y \in F$, $x \neq y$ implies $\rho(x, y) > \varepsilon$. Let $s(F, \varepsilon, \rho, K)$ denote the largest cardinality of any (F, ε, ρ)-separated set of K. In particular, if $K = X$, then for simplicity, denote $r(E, \varepsilon, \rho, K)$ and $s(F, \varepsilon, \rho, K)$ by $r(E, \varepsilon, \rho)$ and $s(F, \varepsilon, \rho)$, respectively.

Let α be a $\mathbb{Z}_+^k$-action on (X, ρ). For a subset $E \subset \mathbb{R}^k$, put a metric by

$$\rho_E(x, y) := \sup\{\rho(\alpha^{\vec{n}}(x), \alpha^{\vec{n}}(y)) | \vec{n} \in E \cap \mathbb{Z}_+^k\},$$

if $E \cap \mathbb{Z}_+^k = \varnothing$, then put $\rho_E(x, y) = 0$.

Given a subset $E \subset \mathbb{R}^k$, $\vec{v} \in \mathbb{R}^k$, let $\|\cdot\|$ denote the Euclid norm on $\mathbb{R}^k$. Then define

$$\text{dist}(\vec{v}, E) := \inf\{\|\vec{v} - \vec{w}\| \big| \vec{w} \in E\}.$$

For any $t > 0$, $s > 0$, put $E^t = \{\vec{v} \in \mathbb{R}^k | \text{dist}(\vec{v}, E) \leq t\}$ and $sE = \{s\vec{v} | \vec{v} \in E\}$.

A d-frame $\Phi = (\vec{v}_1, \ldots, \vec{v}_d)$ is a d-tuple of linearly independent vectors in $\mathbb{R}^k$. Let $\mathbb{F}_d$ be the set of all d-frames. Denote the line segment in $\mathbb{R}^k$ with endpoints $\vec{v}$, $\vec{w}$ by $[\vec{v}, \vec{w}]$. Then we let $Q_\Phi = [\vec{0}, \vec{v}_1] \oplus \cdots \oplus [\vec{0}, \vec{v}_d]$ denote the parallelepiped spanned by Φ.

Definition 2.1 *Let α be a $\mathbb{Z}_+^k$-action on (X, ρ), a d-frame $\Phi \in \mathbb{F}_d$. Define the d-dimensional topological directional entropy of Φ as*

$$h_d(\alpha, \Phi) = \limsup_{\varepsilon \to 0} \, \overline{\lim_{t > 0}} \, \lim_{s \to \infty} \frac{\log r_\alpha((sQ_\Phi)^t, \varepsilon, \rho_{(sQ_\Phi)^t})}{s^d}$$

$$= \limsup_{\varepsilon \to 0} \, \overline{\lim_{t > 0}} \, \lim_{s \to \infty} \frac{\log s_\alpha((sQ_\Phi)^t, \varepsilon, \rho_{(sQ_\Phi)^t})}{s^d}.$$

Now, we consider the case that $d = 1$, then $\Phi = (\vec{v})$, $Q_\Phi = [\vec{0}, \vec{v}]$, so we define the topological directional entropy of α in $\vec{v}$ as

$$h(\alpha, \vec{v}) = \limsup_{\varepsilon \to 0} \, \overline{\lim_{t > 0}} \, \lim_{s \to \infty} \frac{\log r_\alpha((s[\vec{0}, \vec{v}])^t, \varepsilon, \rho_{(s[\vec{0}, \vec{v}])^t})}{s}$$

$$= \limsup_{\varepsilon \to 0} \, \overline{\lim_{t > 0}} \, \lim_{s \to \infty} \frac{\log s_\alpha((s[\vec{0}, \vec{v}])^t, \varepsilon, \rho_{(s[\vec{0}, \vec{v}])^t})}{s}.$$

Before giving the definition of directional preimage entropy for $\mathbb{Z}_+^k$-actions we give some notations and definitions.

Given $\vec{v} \in \mathbb{R}^k$, $t > 0$, for all elements in $\bigcup_{s=1}^{\infty}((s[\vec{0}, \vec{v}])^t \cap \mathbb{Z}_+^k)$, an order $\leq$ is compatible with the alphabetical order. And the set of the such orders is denoted by $\mathcal{O}_{s,t}$. When an order $\leq \in \mathcal{O}_{s,t}$ is taken, we represent the elements in $\bigcup_{s=1}^{\infty}((s[\vec{0}, \vec{v}])^t \cap \mathbb{Z}_+^k)$ as $\{\vec{u_1}, \vec{u_2}, \ldots\}$ and put

$$g_1 = \alpha^{\vec{u_1}} : X \to X, \quad i = 1,$$

$$g_i = \alpha^{\vec{u_i} - \vec{u}_{i-1}} : X \to X, \quad i > 1,$$

where $\{g_i\}_{i=1}^{\infty}$ is a sequence of continuous maps on X.

Remark 2.2 *In fact, g_i is a combination of some elements in $\{f_j\}_{j=1}^k$.*

Let $\{g_i\}_{i=1}^{\infty}$ be a sequence of continuous maps on X. Put

$$g_i^0 = \text{id},$$

$$g_i^n = g_{i+(n-1)} \circ \cdots \circ g_{i+1} \circ g_i,$$

$$g_i^{-n} = g_i^{-1} \circ g_{i+1}^{-1} \circ \cdots \circ g_{i+(n-1)}^{-1}.$$

Define a metric $\rho_{s,t,\leq}$ on X by $\rho_{s,t,\leq}(x,x') = \max\limits_{0\leq i\leq n-1}\{g_1^i(x), g_1^i(x')\}$.

Let $H = \{h_i\}_{i=1}^\infty$ be a sequence of continuous maps on X. For $k = 0, 1, 2, \ldots$, the k-th preimage set of x is the subset of X as follows

$$h^{-k}(x) := \{z \in X | h_1^k(z) = h_k \circ h_{k-1} \circ \cdots \circ h_1(z) = x\}.$$

The n-th preimage tree of x under H is the set $T^n(x, H) := \bigcup_{k=0}^n h^{-k}(x) \times \{k\}$.

The k-th level of the tree $T^n(x, H)$ is the subset $h^{-k}(x) \times \{k\}$ and its branches are defined as $\beta := [z_k, z_{k-1}, \ldots, z_1, z_0 = x]$, where $h_{n-(j-1)}(z_j) = z_{j-1}$, $j = 1, \ldots, k \leq n$, and this k is called the order of the branch β. Let $B^n(x, H)$ be the set of branches whose order is n in $T^n(x, H) \setminus \{x\} \times \{0\}$.

For $\{g_i\}_{i=1}^\infty$ above, we denote the two notations about the n-th preimage tree as $T_{s,t,\leq}^n(x, \alpha, \vec{v})$ and $B_{s,t,\leq}^n(x, \alpha, \vec{v})$, respectively. Obviously, n is related to s and t and a more rigorous notation would be $n(s,t)$. Since it does not cause confusion, we abbreviate it to n here.

Define a metric on $B_{s,t,\leq}^n(x, \alpha, \vec{v})$ by $\rho_{s,t,\leq}(\beta_i, \beta_j) = \max\limits_{1\leq k\leq n}\rho(z_k^i, z_k^j)$ for any branch $\beta_i = [z_n^i, z_{n-1}^i, \ldots, z_0 = x]$ and $\beta_j = [z_n^j, z_{n-1}^j, \ldots, z_0 = x] \in T_{s,t,\leq}^n(x, \alpha, \vec{v})$.

Definition 2.3 *Define the pointwise directional preimage entropies of $\mathbb{Z}_+^k$-action to be*

$$h_p(\alpha, \vec{v}) = \sup_{x\in X}\limsup_{\varepsilon\to 0}\overline{\lim_{t>0}}\lim_{s\to\infty} \frac{\log \sup\limits_{\leq\in\mathcal{O}_{s,t}} r_\alpha((s[\vec{0},\vec{v}])^t, \varepsilon, \rho_{s,t,\leq}, B_{s,t,\leq}^n(x, \alpha, \vec{v}))}{s}$$

$$= \sup_{x\in X}\limsup_{\varepsilon\to 0}\overline{\lim_{t>0}}\lim_{s\to\infty} \frac{\log \sup\limits_{\leq\in\mathcal{O}_{s,t}} s_\alpha((s[\vec{0},\vec{v}])^t, \varepsilon, \rho_{s,t,\leq}, B_{s,t,\leq}^n(x, \alpha, \vec{v}))}{s},$$

$$h_m(\alpha, \vec{v}) = \limsup_{\varepsilon\to 0}\overline{\lim_{t>0}}\lim_{s\to\infty}\sup_{x\in X} \frac{\log \sup\limits_{\leq\in\mathcal{O}_{s,t}} r_\alpha((s[\vec{0},\vec{v}])^t, \varepsilon, \rho_{s,t,\leq}, B_{s,t,\leq}^n(x, \alpha, \vec{v}))}{s}$$

$$= \limsup_{\varepsilon\to 0}\overline{\lim_{t>0}}\lim_{s\to\infty}\sup_{x\in X} \frac{\log \sup\limits_{\leq\in\mathcal{O}_{s,t}} s_\alpha((s[\vec{0},\vec{v}])^t, \varepsilon, \rho_{s,t,\leq}, B_{s,t,\leq}^n(x, \alpha, \vec{v}))}{s}.$$

Let (X, ρ) be a compact metric space. For any order $\leq\in\mathcal{O}_{s,t}$, there is a certain branch distance. Put $\beta = [z_k, z_{k-1}, \ldots, z_1, z_0 = x] \in T_{s,t,\leq}^n(x, \alpha, \vec{v})$ and $\beta' = [z_l', z_{l-1}', \ldots, z_1', z_0' = x'] \in T_{s,t,\leq}^n(x', \alpha, \vec{v})$, define

$$\rho_{s,t,\leq}^b(\beta, \beta') = \begin{cases} \max\limits_{1\leq k\leq n}\rho(z_k, z_l'), & l = k; \\ \mathrm{diam}(X), & l \neq k. \end{cases}$$

Then, for any positive integer n and $x, x' \in X$, define $\rho_{s,t,\leq}^{n,b}(x, x') < \varepsilon$ if for every branch $\beta \in T_{s,t,\leq}^n(x, \alpha, \vec{v})$, there exists a branch $\beta' \in T_{s,t,\leq}^n(x', \alpha, \vec{v})$ such that $\rho_{s,t,\leq}^b(\beta, \beta') < \varepsilon$ and vice-versa (for branch of $T_{s,t,\leq}^n(x', \alpha, \vec{v})$).

Definition 2.4 *Define the branch directional preimage entropy of $\mathbb{Z}_+^k$-action to be*

$$h_i(\alpha, \vec{v}) = \limsup_{\varepsilon\to 0}\overline{\lim_{t>0}}\lim_{s\to\infty} \frac{\log \sup\limits_{\leq\in\mathcal{O}_{s,t}} r_\alpha((s[\vec{0},\vec{v}])^t, \varepsilon, \rho_{s,t,\leq}^{n,b}, X)}{s}$$

$$= \limsup_{\varepsilon \to 0} \overline{\lim_{t > 0}} \, \overline{\lim_{s \to \infty}} \, \frac{\log \sup_{\leq \in \mathcal{O}_{s,t}} s_\alpha((s[\vec{0}, \vec{v}])^t, \varepsilon, \rho_{s,t,\leq}^{n,b}, X)}{s}.$$

Let us call the above definitions as Definition 1. Since it is a little complex, we will give another definition, which will be called as Definition 2.

Denote the lattice point sequence in $\mathbb{Z}_+^k$ as $\{\vec{n_i}\}_{i \in \mathbb{Z}^+}$: $\vec{n_i}$ is the integer in the set $\{\vec{n_i} \| \vec{n_i} - i\vec{v_e}\| = \min_{\vec{m} \in \mathbb{Z}_+^k} |\vec{m} - i\vec{v_e}|\}$ with the smallest norm. Note that the choice of $\vec{n_i}$ is not unique, but the following definitions are independent of the choice of $\vec{n_i}$.

Put

$$\tilde{g}_1 = \alpha^{\vec{n}_1} : X \to X, \quad i = 1,$$
$$\tilde{g}_i = \alpha^{\vec{n}_i - \vec{n}_{i-1}} : X \to X, \quad i > 1,$$

obviously, $\tilde{g}_i \in \{\text{id}, f_1, f_2, \ldots, f_k\}$ and $\{\tilde{g}_i\}_{i=1}^\infty$ is a sequence of continuous maps on X. $(X, \tilde{g}_{1,\infty})$ is called a nonautonomous dynamical system along the direction $\vec{v}$, where $\tilde{g}_{1,\infty} = \{\tilde{g}_i\}_{i=1}^\infty$.

For any positive integer n, define a metric on X by $\rho^n(x, y) = \max_{0 \leq i \leq n-1} \rho(\tilde{g}_1^i(x), \tilde{g}_1^i(y))$.

Then the following types of entropy are defined for $\tilde{g}_{1,\infty}$.

Definition 2.5 *Define the topological directional entropy for $\tilde{g}_{1,\infty}$ to be*

$$h(\alpha, \vec{v}_e) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log r_\alpha(\varepsilon, \rho^n)}{n} = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log s_\alpha(\varepsilon, \rho^n)}{n},$$

where $r_\alpha(\varepsilon, \rho^n)$ is the smallest cardinality of any (ε, ρ^n)-spanning set in $\mathbb{Z}_+^k$ (Similarly, $s_\alpha(\varepsilon, \rho^n)$ is the largest cardinality of any (ε, ρ^n)-separated set in $\mathbb{Z}_+^k$).

Definition 2.6 *Define the pointwise directional preimage entropies for $\tilde{g}_{1,\infty}$ to be*

$$h_p(\alpha, \vec{v}_e) = \sup_{x \in X} \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log r_\alpha(\varepsilon, \rho^n, \tilde{g}_1^{-n}(x))}{n}$$

$$= \sup_{x \in X} \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \frac{\log s_\alpha(\varepsilon, \rho^n, \tilde{g}_1^{-n}(x))}{n},$$

$$h_m(\alpha, \vec{v}_e) = \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \sup_{x \in X} \frac{\log r_\alpha(\varepsilon, \rho^n, \tilde{g}_1^{-n}(x))}{n}$$

$$= \lim_{\varepsilon \to 0} \overline{\lim_{n \to \infty}} \sup_{x \in X} \frac{\log s_\alpha(\varepsilon, \rho^n, \tilde{g}_1^{-n}(x))}{n}.$$

Let $\beta = [z_k, z_{k-1}, \ldots, z_1, z_0 = x] \in T^n(x)$, $\beta' = [z'_l, z'_{l-1}, \ldots, z'_1, z'_0 = x'] \in T^n(x')$. Define

$$\rho^b(\beta, \beta') = \begin{cases} \max_{0 \leq i \leq n-1} \rho(z_k, z'_l), & l = k, \\ \text{diam}(X), & l \neq k. \end{cases}$$

Then, define branch distance $\rho^{n,b}(x, x') < \varepsilon$ for any positive integer n if for every branch $\beta \in T^n(x)$, there exists a branch $\beta' \in T^n(x')$ such that $\rho^b(\beta, \beta') < \varepsilon$ and vice-versa (for branch of $T^n(x')$).

Definition 2.7 *Define the branch directional preimage entropy for $\tilde{g}_{1,\infty}$ to be*

$$h_i(\alpha, \vec{v}_e) = \lim_{\varepsilon \to 0} \varlimsup_{n \to \infty} \frac{\log r_\alpha(\varepsilon, \rho^{n,b}, X)}{n} = \lim_{\varepsilon \to 0} \varlimsup_{n \to \infty} \frac{\log s_\alpha(\varepsilon, \rho^{n,b}, X)}{n}.$$

Obviously, the order of the elements in $\tilde{g}_{1,\infty}$ is compatible with the alphabetical order. Then $h(\alpha, \vec{v}_e) \leq h(\alpha, \vec{v})$ and the following preimage entropies also satisfy the same relationship, namely $h_j(\alpha, \vec{v}_e) \leq h_j(\alpha, \vec{v})$, where $j = p$, m or i.

3. Properties of directional preimage entropy for $\mathbb{Z}_+^k$-actions

In this section, properties of directional preimage entropy for $\mathbb{Z}_+^k$-actions are investigated based on the definition 2.

Proposition 3.1 *For $\tilde{g}_{1,\infty} = \{\tilde{g}_i\}_{i=1}^\infty$ defined above, we have*

$$0 \leq h_p(\alpha, \vec{v}_e) \leq h_m(\alpha, \vec{v}_e) \leq h(\alpha, \vec{v}_e).$$

Proof The result is obtained by definitions of $h_p(\alpha, \vec{v}_e)$, $h_m(\alpha, \vec{v}_e)$ and $h(\alpha, \vec{v}_e)$ directly. $\square$

Proposition 3.2 *For $\tilde{g}_{1,\infty} = \{\tilde{g}_i\}_{i=1}^\infty$ defined above, we have*

$$h(\alpha, \vec{v}_e) \leq h_m(\alpha, \vec{v}_e) + h_i(\alpha, \vec{v}_e).$$

Proof Define

$$\tilde{g}_1 = \alpha^{\vec{n}_1}, \qquad i = 1,$$
$$\tilde{g}_i = \alpha^{\vec{n}_i - \vec{n}_{i-1}}, i > 1,$$

where $\vec{n}_i$ is the integer in $\{\vec{n}_i \,||\, \vec{n}_i - i\vec{v}_e| = \min\limits_{\vec{m} \in \mathbb{Z}_+^k} |\vec{m} - i\vec{v}_e|\}$ with the smallest norm. Obviously, $\tilde{g}_i \in \{\text{id}, f_1, f_2, \ldots, f_k\}$, $\tilde{g}_{1,\infty} = \{\tilde{g}_i\}_{i=1}^\infty$ is a sequence of continuous maps on X. Put

$$\tilde{g}_1^n = \tilde{g}_n \circ \cdots \circ \tilde{g}_2 \circ \tilde{g}_1,$$
$$\tilde{g}_1^{-n} = \tilde{g}_1^{-1} \circ \tilde{g}_2^{-1} \circ \cdots \circ \tilde{g}_n^{-1}.$$

For any $0 < \varepsilon < 3\operatorname{diam}(X)$, $n \geq 1$, let R be a maximal $(\frac{\varepsilon}{3}, \rho^{n,b})$-spanning set of X. For any $x \in X$, let $S(x)$ be a maximal $(\frac{\varepsilon}{3}, \rho^n)$-spanning set of $\tilde{g}_1^{-n}(x)$. If $S = \bigcup_{x' \in R} S(x')$, we only have to prove for any $z_n \in X$ there is $z \in S$ such that $\rho^n(z_n, z) < \varepsilon$, then we can get that S is a (ε, ρ^n)-generated set of X.

Assume $x = \tilde{g}_1^n(z_n)$ for some $z_n \in X$, because R is a maximal $(\frac{\varepsilon}{3}, \rho^{n,b})$-spanning set of X, R is a $(\frac{\varepsilon}{3}, \rho^{n,b})$-spanning set of X. Then there are two cases:

(1) $z_n \in R$, then $x = \tilde{g}_1^n(z_n)$ implies $z_n \in \tilde{g}_1^{-n}(x)$. Because $S(x)$ is a maximal $(\frac{\varepsilon}{3}, \rho^n)$-spanning set of $\tilde{g}_1^{-n}(x)$, $S(x)$ is a $(\frac{\varepsilon}{3}, \rho^n)$-spanning set of the set $\tilde{g}_1^{-n}(x)$, so for any $z_n \in \tilde{g}_1^{-n}(x)$, there exists $z_n' \in S(x)$ such that $\rho^{n,b}(z_n, z_n') \leq \frac{\varepsilon}{3} < \varepsilon$.

(2) $z_n \notin R$, then there is $z_n' \in R$ such that $\rho^{n,b}(z_n, z_n') \leq \frac{\varepsilon}{3}$. Thus for the branch $\beta = [z_n, \tilde{g}_1(z_n), \tilde{g}_1^2(z_n), \ldots, \tilde{g}_1^n(z_n) = x] \in T^n(x, \alpha, \vec{v}_e)$ with the endpoint z_n, we can find β' in another preimage tree $T^n(x', \alpha, \vec{v}_e)$ such that $\rho^b(\beta, \beta') \leq \frac{\varepsilon}{3}$. We know that β and β' have the same order by the definition of branch distance due to $\frac{\varepsilon}{3} < \operatorname{diam}(X)$. Suppose $\beta' = [z_n', z_{n-1}', \ldots, z_1', z_0' =$

x'], then $\rho^n(z_n, z'_n) \leq \frac{\varepsilon}{3}$. Because $S(x')$ is a maximal $(\frac{\varepsilon}{3}, \rho^n)$-spanning set of $\tilde{g}_1^{-n}(x')$, $S(x')$ is a $(\frac{\varepsilon}{3}, \rho^n)$-spanning set of $\tilde{g}_1^{-n}(x')$, and $z'_n \in \tilde{g}_1^{-n}(x')$, so there exists $z \in S(x')$ such that $\rho^n(z'_n, z) \leq \frac{\varepsilon}{3}$, therefore

$$\rho^n(z_n, z) \leq \rho^n(z_n, z'_n) + \rho^n(z'_n, z) \leq \frac{2\varepsilon}{3} < \varepsilon,$$

namely, S is a (ε, ρ^n)-spanning set of X.

Thus $r_\alpha(\varepsilon, \rho^n, X) \leq |S|$, where $|S|$ is the cardinality of the set S.

According to the choice of R and S, it is easy to get

$$r_\alpha(\varepsilon, \rho^n, X) \leq |R| \sup_{x' \in R} |S(x')| \leq s_\alpha(\frac{\varepsilon}{3}, \rho^{n,b}, X) \cdot \sup_{x \in X} s_\alpha(\frac{\varepsilon}{3}, \rho^n, \tilde{g}_1^{-n}(x)).$$

Then we get

$$\varlimsup_{n \to \infty} \frac{\log r_\alpha(\varepsilon, \rho^n, X)}{n} \leq \varlimsup_{n \to \infty} \frac{\log s_\alpha(\frac{\varepsilon}{3}, \rho^{n,b}, X))}{n} + \varlimsup_{n \to \infty} \frac{\log \sup_{x \in X} s_\alpha(\frac{\varepsilon}{3}, \rho^n, \tilde{g}_1^{-n}(x))}{n}.$$

Let $\varepsilon \to 0$. Then

$$h(\alpha, \vec{v}_e) \leq h_m(\alpha, \vec{v}_e) + h_i(\alpha, \vec{v}_e). \quad \square$$

And the topological invariance is easy to prove.

Definition 3.3 *Suppose that (X_1, ρ_1) and (X_2, ρ_2) are compact metric spaces, $\tilde{g}_{1,\infty}^{(1)} = \{\tilde{g}_i^{(1)}\}_{i=1}^\infty$ and $\tilde{g}_{1,\infty}^{(2)} = \{\tilde{g}_i^{(2)}\}_{i=1}^\infty$ are continuous map sequences for $\mathbb{Z}_+^k$-actions $\alpha^{(1)}$ and $\alpha^{(2)}$ on X_1 and X_2, respectively. If there exists a sequence of continuous homeomorphisms $\pi_{1,\infty} = \{\pi_i\}_{i=1}^\infty \colon X_1 \to X_2$ such that $\pi_{i+1} \circ \tilde{g}_i^{(1)} = \tilde{g}_i^{(2)} \circ \pi_i$ for any $i \geq 1$, then $\tilde{g}_{1,\infty}^{(1)}$ and $\tilde{g}_{1,\infty}^{(2)}$ are said to be topologically conjugate.*

Definition 3.4 *Assume that every element g_i of $g_{1,\infty} = \{g_i\}_{i=1}^\infty$ is continuous. Then $g_{1,\infty} = \{g_i\}_{i=1}^\infty$ is said to be equicontinuous if for every $\varepsilon > 0$, $x \in X$, $i = 1, 2, \ldots$, there exists neighborhood $V(x)$ such that for any $y \in V(x)$, $\rho(g_i^n(x), g_i^n(y)) < \varepsilon$.*

Since the sets $\{\tilde{g}_1^{(1)}, \tilde{g}_2^{(1)}, \ldots\}$ and $\{\tilde{g}_1^{(2)}, \tilde{g}_2^{(2)}, \ldots\}$ are finite, there exists a sequence $\pi_{1,\infty}$ such that $\{\pi_1, \pi_2, \ldots\}$ is finite, namely $\pi_{1,\infty}$ is equicontinuous.

Proposition 3.5 *If $\tilde{g}_{1,\infty}^{(1)}$ and $\tilde{g}_{1,\infty}^{(2)}$ are topologically conjugate, then*

$$h_j(\alpha^{(1)}, \vec{v}_e) = h_j(\alpha^{(2)}, \vec{v}_e), \quad j = p, m \text{ or } i.$$

Proof (1) Since $\tilde{g}_{1,\infty}^{(1)}$ and $\tilde{g}_{1,\infty}^{(2)}$ are topologically conjugate, there exists a sequence of continuous homeomorphisms $\pi_{1,\infty} = \{\pi_i\}_{i=1}^\infty \colon X_1 \to X_2$ such that for every $i \geq 1$, $\pi_{i+1} \circ \tilde{g}_i^{(1)} = \tilde{g}_i^{(2)} \circ \pi_i$. Thus for any $\varepsilon > 0$, $x_1, x_2 \in X_1$, there exists $\delta(\varepsilon) > 0$ such that for some $i \geq 1$, $\rho_2(\pi_i(x_1), \pi_i(x_2)) > \varepsilon$ implies $\rho_1(x_1, x_2) > \delta(\varepsilon)$. So if $E \subset X_2$ is a $(\tilde{g}_{1,\infty}^{(2)}, \varepsilon, \rho_2^n)$-separated set of $(g_1^{(2)})^{-n}(y)$, then $\pi_1^{-1}(E)$ is a $(\tilde{g}_{1,\infty}^{(1)}, \delta(\varepsilon), \rho_1^n)$-separated set of $(\tilde{g}_1^{(1)})^{-n}(\pi_{n+1}^{-1}(y))$, therefore

$$s_{\alpha^{(1)}}(\tilde{g}_{1,\infty}^{(1)}, \delta(\varepsilon), \rho_1^n, (\tilde{g}_1^{(1)})^{-n}(\pi_{n+1}^{-1}(y))) \geq s_{\alpha^{(2)}}(\tilde{g}_{1,\infty}^{(2)}, \delta(\varepsilon), \rho_2^n, (\tilde{g}_1^{(2)})^{-n}(y)).$$

Hence by the definition of pointwise directional preimage entropies, we get

$$h_p(\alpha^{(1)}, \vec{v}_e) \geq h_p(\alpha^{(2)}, \vec{v}_e) \text{ and } h_m(\alpha^{(1)}, \vec{v}_e) \geq h_m(\alpha^{(2)}, \vec{v}_e).$$

Furthermore, changing the positions of $\tilde{g}_{1,\infty}^{(1)}$ and $\tilde{g}_{1,\infty}^{(2)}$, we have

$$h_p(\alpha^{(1)}, \vec{v}_e) \leq h_p(\alpha^{(2)}, \vec{v}_e) \ \text{ and } \ h_m(\alpha^{(1)}, \vec{v}_e) \leq h_m(\alpha^{(2)}, \vec{v}_e).$$

Therefore, we have

$$h_p(\alpha^{(1)}, \vec{v}_e) = h_p(\alpha^{(2)}, \vec{v}_e) \ \text{ and } \ h_m(\alpha^{(1)}, \vec{v}_e) = h_m(\alpha^{(2)}, \vec{v}_e).$$

(2) Given ε and $\delta(\varepsilon)$ as those in (1). If $E \subset X_2$ is a $(\tilde{g}_{1,\infty}^{(2)}, \varepsilon, \rho_2^{n,b})$-separated set of X_2, then $\pi_{n+1}^{-1}(E)$ is a $(\tilde{g}_{1,\infty}^{(1)}, \delta(\varepsilon), \rho_1^{n,b})$-separated set of X_1, then

$$s_{\alpha^{(1)}}(\tilde{g}_{1,\infty}^{(1)}, \delta(\varepsilon), \rho_1^{n,b}, X_1) \geq s_{\alpha^{(2)}}(\tilde{g}_{1,\infty}^{(2)}, \delta(\varepsilon), \rho_2^{n,b}, X_2).$$

Hence by the definition of branch directional preimage entropies, we have

$$h_i(\alpha^{(1)}, \vec{v}_e) \geq h_i(\alpha^{(2)}, \vec{v}_e),$$

and then changing the positions of $\tilde{g}_{1,\infty}^{(1)}$ and $\tilde{g}_{1,\infty}^{(2)}$, we get

$$h_i(\alpha^{(1)}, \vec{v}_e) \leq h_i(\alpha^{(2)}, \vec{v}_e).$$

Therefore, we have $h_i(\alpha^{(1)}, \vec{v}_e) = h_i(\alpha^{(2)}, \vec{v}_e)$. $\square$

4. Some systems with zero directional preimage entropy

This section is devoted to proving that $h_i(\alpha, \vec{v}_e) = 0$ for the following three systems generated by:

(1) Forward-expansive covering maps;

(2) Actions on a finite graph;

(3) Actions on an infinite graph with some additional conditions.

4.1. $\mathbb{Z}_+^k$-action generated by forward-expansive covering maps

First we give some basic definitions before proving that $h_i(\alpha, \vec{v}_e) = 0$ for forward-expansive covering maps.

Definition 4.1 ([12]) *A continuous map $f : X \to X$ is called forward expansive if there is an $\varepsilon > 0$ such that whenever $x, y \in X$ and $x \neq y$ there exists $n \geq 0$ with $\rho(f^n(x), f^n(y)) \geq \varepsilon$. In this case, ε is called an expansive constant for f.*

Definition 4.2 ([12]) *A continuous map $f : X \to X$ is called a covering map if for any $x \in X$, there is an open neighborhood U_x whose preimage $f^{-1}(U_x)$ is a disjoint union of open sets, that is $f^{-1}(U_x) = V_1(x) \cup V_2(x) \cup \cdots \cup V_k(x)$, and $V_s(x) \cap V_t(x) = \varnothing$ $(s \neq t; s, t = 1, 2, \ldots, k)$, and f is a homeomorphism on each $V_i(x)$.*

The expanding map is an example of forward-expansive covering maps.

Definition 4.3 ([12]) *Suppose that M is a closed Riemannian manifold, $\|\cdot\|$ is the norm on the tangential TM induced by the Riemannian inner product $\langle \cdot, \cdot \rangle$ on M, $d(\cdot, \cdot)$ is the metric on M, and $f \colon M \to M$ is a mapping on M. If there exists $C > 0$ and $\lambda > 1$ such that for any $v \in T_x M$,*

$\|Tf^n(v)\| \geq C\lambda^n\|v\|$, $n = 1, 2, \ldots$, then $f \colon M \to M$ is called an expanding map. In this case, λ is called an expanding constant for f.

Theorem 4.4 *Let α be a $\mathbb{Z}_+^k$-action on a closed Riemannian manifold M whose generators are expanding maps. Then $h_i(\alpha, \vec{v}_e) = 0$.*

Proof Actually, for the sequence $\tilde{g}_{1,\infty} = \{\tilde{g}_i\}_{i=1}^\infty$ which is induced by α and $\vec{v}_e$, because the set $\{\tilde{g}_1, \tilde{g}_2, \ldots\}$ is finite and the generators f_i are expanding maps, the covering property and compactness of M yield that there exists $\varepsilon_0 > 0$ such that for any $x \in X$, $i = 1, 2, \ldots$,

$$\tilde{g}_i^{-1}(B(x, \varepsilon_0)) = V_{i,1}(x) \cup V_{i,2}(x) \cup \cdots \cup V_{i,k(x)}(x),$$

where $V_{i,s}(x) \cap V_{i,t}(x) = \varnothing$ ($s \neq t$; $s, t = 1, 2, \ldots, k(x)$), and on each $V_{i,s}(x)$, f is a homeomorphism.

By the expansion of $\tilde{g}_i$, for any $x_1, x_2 \in V_{i,s}(x)$, $\lambda > 1$, we can get

$$\rho(\tilde{g}_i(x_1), \tilde{g}_i(x_2)) \geq \lambda\rho(x_1, x_2),$$

where ρ is the metric induced by the Riemannian structure on M. By $\tilde{g}_i(V_{i,s}(x)) = B(x, \varepsilon_0)$ and the expansion of $\tilde{g}_i$ on $V_{i,s}(x)$, we have $\text{diam}(V_{i,s}(x)) < \varepsilon_0$.

Now given $x' \in B(x, \varepsilon_0)$ and a branch $\beta = [z_k, z_{k-1}, \ldots, z_1, z_0 = x] \in T^n(x, \alpha, \vec{v}_e)$, we can find a branch $\beta' = [z_k', z_{k-1}', \ldots, z_1', z_0' = x'] \in T^n(x', \alpha, \vec{v}_e)$, such that $\rho^{n,b}(x, x') = \rho(x, x')$. Pick $z_0' = x'$, because

$$\tilde{g}_n(z_1) = z_0 = x,$$
$$\tilde{g}_n^{-1}(B(x, \varepsilon_0)) = V_{n,1}(x) \cup V_{n,2}(x) \cup \cdots \cup V_{n,k(x)}(x),$$

where $V_{n,s}(x) \cap V_{n,t}(x) = \varnothing$ ($s \neq t$; $s, t = 1, 2, \ldots, k(x)$), and on each $V_{n,s}(x)$ ($s = 1, 2, \ldots, k(x)$)$\tilde{g}_n$ is a homeomorphism, there exists a unique $V_{n,s}(x)$ such that $z_1 \in V_{n,s}(x)$. Since $x' \in B(x, \varepsilon_0)$, there must be a point $z_1' \in V_{n,s}(x)$ such that $\tilde{g}_n(z_1') = z_0' = x'$.

Because $z_1, z_1' \in V_{n,s}(x)$ and $\text{diam}(V_{n,s}(x)) < \varepsilon_0$, we can start from $V_{n,s}(x)$ and regard z_1, z_1', $V_{i,s}(x)$ as the former x, x', $B(x, \varepsilon_0)$. According to the construction above, we can get z_2' such that $\rho(z_2, z_2') < \varepsilon_0$ and $\tilde{g}_{n-1}(z_2') = z_1'$. And so on, we can get the branch $\beta' = [z_k', z_{k-1}', \ldots, z_1', z_0' = x'] \in T^n(x', \alpha, \vec{v}_e)$ satisfying $\rho^b(\beta, \beta') = \rho(x, x')$. By the arbitrariness of β, we have $\rho^{n,b}(x, x') = \rho(x, x')$. So for any $0 < \varepsilon < \varepsilon_0$, $r(\tilde{g}_i, \varepsilon, \rho^{n,b})$ is independent of n, then $h_i(\alpha, \vec{v}_e) = 0$. $\square$

4.2. $\mathbb{Z}_+^k$-action on finite graphs

Assume that X is a graph. X is a finite graph if there exists a distinguished finite set of points (vertices) $V = \{v_1, v_2, \ldots, v_n\}$ such that $X \setminus V$ has finitely many components (edges), each homeomorphic to the unit interval $(0, 1)$.

The metric in which each edge has length 1 is called the geodesic metric, and the distance between two points is the length of the shortest path joining them. Any in the graph homeomorphic image of $[0, 1]$ in X is said to be a closed interval and the interior of a closed interval is

called an open interval.

Lemma 4.5 *Suppose X is a finite graph, $f\colon X \to X$ a continuous map, and I is a closed interval in X. Then the number of points in the frontier of $f(I)$ does not exceed four times the number of edges in X.*

Proof For the proof, the reader can refer to [12]. □

Definition 4.6 ([12]) *Let P be a finite set of points in X. If each component of $X \setminus P$ is an open interval in X and two distinct open intervals share at most one common endpoint, then we call P a division of X. The closures of these intervals are called the atoms of P.*

Note that when $|P| = N$, a division P has at most $2N$ distinct atoms.

Suppose that $g_{1,\infty} = \{g_i\}_{i=1}^{\infty}$ is a sequence of continuous selfmaps on finite graph X. Given a division P of X, define a sequence of divisions $P_i(g_{1,\infty})$, $i = 0, 1, 2, \dots$ by

$$
\begin{aligned}
P_0(g_{1,\infty}) =& P_0, \\
P_{i+1}(g_{1,\infty}) =& P_i(g_{1,\infty}) \cup g_{i+1}(P_i(g_{1,\infty})) \cup \\
& \{\text{frontier pts of } g_{i+1}(I) | I \text{ an atom of } P_i(g_{1,\infty})\}.
\end{aligned}
$$

Then we have the following lemma, whose proof is similar to that in [12].

Lemma 4.7 *Suppose that X is a finite graph, $g_{1,\infty} = \{g_i\}_{i=1}^{\infty}$ is a sequence of continuous selfmaps on X, $P_i(g_{1,\infty})$ ($i = 0, 1, 2, \dots$) is a sequence of divisions as above. If x and x' are both interior to the same atom of $P_n(g_{1,\infty})$ and $\beta = [z_k, z_{k-1}, \dots, z_1, z_0 = x] \in T^n(x, \alpha, \vec{v_e})$, then there exists a branch $\beta' = [z'_k, z'_{k-1}, \dots, z'_1, z'_0 = x'] \in T^n(x', \alpha, \vec{v_e})$ such that for $i = 0, 1, 2, \dots, k, z_i$ and z'_i are both interior to the same atom of $P_{n-i}(g_{1,\infty})$.*

Then we have the following theorem.

Theorem 4.8 *Suppose that X is a finite graph and α is a $\mathbb{Z}_+^k$-action on (X, ρ), then $h_i(\alpha, \vec{v_e}) = 0$.*

Proof Since the set $\{\tilde{g}_1, \tilde{g}_2, \dots\}$ is finite, $\tilde{g}_{1,\infty}$ is equicontinuous. By the equicontinuity of $\tilde{g}_{1,\infty}$, we have that for any positive integer k and $\varepsilon > 0$, there exists $\delta > 0$ such that $\rho(x, x') \leq \delta$ implies

$$
\rho(\tilde{g}_i^j(x), \tilde{g}_i^j(x')) \leq \varepsilon, \tag{4.1}
$$

where $i = 1, 2, \dots$ and $j = 0, 1, 2, \dots, k$.

Pick a division P of X such that the length of each atom of P is at most δ. Let $P_0 = P$. By the above construction method, we get $P_0(\tilde{g}_{1,\infty})$, $\dots$, $P_k(\tilde{g}_{1,\infty})$. Define $\bar{g}_{1,\infty} = \{\bar{g}_i\}_{i=1}^{\infty}$, where $\bar{g}_i = \tilde{g}_{ik}^k$ and then let $P_0 = P_k(\tilde{g}_{1,\infty})$. We obtain $P_0(\bar{g}_{1,\infty})$, $P_1(\bar{g}_{1,\infty})$, $\dots$. By the way in [12], Lemma 4.7 and (4.1), for any integer $n > 0$, if x and x' are both interior to the same atom of $P_n(\bar{g}_{1,\infty})$, then $\rho^{(n+1)k,b}(x, x') \leq \varepsilon$. So a $(\tilde{g}_{1,\infty}, \varepsilon, \rho^{nk,b})$-spanning set for X can be formed by adding one inner point to each atom of $P_n(\bar{g}_{1,\infty})$ to the division points of $P_n(\bar{g}_{1,\infty})$.

Denote by E the number of edges in X, by $N(P_n(\bar{g}_{1,\infty}))$ the number of points in $P_n(\bar{g}_{1,\infty})$,

and by $A(P_n(\bar{g}_{1,\infty}))$ the number of atoms for $P_n(\bar{g}_{1,\infty})$. Since

$$A(P_n(\bar{g}_{1,\infty})) \leq 2 \cdot N(P_n(\bar{g}_{1,\infty})),$$

we have

$$r(\tilde{g}_{1,\infty}, \varepsilon, \rho^{nk,b}, X) \leq A(P_n(\bar{g}_{1,\infty})) + N(P_n(\bar{g}_{1,\infty})) \leq 3 \cdot N(P_n(\bar{g}_{1,\infty})).$$

By construction of $P_n(\bar{g}_{1,\infty})$ and Lemma 4.5, we have

$$\begin{aligned}
N(P_n(\bar{g}_{1,\infty})) &\leq 4E \cdot A(P_{n-1}(\bar{g}_{1,\infty})) + 2 \cdot N(P_{n-1}(\bar{g}_{1,\infty})) \\
&\leq 4E \cdot 2 \cdot N(P_{n-1}(\bar{g}_{1,\infty})) + 2 \cdot N(P_{n-1}(\bar{g}_{1,\infty})) \\
&\leq (8E + 2) \cdot N(P_{n-1}(\bar{g}_{1,\infty})),
\end{aligned}$$

thus

$$N(P_n(\bar{g}_{1,\infty})) \leq (8E + 2)^n \cdot N(P_0(\bar{g}_{1,\infty})).$$

Also, for $nk < m < (n+1)k$,

$$r(\tilde{g}_{1,\infty}, \varepsilon, \rho^{nk,b}, X) \leq r(\tilde{g}_{1,\infty}, \varepsilon, \rho^{m,b}, X) \leq r(\tilde{g}_{1,\infty}, \varepsilon, \rho^{(n+1)k,b}, X),$$

so

$$\begin{aligned}
\varlimsup_{m\to\infty} \frac{\log r_\alpha(\tilde{g}_{1,\infty}, \varepsilon, \rho^{m,b}, X)}{m} &\leq \varlimsup_{n\to\infty} \frac{\log r_\alpha(\tilde{g}_{1,\infty}, \varepsilon, \rho^{(n+1)k,b}, X)}{nk} \\
&\leq \frac{1}{k} \varlimsup_{n\to\infty} \frac{\log 3 \cdot N(P_{n+1}(\bar{g}_{1,\infty}))}{n} \leq \frac{1}{k} \varlimsup_{n\to\infty} \frac{\log 3(8E+2)^{n+1} \cdot N(P_0(\bar{g}_{1,\infty}))}{n} \\
&= \frac{1}{k} \log(8E + 2).
\end{aligned}$$

Note that ε is arbitrary, so we have $h_i(\alpha, \vec{v}_e) \leq \frac{1}{k} \log(8E + 2)$. By the arbitrariness of k, we have $h_i(\alpha, \vec{v}_e) = 0$. $\square$

4.3. $\mathbb{Z}_+^k$-action on some infinite graphs

Suppose $G(V, E)$ (denoted as G for convenience) is a graph satisfying:

(1) The set V of vertices and the set E of edges are infinite;

(2) There are finite accumulation points in V, namely $V' = \{a_1, a_2, \dots, a_k\}$, and

$$f_j^{-1}\Big(\bigcup_{i=1}^k U(a_i, \varepsilon) \cap G\Big) \subset \bigcup_{i=1}^k U(a_i, \varepsilon) \cap G$$

for any $\varepsilon > 0$, $1 \leq i \leq k$, $1 \leq j \leq l$;

(3) For $\varepsilon > 0$, put $A_1 = \bigcup_{i=1}^k U(a_i, \varepsilon) \cap V$, $A_2 = (\bigcup_{i=1}^k U(a_i, \varepsilon))^c \cap V$ and $E_\varepsilon = \{(x, y)|(x, y) \in E,$ where $x \in A_1, y \in A_2\}$, $V_\varepsilon = \{v | v$ is vertex of edge in $E_\varepsilon\}$.

We require that E_ε is a finite set. It is clear that A_1 is an infinite set and A_2 is finite. Suppose P is an infinite set of points of G. If each component of $G \setminus P$ is an open interval in G and two distinct open intervals share at most one common endpoint, then we call P a division of G. The closures of these intervals are called the atoms of P.

Suppose $g_{1,\infty} = \{g_i\}_{i=1}^{\infty}$ is a sequence of continuous selfmaps of G. Given a division P_0 of G, define a sequence of divisions $P_i(g_{1,\infty})$, $i = 0, 1, 2, \ldots$ by

$$P_0(g_{1,\infty}) = P_0,$$
$$P_{i+1}(g_{1,\infty}) = P_i(g_{1,\infty}) \cup g_{i+1}(P_i(g_{1,\infty})) \cup$$
$$\{\text{frontier pts of } g_{i+1}(I) \,|\, I \text{ is an atom of } P_i(g_{1,\infty})\}.$$

Then we have the following lemma whose proof is similar to Lemma 4.7.

Lemma 4.9 *Let G be a infinite graph, $g_{1,\infty} = \{g_i\}_{i=1}^{\infty}$ a sequence of continuous selfmaps on G, and $P_i(g_{1,\infty})$ the sequence of divisions as above. For any integer $n > 0$, if x and x' are both interior to the same atom of $P_n(g_{1,\infty})$, and $\beta = [z_k, z_{k-1}, \ldots, z_1, z_0 = x] \in T^n(x, \alpha, \vec{v_e})$, then there exists a branch $\beta' = [z_k', z_{k-1}', \ldots, z_1', z_0' = x'] \in T^n(x', \alpha, \vec{v_e})$, such that for $i = 0, 1, 2, \ldots, k$, z_i and z_i' are both interior to the same atom of $P_{n-i}(g_{1,\infty})$.*

Take $\varepsilon > 0$, we need to construct a new sequence partition of partitions

$$P_{\varepsilon,i}(g_{1,\infty}) = P_i(g_{1,\infty}) \cap G(V_{\varepsilon/2}, E_{\varepsilon/2}), \quad i = 0, 1, 2, \ldots.$$

For the sequence of partitions $P_{\varepsilon,i}(g_{1,\infty})$, we can get the following lemma.

Lemma 4.10 *Suppose that G is an infinite graph, $g_{1,\infty} = \{g_i\}_{i=1}^{\infty}$ is a sequence of continuous maps on G, each atom in P_0 is less than or equal to ε, $P_{\varepsilon,i}(g_{1,\infty})$ is the sequence of divisions as above. For any integer $n > 0$, if x and x' are both interior to the same atom of $P_{\varepsilon,n}(g_{1,\infty})$, and $\beta = [z_k, z_{k-1}, \ldots, z_1, z_0 = x] \in T^n(x, \alpha, \vec{v_e})$, then there exists a branch $\beta' = [z_k', z_{k-1}', \ldots, z_1', z_0' = x'] \in T^n(x', \alpha, \vec{v_e})$ such that $\rho^{n,b}(x, x') \leq \varepsilon$.*

Proof According to the construction of $P_{\varepsilon,i}(\tilde{g}_{1,\infty})$, we can get $P_{\varepsilon,i}(\tilde{g}_{1,\infty}) \subset P_i(g_{1,\infty})$, so if x and x' are both interior to the same atom of $P_{\varepsilon,n}(\tilde{g}_{1,\infty})$, then x and x' are both interior to the same atom of $P_n(\tilde{g}_{1,\infty})$. By Lemma 4.9 there exists a branch $\beta' = [z_k', z_{k-1}', \ldots, z_1', z_0' = x'] \in T^n(x', \alpha, \vec{v_e})$ such that for $i = 0, 1, 2, \ldots, k$, z_i and z_i' are both interior to the same atom of $P_{n-i}(g_{1,\infty})$. Thus we get $\rho^{n,b}(x, x') \leq \varepsilon$. $\square$

Theorem 4.11 *Assume that G is the above infinite graph, and α is a $\mathbb{Z}_+^k$-action on G, then $h_i(\alpha, \vec{v_e}) = 0$.*

Proof Because the set $\{\tilde{g}_1, \tilde{g}_2, \ldots\}$ is finite, $\tilde{g}_{1,\infty}$ is equicontinuous. By equicontinuity of $\tilde{g}_{1,\infty}$, for any positive integer k and $\varepsilon > 0$ there is $\delta > 0$ such that $\rho(x, x') \leq \delta$ implies

$$\rho(\tilde{g}_i^j(x), \tilde{g}_i^j(x')) \leq \varepsilon, \tag{4.2}$$

where $i = 1, 2, \ldots$ and $j = 0, 1, 2, \ldots, k$.

Suppose that there is only one accumulation point a on the infinite graph G, then for any $\varepsilon > 0$, there are only finite points in V out of $U(a, \frac{\varepsilon}{2})$. The meaning of A_1, A_2, and $E_{\varepsilon/2}$ is described as above. To simplify the notation, denote $G(V_{\varepsilon/2}, E_{\varepsilon/2})$ by $G_{\varepsilon/2}$.

A partition P of G is taken whose length of each atom on the partition P is less than or equal to δ. According to the above construction, we start constructing the sequence of partitions

from $P_{\varepsilon,0} = P \cap G_{\varepsilon/2}$ as follows

$$P_{\varepsilon,0}(\tilde{g}_{1,\infty}), \ldots, P_{\varepsilon,k}(\tilde{g}_{1,\infty}).$$

Define sequence $\bar{g}_{1,\infty} = \{\bar{g}_i\}_{i=1}^{\infty}$, where $\bar{g}_i = \tilde{g}_{ik}^k$, then we continue to construct the sequence of partitions from $P_{\varepsilon,k}(\tilde{g}_{1,\infty})$ as the new $P_{\varepsilon,0}$. Construct the partition sequence

$$P_{\varepsilon,0}(\bar{g}_{1,\infty}), P_{\varepsilon,1}(\bar{g}_{1,\infty}), \ldots.$$

In fact, if $x \in G_{\varepsilon/2}$, by Lemma 4.10 and (4.2) as that for a finite graph, for any given integer $n > 0$, we can find $x' \in P_{\varepsilon,n}(\bar{g}_{1,\infty})$ such that $\rho^{(n+1)k,b}(x, x') \leq \varepsilon$. If $x \in G \backslash G_{\varepsilon/2}$, it is easy to find x' such that $\rho^{(n+1)k,b}(x, x') \leq \varepsilon$. A new set will be given if we take any inner point from each atom of $P_{\varepsilon,n}(\bar{g}_{1,\infty})$ and add it to partition $P_{\varepsilon,n}(\bar{g}_{1,\infty})$. The new set is a $(\tilde{g}_{1,\infty}, \varepsilon, \rho^{nk,b})$-spanning set of G.

Ley $N(P_{\varepsilon,n}(\bar{g}_{1,\infty}))$ be the number of point in $P_{\varepsilon,n}(\bar{g}_{1,\infty})$, $A(P_{\varepsilon,n}(\bar{g}_{1,\infty}))$ be the number of atom in $P_{\varepsilon,n}(\bar{g}_{1,\infty})$. Since

$$A(P_{\varepsilon,n}(\bar{g}_{1,\infty})) \leq 2 \cdot N(P_{\varepsilon,n}(\bar{g}_{1,\infty})),$$

we have

$$r(\tilde{g}_{1,\infty}, \varepsilon, \rho^{nk,b}, G) \leq A(P_{\varepsilon,n}(\bar{g}_{1,\infty})) + N(P_{\varepsilon,n}(\bar{g}_{1,\infty})) \leq 3 \cdot N(P_{\varepsilon,n}(\bar{g}_{1,\infty})).$$

According to the construction of $P_{\varepsilon,n}(\bar{g}_{1,\infty})$ and Lemma 4.5, we obtain

$$\begin{aligned}
N(P_{\varepsilon,n}(\bar{g}_{1,\infty})) &\leq 4E_{\varepsilon/2} \cdot A(P_{\varepsilon,n-1}(\bar{g}_{1,\infty})) + 2 \cdot N(P_{\varepsilon,n-1}(\bar{g}_{1,\infty})) \\
&\leq 4E_{\varepsilon/2} \cdot 2 \cdot N(P_{\varepsilon,n-1}(\bar{g}_{1,\infty})) + 2 \cdot N(P_{\varepsilon,n-1}(\bar{g}_{1,\infty})) \\
&= (8E_{\varepsilon/2} + 2) \cdot N(P_{\varepsilon,n-1}(\bar{g}_{1,\infty})),
\end{aligned}$$

therefore

$$N(P_{\varepsilon,n}(\bar{g}_{1,\infty})) \leq (8E_{\varepsilon/2} + 2)^n \cdot N(P_{\varepsilon,0}(\bar{g}_{1,\infty})).$$

When $nk < m < (n+1)k$, we have

$$r(\tilde{g}_{1,\infty}, \varepsilon, \rho^{nk,b}, G) \leq r(\tilde{g}_{1,\infty}, \varepsilon, \rho^{m,b}, G) \leq r(\tilde{g}_{1,\infty}, \varepsilon, \rho^{(n+1)k,b}, G),$$

thus

$$\begin{aligned}
\varlimsup_{m \to \infty} \frac{\log r_\alpha(\tilde{g}_{1,\infty}, \varepsilon, \rho^{m,b}, G)}{m} &\leq \varlimsup_{n \to \infty} \frac{\log r_\alpha(\tilde{g}_{1,\infty}, \varepsilon, \rho^{(n+1)k,b}, G)}{nk} \\
&\leq \frac{1}{k} \varlimsup_{n \to \infty} \frac{\log 3 \cdot N(P_{\varepsilon,n+1}(\bar{g}_{1,\infty}))}{n} \\
&\leq \frac{1}{k} \varlimsup_{n \to \infty} \frac{\log 3(8E_{\varepsilon/2} + 2)^{n+1} \cdot N(P_{\varepsilon,0}(\bar{g}_{1,\infty}))}{n} \\
&= \frac{1}{k} \log(8E_{\varepsilon/2} + 2).
\end{aligned}$$

By arbitrariness of k, we have

$$\varlimsup_{n \to \infty} \frac{\log r_\alpha(\tilde{g}_{1,\infty}, \varepsilon, \rho^{m,b}, G)}{n} \leq 0,$$

then by arbitrariness of ε, we get $h_i(\alpha, \vec{v}_e) = 0$.

Similar results can be obtained when there are a finite accumulation points on the infinite graph G. $\square$

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