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Indefinite Least Squares Problem with Quadratic Constraint and Its Condition Numbers

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Abstract In this paper, we consider the indefinite least squares problem with quadratic constraint and its condition numbers. The conditions under which the problem has the unique solution are first presented. Then, the normwise, mixed, and componentwise condition numbers for solution and residual of this problem are derived. Numerical example is also provided to illustrate these results.

Keywords indefinite least squares problem; quadratic constraint; normwise condition number; mixed condition number; componentwise condition number

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1. Introduction

The indefinite least squares problem with quadratic constraint (ILSQC) can be stated as follows:

$$\min_{x\in\mathbb{R}^n} (b-Ax)^T J(b-Ax), \text{ subject to } \|Cx-d\|_2=\gamma, \tag{1.1}$$

where $A\in\mathbb{R}^{m\times n}$ with $m\ge n$, $C\in\mathbb{R}^{s\times n}$, $b\in\mathbb{R}^m$, $d\in\mathbb{R}^s$, $\gamma>0$ and J is a signature matrix defined by

$$J=\begin{bmatrix} I_p & 0\\ 0 & -I_q\end{bmatrix},\quad p+q=m.$$

The ILSQC problem can be reduced to the general least squares problem with quadratic constraint (LSQC) by setting $J=I_m$, which can be arise in a variety of applications, such as smoothing of noisy data, the solution of discretized ill-posed problems from inverse problem, and in trust region methods for nonlinear least squares problems [1]. The ILSQC problem can also be converted to the indefinite least squares (ILS) problem by removing the quadratic constraint.

The LSQC problem was first investigated by Å. Björck [1]. Gander [2] presented the conditions under which the problem has the unique solution. Later, some scholars also considered this problem and its varants. For example, Golub and von Matt [3] discussed this problem from

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the view of the theory of Gauss quadrature; Schöne and Hanning [4] studied the least squares problem with absolute quadratic constraints and its applications; Chan et al. [5] presented an algorithm for solving LSQC problem and derived a formula for estimating the Lagrange multiplier; Mead and Renaut [6] discussed the least squares problem with inequality constraints as quadratic constraints. Recently, Diao [7] considered the condition numbers for the least squares problem with quadratic inequality constraint and presented the expressions of normwise, mixed and componentwise condition numbers. It is worth to mention that the systematic theory for normwise condition number was first given by Rice [8] and the terminologies of mixed and componentwise condition numbers were first introduced by Gohberg and Koltracht [9].

The ILS problem was first introduced by Chandrasekaran et al. [10], which has many important applications. For example, it can be used to solve the total least squares problem [11]. Later many researchers have paid attention to the perturbation analysis and the condition numbers for the total least squares problem; see [12–14] and the references therein. In literature, some scholars investigated the numerical algorithms, stability of algorithms, and perturbation analysis of ILS problem [15–18]. Bojanczyk et al. [19] and Grcar [20] discussed its normwise condition number and Li et al. [21] considered its mixed and componentwise condition numbers. Recently, Li and Wang [22] obtained the partial unified condition numbers for the ILS problem. Some results of the paper were recovered by Diao and Zhou [23] by using the dual techniques of condition number theory, and some results were extended to the equality constrained indefinite least squares problem by Wang and Yang [24].

However, to our best knowledge, there is no work on the solution and condition numbers of ILSQC problem so far. In this paper, we will study the solution of ILSQC problem and its condition numbers. Specifically, we will discuss the condition of the uniqueness of the solution of this problem in Section 3 and provide the expressions of normwise, mixed and componentwise condition numbers for solution in Section 4. The expressions of normwise, mixed and componentwise condition numbers for residual are given in Section 5. In addition, Section 2 presents some preliminaries and Section 6 gives a numerical example to illustrate the obtained results.

2. Notations and preliminaries

In this section, we first introduce the definitions of the three condition numbers mentioned in Section 1. To this end, we need the following notations. The first one is the entry-wise division [25] between the vectors $a \in \mathbb{R}^p$ and $b = [b_1, \ldots, b_p] \in \mathbb{R}^p$ defined by

$$\frac{a}{b} = \mathrm{diag}(b^{\ddagger})a,$$

where $\mathrm{diag}(b^{\ddagger})$ is diagonal with diagonal elements $b_1^{\ddagger}, \ldots, b_p^{\ddagger}$. Here, for a number $c \in \mathbb{R}, c^{\ddagger}$ is defined by

$$c^{\ddagger} = \begin{cases} \frac{1}{c}, & \text{if } c \neq 0, \\ 1, & \text{if } c = 0. \end{cases}$$

Thus, we can define the relative distance between a and b as

$$d(a,b) = \|\frac{a-b}{b}\|_\infty = \max_{1\le i\le p}\{\frac{|a_i-b_i|}{|b_i|}\}.$$

When $d(a,b) < \infty$, $d(a,b)$ can be written as

$$d(a,b) = \min\{\delta \ge 0 \mid |a_i - b_i| \le \delta|b_i|, \quad i = 1,\dots,p\}.$$

In addition, for $\varepsilon > 0$, we denote $B^\circ(a,\varepsilon) = \{x \mid d(x,a) \leqslant \varepsilon\}$ and $B(a,\varepsilon) = \{x \mid \|x-a\|_2 \leqslant \varepsilon\|a\|_2\}$.

Definition 2.1 ([9, 25, 26]) *Let $F : \mathbb{R}^p \to \mathbb{R}^q$ be a continuous mapping defined on an open set $\mathrm{Dom}(F) \subset \mathbb{R}^p$ with $\mathrm{Dom}(F)$ denoting the domain of definition of function F and $a \in \mathrm{Dom}(F)$ satisfy $a \neq 0$ and $F(a) \neq 0$.*

(i) The normwise condition number of F at a is defined by

$$\kappa(F,a) = \lim_{\varepsilon\to 0}\sup_{\substack{x\in B(a,\varepsilon)\\ x\neq a}}\left(\frac{\|F(x)-F(a)\|_2}{\|F(a)\|_2}/\frac{\|x-a\|_2}{\|a\|_2}\right).$$

(ii) The mixed condition number of F at a is defined by

$$m(F,a) = \lim_{\varepsilon\to 0}\sup_{\substack{x\in B^o(a,\varepsilon)\\ x\neq a}}\frac{\|F(x)-F(a)\|_\infty}{\|F(a)\|_\infty}\frac{1}{d(x,a)}.$$

(iii) The componentwise condition number of F at a is defined by

$$c(F,a) = \lim_{\varepsilon\to 0}\sup_{\substack{x\in B^o(a,\varepsilon)\\ x\neq a}}\frac{d(F(x),F(a))}{d(x,a)}.$$

In order to give the expressions of the mixed and componentwise condition numbers, the following definition of the Fréchet derivative is necessary.

Definition 2.2 ([27]) *Suppose that F is a mapping, $F : U \in R^p \to R^q$ with U being an open set. Then F is said to be Fréchet differentiable at $a \in U$ if there exists a bounded linear operator $DF : R^p \to R^q$ such that*

$$\lim_{h\to 0}\frac{\|F(a+h) - F(a) - DF(h)\|}{\|h\|} = 0.$$

When F is Fréchet differentiable at a, we use the notation $DF(a)$ to denote the Fréchet derivative or derivative of F at a.

With the Fréchet derivative, the following lemma gives the explicit representations of these three condition numbers.

Lemma 2.3 ([25]) *With the same assumptions as in Definition 2.1, and supposing that F is Fréchet differentiable at a, we have*

$$\kappa(F,a) = \frac{\|DF(a)\|_2\|a\|_2}{\|F(a)\|_2},$$

$$m(F,a) = \frac{\|DF(a)\mathrm{diag}(a)\|_\infty}{\|F(a)\|_\infty} = \frac{\||DF(a)||a|\|_\infty}{\|F(a)\|_\infty},$$

$$c(F,a) = \|\mathrm{diag}^{\ddagger}(F(a))DF(a)\mathrm{diag}(a)\|_\infty = \|\frac{|DF(a)||a|}{|F(a)|}\|_\infty,$$

where $DF(a)$ is the Fréchet derivative of F at a and $|a|$ is to take the absolute value of elements in a.

Recall that for any matrix $A = [a_1, \ldots, a_n] \in \mathbb{R}^{m\times n}$ with $a_i \in \mathbb{R}^m$, the operator vec is defined by

$$\mathrm{vec}(A) = [a_1^T, \ldots, a_n^T]^T \in \mathbb{R}^{mn},$$

and the Kronecker product between $A = (a_{ij}) \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is defined by $A \otimes B = [a_{ij}B] \in \mathbb{R}^{mp\times nq}$.

To obtain the explicit expressions of the above condition numbers, we need some properties of Kronecker product , we need some properties of the operator vec and Kronecker product [28–31]

$$|A \otimes B| = |A| \otimes |B|, \tag{2.1}$$

$$\mathrm{vec}(AXB) = (B^T \otimes A)\mathrm{vec}(X), \tag{2.2}$$

$$\Pi_{mn}\mathrm{vec}(A) = \mathrm{vec}(A^T), \tag{2.3}$$

where the notation $|A|$ is a matrix whose components are the absolute values of the corresponding components of A, and $X \in \mathbb{R}^{n\times p}$, and $\Pi_{st} \in \mathbb{R}^{st\times st}$ is the vec-permutation matrix which depends only on the dimensions s and t.

In addition, we also need the following three lemmas.

Lemma 2.4 ([26]) *For any matrices U, V, C, D, R and S with dimensions making the following well defined*

$$[U \otimes V + (C \otimes D)\Pi]\mathrm{vec}(R),$$
$$\frac{[U \otimes V + (C \otimes D)\Pi]\mathrm{vec}(R)}{S},$$
$$VRU^T \text{ and } DR^TC^T,$$

we have

$$\||[U \otimes V + (C \otimes D)\Pi]|\mathrm{vec}(|R|)\|_\infty \le \|\mathrm{vec}(|V||R||U|^T + |D||R|^T|C|^T)\|_\infty$$

and

$$\|\frac{|[U \otimes V + (C \otimes D)\Pi]|\mathrm{vec}(|R|)}{|S|}\|_\infty \le \|\frac{\mathrm{vec}(|V||R||U|^T + |D||R|^T|C|^T)}{|S|}\|_\infty.$$

In the following, we will define the the product norm to measure the input data $[A, b]$. Let α and β be two positive real numbers, for the data space $\mathbb{R}^{m\times n} \times \mathbb{R}^m$, then

$$\|(A, b)\|_F = \sqrt{\alpha^2\|A\|_F^2 + \beta^2\|b\|_F^2}. \tag{2.4}$$

Lemma 2.5 ([7]) *Let $V \in \mathbb{R}^{m\times n}$, $X \in \mathbb{R}^{n\times m}$, $Y \in \mathbb{R}^{n\times n}$, $s \in \mathbb{R}^n$, $t \in \mathbb{R}^m$, $u \in \mathbb{R}^n$, and define the linear operator l by*

$$l(V, u) := -XVs + YV^Tt + Xu. \tag{2.5}$$

where α and β are positive real numbers. Then, the spectral norm of l is

$$\|l\|_2 = \sup_{V\neq 0, u\neq 0} \frac{\|l(V,u)\|_2}{\|(V,u)\|_F} = \left\| \begin{bmatrix} -\frac{1}{\beta}\|s\|_2 X, & \frac{1}{\alpha}\|t\|_2 Y \end{bmatrix} \begin{bmatrix} c_1 I_m - c_2 \frac{tt^T}{{}_o\|t\|_2^2} & \frac{\alpha}{\beta}\frac{ts^T}{\|t\|_2\|s\|_2} \\ 0 & I_n \end{bmatrix} \right\|_2, \quad (2.6)$$

where $c_1 = \sqrt{\frac{\beta^2}{\alpha^2} + \frac{1}{\|s\|_2^2}}$ and $c_2 = c_1 + \frac{1}{\|s\|_2}$.

Lemma 2.6 ([32, Page 171, Theorem 3]) *Let T be the set of non-singular real $m \times m$ matrices, and S be an open subset of $\mathbb{R}^{n\times q}$. If the matrix function $G : S \to T$ is k times (continuously) differentiable on S, then so is the matrix function $G^{-1} : S \to T$ defined by $G^{-1}(X) = (G(X))^{-1}$, and*

$$dG^{-1} = -G^{-1}(dG)G^{-1},$$

where dG is the differential of G.

3. Solution to ILSQC problem

We first show that the solution x to ILSQC problem (1.1) is the same as the one to the generalized normal equation (3.1) as done in [33, Theorem 2.6.1].

Theorem 3.1 *Let x be the solution to ILSQC problem (1.1) and x_λ be the solution to the following generalized normal equation*

$$(A^T JA + \lambda C^T C)x_\lambda = A^T Jb + \lambda C^T d, \quad (3.1)$$

Then $x = x_\lambda$, where the parameter $\lambda > 0$ is determined by the secular equation

$$\|Cx_\lambda - d\|^2 = \gamma^2.$$

Proof Using the method of Lagrange multipliers, we consider the function

$$L(x,\lambda) := (b - Ax)^T J(b - Ax) + \lambda\{\|Cx - d\|^2 - \gamma^2\},$$

where $\lambda > 0$ is a Lagrange multiplier. Setting the gradient of $L(x,\lambda)$ with respect to x to be zero gives (3.1) where λ is obtained by solving the secular equation. So the solutions of ILSQC problem (1.1) and the generalized normal equation (3.1) are the same. □

In the following, we present two properties of the solution to the generalized normal equation (3.1), from which we can obtain the condition under which the ILSQC problem (1.1) has the unique solution.

Lemma 3.2 *If (x_1, λ_1) and (x_2, λ_2) are two solutions of the normal equation (3.1), then*

$$(b - Ax_2)^T J(b - Ax_2) - (b - Ax_1)^T J(b - Ax_1) = \frac{\lambda_1 - \lambda_2}{2}\|C(x_1 - x_2)\|^2. \quad (3.2)$$

Proof Since (x_1, λ_1) and (x_2, λ_2) are solutions of (3.1), we have

$$A^T JAx_1 - A^T Jb = -\lambda_1 C^T Cx_1 + \lambda_1 C^T d, \quad (3.3)$$

and

$$A^T JAx_2 - A^T Jb = -\lambda_2 C^T Cx_2 + \lambda_2 C^T d. \quad (3.4)$$

If we multiply (3.4) by x_2^T and (3.3) by x_1^T, and subtract the resulting second equation from the first one, we obtain

$$x_2^T A^T JAx_2 - x_1^T A^T JAx_1 - b^T JA(x_2 - x_1) = \lambda_1[\|Cx_1\|^2 - d^T Cx_1] - \lambda_2[\|Cx_2\|^2 - d^T Cx_2]. \tag{3.5}$$

Similarly, by multiplying (3.3) by x_2^T and (3.4) by x_1^T, and subtracting the resulting second equation from the first one, we get

$$-b^T JA(x_2 - x_1) = \lambda_1(-x_2^T C^T Cx_1 + d^T Cx_2) - \lambda_2(-x_1^T C^T Cx_2 + d^T Cx_1). \tag{3.6}$$

Observe that

$$\begin{aligned}&(b - Ax_2)^T J(b - Ax_2) - (b - Ax_1)^T J(b - Ax_1)\\ &\quad = x_2^T A^T JAx_2 - x_1^T A^T JAx_1 - 2b^T JA(x_2 - x_1).\end{aligned}$$

Thus putting (3.5) and (3.6) together leads to

$$\begin{aligned}&(b - Ax_2)^T J(b - Ax_2) - (b - Ax_1)^T J(b - Ax_1)\\ &\quad = \lambda_1\{\|Cx_1\|^2 - d^T Cx_1 - x_2^T C^T Cx_1 + d^T Cx_2\} -\\ &\qquad \lambda_2\{\|Cx_2\|^2 - d^T Cx_2 - x_1^T C^T Cx_2 + d^T Cx_1\}.\end{aligned} \tag{3.7}$$

On the other hand, we also have

$$\|Cx_1 - d\|^2 = \|Cx_2 - d\|^2,$$

which yields

$$\|Cx_1\|^2 - d^T Cx_1 + d^T Cx_2 = \|Cx_2\|^2 - d^T Cx_2 + d^T Cx_1. \tag{3.8}$$

From (3.8), we obtain that the multiplier factors of λ_1 and λ_2 in (3.7) are the same, which is also equal to their arithmetic mean:

$$\frac{1}{2}\{\|Cx_1\|^2 - 2x_1^T C^T Cx_2 + \|Cx_2\|^2\} = \frac{1}{2}\|C(x_1 - x_2)\|^2. \tag{3.9}$$

From equations (3.7) and (3.9), we obtain our required result (3.2). □

Lemma 3.3 *If (x_1, λ_1) and (x_2, λ_2) are two solutions of the normal equation (3.1), then*

$$\begin{aligned}&(\lambda_1 + \lambda_2)\{(b - Ax_2)^T J(b - Ax_2) - (b - Ax_1)^T J(b - Ax_1)\}\\ &\quad = (\lambda_2 - \lambda_1)(x_2 - x_1)^T A^T JA(x_2 - x_1).\end{aligned} \tag{3.10}$$

Proof Since (x_1, λ_1) and (x_2, λ_2) are solutions of the normal equation (3.1), we have

$$\lambda_1 C^T Cx_1 - \lambda_1 C^T d = -A^T JAx_1 + A^T Jb \tag{3.11}$$

and

$$\lambda_2 C^T Cx_2 - \lambda_2 C^T d = -A^T JAx_2 + A^T Jb. \tag{3.12}$$

If we multiply (3.11) by $\lambda_1 x_1^T$ and (3.12) by $\lambda_2 x_2^T$ and subtract the resulting second equation from the first one, we obtain

$$\lambda_1\lambda_2(x_2 - x_1)^T Cd^T = (\lambda_2 - \lambda_1)x_1^T A^T JAx_2 + (\lambda_1 x_1 - \lambda_2 x_2)^T A^T Jb. \tag{3.13}$$

Similarly by multiplying (3.11) by $\lambda_1 x_2^T$ and (3.12) by $\lambda_2 x_1^T$ and subtracting the resulting second equation from the first one, we get

$$\begin{aligned}&\lambda_1\lambda_2\{\|Cx_2\|^2 - \|Cx_1\|^2 + (x_1 - x_2)^T Cd^T\}\\ &= \lambda_2 x_1^T A^T JAx_1 - \lambda_1 x_2^T A^T JAx_2 + (\lambda_1 x_2 - \lambda_2 x_1)^T A^T Jb. \end{aligned} \tag{3.14}$$

Observe that $0 = \|Cx_2 - d\|^2 - \|Cx_2 - d\|^2 = \|Cx_2\|^2 - \|Cx_1\|^2 + 2(x_1 - x_2)^T C^T d$. Thus, subtracting (3.13) from (3.14), we obtain

$$\begin{aligned}&\lambda_1(x_2^T A^T JAx_2 - x_2^T A^T Jb + x_1^T A^T Jb - x_1^T A^T JAx_2)\\ &= \lambda_2(x_1^T A^T JAx_1 - x_1^T A^T Jb - x_2^T A^T Jb - x_1^T A^T JAx_2). \end{aligned} \tag{3.15}$$

Note that the left hand side of (3.15) can be rewritten as

$$\frac{1}{2}\{(b - Ax_2)^T J(b - Ax_2) - (b - Ax_1)^T J(b - Ax_1) + (x_2 - x_1)^T A^T JA(x_2 - x_1)\}\lambda_1.$$

The case for the right hand side of (3.15) is similar. Putting them together, we obtain (3.10). □

With the help of the results in the above two lemmas, we now give the condition of the uniqueness of the solution of the ILSQC problem (1.1).

Theorem 3.4 *If the following condition holds*

$$\operatorname{rank}\begin{pmatrix} A^T JA \\ C \end{pmatrix} = n, \tag{3.16}$$

then the solution x to the ILSQC problem (1.1) is unique.

Proof Assume (x_1, λ_1) and (x_2, λ_2) are solutions of the normal equation (3.1) which also solve the ILSQC problem (1.1). If $\lambda_1 \neq \lambda_2$, then we have

$$(b - Ax_1)^T J(b - Ax_1) = (b - Ax_2)^T J(b - Ax_2) = \min_{x\in\mathbb{R}^n}(b - Ax)^T J(b - Ax).$$

By Eq. (3.2), we obtain

$$\|C(x_1 - x_2)\|^2 = 0. \tag{3.17}$$

While Eq. (3.10) implies that

$$(x_2 - x_1)^T A^T JA(x_2 - x_1) = 0. \tag{3.18}$$

Eqs. (3.17) and (3.18) are equivalent to

$$\begin{pmatrix} A^T JA \\ C \end{pmatrix}(x_2 - x_1) = 0. \tag{3.19}$$

If $x_1 \neq x_2$ then (3.17) and (3.18) shows $A^T JA$ and C have non-trivially intersecting null space, which is a contradiction. If $\lambda_1 = \lambda_2 = \lambda$ than from (3.1), we have

$$\begin{aligned}(A^T JA + \lambda C^T C)x_1 &= A^T Jb + \lambda C^T d,\\ (A^T JA + \lambda C^T C)x_2 &= A^T Jb + \lambda C^T d.\end{aligned}$$

Subtracting the above equations, we obtain

$$(A^TJA + \lambda C^TC)(x_1 - x_2) = 0.$$

If $x_1 \neq x_2$ then $\lambda = -\lambda'$, which is also a contradiction. Therefore we must have $x_1 = x_2$ and $\lambda_1 = \lambda_2$. Hence A^TJA and C have a trivially intersection of their nullspaces because the condition (3.16). □

4. Condition numbers for ILSQC problem

We first present the explicit asseveration for the Fréchet derivative of the mapping ϕ defined by:

$$(a, c, b, d) \to \phi(a, c, b, d) = x(a, c, b, d) = Q(A, C)(A^TJb + \lambda C^Td), \tag{4.1}$$

where $(A^TJA + \lambda C^TC)$ is nonsingular and $Q(A, C) = (A^TJA + \lambda C^TC)^{-1}$ with $a = \mathrm{vec}(A)$, and $c = \mathrm{vec}(C)$.

Lemma 4.1 *The Fréchet derivative of the mapping ϕ at (a, c, b, d) has the following matrix expression*

$$d\phi(a, c, b, d) = [F(A, C, b, d), G(A, C, b, d), Q(A, C)KA^TJ, Q(A, C)(\lambda KC^T + l)],$$

where

$$\begin{aligned}
&F(A, C, b, d) = Q(A, C)K(I_n \otimes (Jr_1)^T - x^T \otimes A^TJ),\\
&G(A, C, b, d) = Q(A, C)(\lambda(K \otimes r_2^T) - (x^T \otimes (\lambda KC^T + l)),\\
&r_1 = b - Ax,\ r_2 = d - Cx,\ K = I_n - \frac{C^Tr_2r_2^TCQ(A, C)}{r_2^TCQ(A, C)C^Tr_2},\ l = \frac{C^Tr_2r_2^T}{r_2^TCQ(A, C)C^Tr_2}.
\end{aligned}$$

Proof It is easy to find that the mapping ϕ is continuous on $\mathbb{R}^{mn} \times \mathbb{R}^{sn} \times \mathbb{R}^m \times \mathbb{R}^s$ and is Fréchet differentiable at (a, c, b, d). In the following, we give the expression of the Fréchet derivative of ϕ at (a, c, b, d). Firstly, we obtain the derivative of λ in $Q(A, C) = (A^TJA + \lambda C^TC)^{-1}$ with respect to (a, c, b, d). Note that

$$\gamma^2 = \|Cx - d\|_2^2 = \|CQ(A, C)(A^TJA + \lambda C^TC)^{-1} - d\|_2^2$$

and γ is constant. Thus, differentiating both sides of the above equation, we can deduce

$$\begin{aligned}
0 &= 2(Cx - d)^T\mathrm{d}(Cx - d)\\
&= 2(Cx - d)^T(\mathrm{d}(C)x + C\mathrm{d}x - \mathrm{d}d)\\
&= -2r_2^T(\mathrm{d}(C)x + \mathrm{Cd}\left(Q(A, C)(A^TJb + \lambda C^Td)\right) - \mathrm{d}d)\\
&= -2r_2^T\left\{\mathrm{d}(C)x + C[-(Q(A, C)\mathrm{d}(A^TJA + \lambda C^TC)x + Q(A, C)\mathrm{d}(A^Tb + \lambda C^Td)] - \mathrm{d}d)\right\}\\
&= -2r_2^T\{\mathrm{d}(C)x + CQ(A, C)[-(\mathrm{d}A^TJA + A^TJ\mathrm{d}A + \mathrm{d}\lambda C^TC + \lambda \mathrm{d}C^TC + \lambda C^T\mathrm{d}C)x]+\\
&\quad CQ(A, C)[\mathrm{d}A^TJb + A^TJ\mathrm{d}b + \mathrm{d}\lambda C^Td + \lambda \mathrm{d}C^Td + \lambda C^T\mathrm{d}d] - \mathrm{d}d\}\\
&= -2r_2^T\{\mathrm{d}(C)x + CQ(A, C)[\mathrm{d}A^TJr_1 + \mathrm{d}\lambda C^Tr_2 + A^TJ(\mathrm{d}b - \mathrm{d}Ax) + \lambda \mathrm{d}C^Tr_2+\\
&\quad \lambda C^T(\mathrm{d}d - \mathrm{d}Cx)] - \mathrm{d}d\}.
\end{aligned}$$

From the above equation, we obtain the expression for $\mathrm{d}\lambda$:

$$\mathrm{d}\lambda = \frac{-r_2^T\{\mathrm{d}(C)x + CQ(A,C)[\mathrm{d}A^TJr_1 + A^TJ(\mathrm{d}b - \mathrm{d}Ax) + \lambda\mathrm{d}C^Tr_2 + \lambda C^T(\mathrm{d}d - \mathrm{d}Cx)] - \mathrm{d}d\}}{r_2^TCQ(A,C)C^Tr_2}. \tag{4.2}$$

Now, differentiating both sides of $\phi(A,C,b,d) = Q(A,C)(A^TJb + \lambda C^Td)$ leads to

$$\begin{aligned}\mathrm{d}\phi(a,c,b,d) &= \mathrm{d}[Q(A,C)(A^TJb + \lambda C^Td)\\ &= Q(A,C)[\mathrm{d}A^TJr_1 + \mathrm{d}\lambda C^Tr_2 + A^TJ(\mathrm{d}b - \mathrm{d}Ax) + \lambda\mathrm{d}C^Tr_2 + \lambda C^T(\mathrm{d}d - \mathrm{d}Cx)],\end{aligned}$$

Bringing together (4.2) into the above equation and after rearranging, we get

$$\begin{aligned}\mathrm{d}\phi(a,c,b,d) =& Q(A,C)K[\mathrm{d}A^TJr_1 + A^TJ(\mathrm{d}b - \mathrm{d}Ax) + \lambda\mathrm{d}C^Tr_2 + \lambda C^T(\mathrm{d}d - \mathrm{d}Cx)] -\\ & Q(A,C)l(\mathrm{d}Cx - \mathrm{d}d)\\ =& Q(A,C)\{K[\mathrm{d}A^TJr_1 + A^TJ(\mathrm{d}b - \mathrm{d}Ax) + \lambda\mathrm{d}C^Tr_2 + \lambda C^T(\mathrm{d}d - \mathrm{d}Cx)] -\\ & l(\mathrm{d}Cx - \mathrm{d}d)\}\\ =& Q(A,C)\{K[\mathrm{d}A^TJr_1 + A^TJ(\mathrm{d}b - \mathrm{d}Ax)] + \lambda K\mathrm{d}C^Tr_2 - (\lambda KC^T + l)\mathrm{d}Cx +\\ & (\lambda KC^T + l)\mathrm{d}d)\}.\end{aligned}$$

Applying vec operator on the both sides of the above equation and using (2.2) and (2.3) gives

$$\begin{aligned}\mathrm{d}\phi =& Q(A,C)[-(x^T\otimes(KA^TJ))\mathrm{vec}(\mathrm{d}A) + (Jr_1^T\otimes K)\mathrm{vec}(\mathrm{d}A^T) - (x^T\otimes(\lambda KC^T + l))\mathrm{vec}(\mathrm{d}C) +\\ & \lambda(r_2^T\otimes K)\mathrm{vec}(\mathrm{d}C^T) + KA^TJ\mathrm{d}b + (KC^T + l)\mathrm{d}d] \quad \text{by (3)}\\ =& Q(A,C)[-x^T\otimes(KA^TJ) + (Jr_1^T\otimes K)\Pi]\mathrm{vec}(\mathrm{d}A) + [-(x^T\otimes(\lambda KC^T + l)) + \lambda(r_2^T\otimes K)\Pi]\mathrm{vec}(\mathrm{d}C) +\\ & KA^TJ\mathrm{d}b + (\lambda KC^T + l)\mathrm{d}d)] \quad \text{by (2.3)}\\ =& Q(A,C)[K\otimes(Jr_1^T) - x^T\otimes(KA^TJ), \lambda(K\otimes r_2^T) - (x^T\otimes(\lambda KC^T + l)), KA^TJ, \lambda KC^T + l]\times\\ & \begin{bmatrix}\mathrm{vec}(\mathrm{d}A)\\ \mathrm{vec}(\mathrm{d}C)\\ \mathrm{d}b\\ \mathrm{d}d\end{bmatrix},\end{aligned}$$

where

$$(K\otimes Jr_1^T) - x^T\otimes(KA^TJ) = K(I_n\otimes(Jr_1)^T - x^T\otimes(A^TJ)).$$

Thus, we have the desired result. □

Now, we define the normwise, mixed, and componentwise condition numbers for ILSQC problem as follows:

$$\kappa^{ILSQC}(A,C,b,d) := \lim_{\varepsilon\to 0} \sup_{\left\|\left(\begin{bmatrix}\Delta A\\ \Delta C\end{bmatrix}, \begin{bmatrix}\Delta b\\ \Delta d\end{bmatrix}\right)\right\|_F \leqslant \varepsilon} \frac{\|\Delta x\|_2 \left\|\left(\begin{bmatrix}A\\ C\end{bmatrix}, \begin{bmatrix}b\\ d\end{bmatrix}\right)\right\|_F}{\varepsilon\|x\|_2}, \tag{4.3}$$

$$m^{ILSQC}(A,C,b,d) := \lim_{\varepsilon\to 0} \sup_{\substack{|\Delta A|\leqslant\varepsilon|A|, |\Delta C|\leqslant\varepsilon|C|\\ |\Delta b|\leqslant\varepsilon|b|, |\Delta d|\leqslant\varepsilon|d|}} \frac{\|\Delta x\|_\infty}{\varepsilon\|x\|_\infty},$$

$$c^{ILSQC}(A,C,b,d) := \lim_{\varepsilon\to 0} \sup_{\substack{|\Delta A|\leqslant\varepsilon|A|,|\Delta C|\leqslant\varepsilon|C| \\ |\Delta b|\leqslant\varepsilon|b|,|\Delta d|\leqslant\varepsilon|d|}} \frac{1}{\varepsilon}\left\|\frac{\Delta x}{x}\right\|_\infty,$$

where $\|\cdot\|_F$ is the product norm defined by (2.4),

$$\Delta x = \phi(a+\delta a, c+\delta c, b+\delta b, d+\delta d) - \phi(a,c,b,d), \tag{4.4}$$

with $\delta a = \mathrm{vec}(\Delta \mathrm{A})$, $\delta c = \mathrm{vec}(\Delta \mathrm{C})$, $\delta b = \Delta b$, and $\delta d = \Delta d$. Using the mapping (4.1) and noting that

$$\Delta x = \mathrm{d}\phi(a,c,b,d).(\delta a^T, \delta c^T, \delta b^T, \delta d^T)^T + O(\epsilon^2) = \mathrm{d}\phi(a,c,b,d)\begin{bmatrix}\delta a\\ \delta c\\ \delta b\\ \delta d\end{bmatrix} + O(\epsilon^2),$$

we have

$$\kappa^{ILSQC}(A,C,b,d) = \kappa(\phi;a,c,b,d), \quad m^{ILSQC}(A,C,b,d) = m(\phi;a,c,b,d),$$

$$c^{ILSQC}(A,C,b,d) = c(\phi;a,c,b,d).$$

In the following theorem, we present the explicit asseverations for normwise, mixed and componentwise condition numbers for the solution x of ILSQC problem (1.1).

Theorem 4.2 *For the solution of ILSQC problem (1.1)*

$$x = (A^TJA + \lambda C^TC)^{-1}(A^TJb + \lambda C^Td),$$

the normwise, mixed and componentwise condition number defined by (4.3) are

$$\kappa^{ILSQC}(A,C,b,d) = \frac{\left\|\left(\begin{bmatrix}A\\ C\end{bmatrix}, \begin{bmatrix}b\\ d\end{bmatrix}\right)\right\|_F}{\|x_\lambda\|_2}\times$$
$$\left\|Q(A,C)\begin{bmatrix}-\frac{1}{\beta}\|x\|_2KA^TJ & -\frac{1}{\beta}\|x\|_2(\lambda KC^T + l) & \frac{1}{\alpha}\|t\|_2K\end{bmatrix}\begin{bmatrix}C_1I_{m+p} - C_2\frac{rr^T}{\|r\|_2^2} & \frac{\beta}{\alpha}\frac{rx^T}{\|x\|_2\|r\|_2}\\ 0 & I_n\end{bmatrix}\right\|_2,$$

$$m^{ILSQC}(A,C,b,d)$$
$$= \frac{\||F(A,C,b,d)|\mathrm{vec}(|A|) + |G(A,C,b,d)|\mathrm{vec}(|C|) + |Q(A,C)KA^TJ||b| + |Q(A,C)(\lambda KC^T + l)||d|\|_\infty}{\|x\|_\infty},$$

$$c^{ILSQC}(A,C,b,d)$$
$$= \left\|\frac{|F(A,C,b,d)|\mathrm{vec}(|A|) + |G(A,C,b,d)|\mathrm{vec}(|C|) + |Q(A,C)KA^TJ||b| + |Q(A,C)(\lambda KC^T + l)||d|}{|x|}\right\|_\infty,$$

where

$$r = \begin{bmatrix}Jr_1\\ \lambda r_2\end{bmatrix}, \ c_1 = \sqrt{\frac{\beta^2}{\alpha^2} + \frac{1}{\|x\|_2^2}}, c_2 = c_1 + \frac{1}{\|x\|_2}.$$

Proof From Lemma 4.1 and Definition 2.2 for the normwise condition number $\kappa^{ILSQC}(A,C,b,d)$, we know that

$$\kappa^{ILSQC}(A,C,b,d) = \sup_{\substack{\mathrm{d}A\neq 0, \mathrm{d}b\neq 0\\ \mathrm{d}C\neq 0, \mathrm{d}d\neq 0}} \frac{\|\mathrm{d}\phi(A,C,b,d)\cdot(\mathrm{d}A,\mathrm{d}C,\mathrm{d}b,\mathrm{d}d)\|_2}{\left\|\left(\begin{bmatrix}\mathrm{d}A\\ \mathrm{d}C\end{bmatrix}, \begin{bmatrix}\mathrm{d}b\\ \mathrm{d}d\end{bmatrix}\right)\right\|_F} \cdot \frac{\left\|\left(\begin{bmatrix}A\\ C\end{bmatrix}, \begin{bmatrix}b\\ d\end{bmatrix}\right)\right\|_F}{\|x_\lambda\|_2}.$$

Using Lemma 4.1 and some algebraic operations, we have

$$\begin{aligned}
&\mathrm{d}\phi(a,c,b,d)\cdot(\mathrm{d}A,\mathrm{d}C,\mathrm{d}b,\mathrm{d}d)\\
&= Q(A,C)\{K[\mathrm{d}A^T Jr_1 + A^T J(\mathrm{d}b - \mathrm{d}Ax)] + \lambda K \mathrm{d}C^T r_2 - (\lambda KC^T + l)\mathrm{d}Cx + (\lambda KC^T + l)\mathrm{d}d)\}\\
&= Q(A,C)\left\{-\begin{bmatrix}KA^TJ & \lambda KC^T + l\end{bmatrix}\begin{bmatrix}\mathrm{d}A\\ \mathrm{d}C\end{bmatrix} x + K\begin{bmatrix}\mathrm{d}A\\ \mathrm{d}C\end{bmatrix}^T \begin{bmatrix}Jr_1\\ \lambda r_2\end{bmatrix} + \begin{bmatrix}KA^TJ & \lambda KC^T + l\end{bmatrix}\begin{bmatrix}\mathrm{d}b\\ \mathrm{d}d\end{bmatrix}\right\}
\end{aligned}$$

From Lemma 2.5 and identifying

$$X = Q(A,C)\begin{bmatrix}KA^TJ & \lambda KC^T + l\end{bmatrix},\ Y = Q(A,C)K,$$

$$V = \begin{bmatrix}\mathrm{d}A\\ \mathrm{d}C\end{bmatrix},\ u = \begin{bmatrix}\mathrm{d}b\\ \mathrm{d}c\end{bmatrix},\ s = x,\ t = \begin{bmatrix}Jr_1\\ \lambda r_2\end{bmatrix},$$

we conclude the following form

$$\begin{aligned}
&\kappa^{ILSQI}(A,C,b,d) = \sup_{\substack{\mathrm{d}A\neq 0, \mathrm{d}b\neq 0\\ \mathrm{d}C\neq 0, \mathrm{d}d\neq 0}} \frac{\|\mathrm{d}\phi(A,C,b,d)\cdot(\mathrm{d}A,\mathrm{d}C,\mathrm{d}b,\mathrm{d}d)\|_2}{\left\|\left(\begin{bmatrix}\mathrm{d}A\\ \mathrm{d}C\end{bmatrix}, \begin{bmatrix}\mathrm{d}b\\ \mathrm{d}d\end{bmatrix}\right)\right\|_F}\\
&= \left\|Q(A,C)\begin{bmatrix}-\frac{1}{\beta}\|x\|_2 KA^TJ & -\frac{1}{\beta}\|x\|_2(\lambda KC^T + l) & \frac{1}{\alpha}\|t\|_2 K\end{bmatrix}\begin{bmatrix}C_1 I_{m+p} - C_2\frac{rr^T}{\|r\|_2^2} & \frac{\beta}{\alpha}\frac{rx^T}{\|x\|_2\|r\|_2}\\ 0 & I_n\end{bmatrix}\right\|_2,
\end{aligned}$$

then we have the explicit asseveration of $\kappa^{ILSQC}(A,C,b,d)$.

Combining Lemma 2.3 with Lemma 4.1, we have the mixed and componentwise condition numbers
$m^{ILSQC}(A,C,b,d)$

$$\begin{aligned}
&= \frac{\left\| |d\phi(a,c,b,d)| \begin{bmatrix}|a|\\ |c|\\ |b|\\ |d|\end{bmatrix}\right\|_\infty}{\|x\|_\infty}\\
&= \frac{\left\| |[F(A,C,b,d), G(A,C,b,d), Q(A,C)KA^TJ, Q(A,C)(\lambda KC^T + l)]| \begin{bmatrix}|a|\\ |c|\\ |b|\\ |d|\end{bmatrix}\right\|_\infty}{\|x\|_\infty}\\
&= \left\|\frac{|F(A,C,b,d)|\mathrm{vec}(|A|) + |G(A,C,b,d)|\mathrm{vec}(|C|) + |Q(A,C)KA^TJ||b| + |Q(A,C)(\lambda KC^T + l)||d|}{|x|}\right\|_\infty,
\end{aligned}$$

and
$c^{ILSQC}(A,C,b,d)$

$$= \left\| \frac{|d\phi(a,c,b,d)| \begin{bmatrix} |a| \\ |c| \\ |b| \\ |d| \end{bmatrix}}{|x|} \right\|_\infty$$

$$= \left\| \frac{|[F(a,c,b,d), G(A,C,b,d), Q(A,C)KA^TJ, Q(A,C)(\lambda KC^T+l)]| \begin{bmatrix} |a| \\ |c| \\ |b| \\ |d| \end{bmatrix}}{|x|} \right\|_\infty$$

$$= \| \frac{|F(A,C,b,d)||\mathrm{vec}(|A|)| + |G(A,C,b,d)||\mathrm{vec}(|C|)| + |Q(A,C)KA^TJ||b| + |Q(A,C)(\lambda KC^T+l)||d|}{|x|} \|_\infty .$$

The next corollary give the easier upper bounds for condition numbers $m^{ILSQC}(A,C,b,d)$ and $c^{ILSQC}(A,C,b,d)$.

Corollary 4.3 *Assume that the conditions of Theorem 4.2 hold. Then*

$$\begin{aligned} m^{ILSQC}(A,C,b,d) \leq & m_{upp}^{ILSQC}(A,C,b,d) \\ = & \| |Q(A,C)K|(|A^T||Jr_1| + |A^TJ||A||x|) + |Q(A,C)(|\lambda||K||C^T||r_2| + \\ & |\lambda KC^T + l||C||x|)| + |Q(A,C)KA^TJ||b| + |Q(A,C)(\lambda KC^T+l)||d|\|_\infty / \|x\|_\infty , \\ c^{ILSQC}(A,C,b,d) \leq & c_{upp}^{ILSQC}(A,C,b,d) \\ = & \| |Q(A,C)K|(|A^T||Jr_1| + |A^TJ||A||x|) + |Q(A,C)(|\lambda||K||C^T||r_2| + \\ & |\lambda KC^T + l||C||x|)| + |Q(A,C)KA^TJ||b| + |Q(A,C)(\lambda KC^T+l)||d|/x\|_\infty . \end{aligned}$$

Proof Applying Lemma 2.4 to $m^{ILSQC}(A,C,b,d)$ and $c^{ILSQC}(A,C,b,d)$ yields

$$\begin{aligned} m_{upp}^{ILSQC}(A,C,b,d) = & \| [Q(A,C)[-x^T \otimes (KA^TJ) + (Jr_1^T \otimes K)\Pi]\mathrm{vec}(\mathrm{d}A) + [-(x^T \otimes (\lambda KC^T + l)) + \\ & \lambda(r_2^T \otimes K)\Pi]\mathrm{vec}(\mathrm{d}C) + KA^TJ\mathrm{d}b + (\lambda KC^T + l)\mathrm{d}d)]\|_\infty / \|x\|_\infty , \\ \leq & \| |Q(A,C)K|(|A^T||Jr_1| + |A^TJ||A||x|) + |Q(A,C)(|\lambda||K||C^T||r_2| + \\ & |\lambda KC^T + l||C||x|)| + |Q(A,C)KA^TJ||b| + |Q(A,C)(\lambda KC^T+l)||d|\|_\infty / \|x\|_\infty . \end{aligned}$$

The proof of the upper bound for $c_{upp}^{ILSQC}(A,C,b,d)$ is similar, so it is omitted. □

5. Condition numbers for residual of ILSQC problem

In this section, we will derive the normwise, mixed and componentwise condition numbers for the residual vector r of ILSQC problem. We first consider the explicit asseveration for the Fréchet derivative of ψ defined at (a,c,b,d). Let $\psi:\mathbb{R}^{mn}\times\mathbb{R}^{sn}\times\mathbb{R}^{m}\times\mathbb{R}^{s}\longrightarrow\mathbb{R}^{m}$ defined by

$$(a,c,b,d)\to\psi(a,c,b,d)=r(a,c,b,d)=b-A(Q(A,C)(A^TJb+\lambda C^Td)), \tag{5.1}$$

where $(A^TJA+\lambda C^TC)$ is nonsingular and $Q(A,C)=(A^TJA+\lambda C^TC)^{-1}$.

Now, we define the normwise, mixed, and componentwise condition numbers for residual of ILSQC problem as follows:

$$\begin{aligned}
\kappa_{res}(A,C,b,d) &:= \lim_{\varepsilon\to0}\sup_{\left\|\left(\begin{bmatrix}\Delta A\\ \Delta C\end{bmatrix},\begin{bmatrix}\Delta b\\ \Delta d\end{bmatrix}\right)\right\|_F\leqslant\varepsilon}\frac{\|\Delta r\|_2\left\|\left(\begin{bmatrix}A\\ C\end{bmatrix},\begin{bmatrix}b\\ d\end{bmatrix}\right)\right\|_F}{\varepsilon\|r\|_2},\\
m_{res}(A,C,b,d) &:= \lim_{\varepsilon\to0}\sup_{\substack{|\Delta A|\leqslant\varepsilon|A|,|\Delta C|\leqslant\varepsilon|C|\\ |\Delta b|\leqslant\varepsilon|b|,|\Delta d|\leqslant\varepsilon|d|}}\frac{\|\Delta r\|_\infty}{\varepsilon\|r\|_\infty},\\
c_{res}(A,C,b,d) &:= \lim_{\varepsilon\to0}\sup_{\substack{|\Delta A|\leqslant\varepsilon|A|,|\Delta C|\leqslant\varepsilon|C|\\ |\Delta b|\leqslant\varepsilon|b|,|\Delta d|\leqslant\varepsilon|d|}}\frac{1}{\varepsilon}\|\frac{\Delta r}{r}\|_\infty,
\end{aligned} \tag{5.2}$$

where $\|\cdot\|_F$ is the product norm defined by (2.4),

$$r+\Delta r=(b+\Delta b)-(A+\Delta A)(x+\Delta x),$$

and ψ is defined by (5.1), using the mapping, we have

$$\kappa_{res}(A,C,b,d)=\kappa(\psi;a,c,b,d),\ m_{res}(A,C,b,d)=m(\psi;a,c,b,d),\ c_{res}(A,C,b,d)=c(\psi;a,c,b,d).$$

In the following theorem, we prove the explicit asseveration for the Fréchet derivative of ψ at (a,c,b,d).

Lemma 5.1 *The function ψ is continuous on $\mathbb{R}^{mn}\times\mathbb{R}^{sn}\times\mathbb{R}^{m}\times\mathbb{R}^{s}$. In addition ψ is Fréchet differentiable at (a,c,b,d) and has the matrix expression*

$$\mathrm{d}\psi(A,C,b,d)=[S(A,C,b,d),H(A,C,b,d),M,-N],$$

where

$$\begin{aligned}
&S(A,C,b,d)=-x^T\otimes M-(Jr_1)^T\otimes(AQ(A,C)K)\Pi_{mn},\\
&H(A,C,b,d)=x^T\otimes N-\lambda(r_2^T\otimes(AQ(A,C)K))\Pi_{pn},\\
&M=(I_m-AQ(A,C)KA^TJ),\quad N=AQ(A,C)(\lambda KC^T+l).
\end{aligned}$$

Proof Differentiating both sides $\psi(A,C,b,d)=b-Ax$, we get

$$\begin{aligned}
\mathrm{d}\psi(a,c,b,d)=&\mathrm{d}[b-Ax]\\
=&\mathrm{d}b-\mathrm{d}Ax-A[Q(A,C)\{K[\mathrm{d}A^TJr_1+A^TJ(\mathrm{d}b-\mathrm{d}Ax)]+\lambda K\mathrm{d}C^Tr_2-\\
&(\lambda KC^T+l)\mathrm{d}Cx+(\lambda KC^T+l)\mathrm{d}d)\}]
\end{aligned}$$

$$=M(\mathrm{d}b-\mathrm{d}Ax)-AQ(A,C)K(\mathrm{d}A^TJr_1+\lambda\mathrm{d}C^Tr_2)+N(\mathrm{d}Cx-\mathrm{d}d).$$

By applying the vec operator, we obtain

$$\begin{aligned}\mathrm{d}\psi =&\mathrm{vec}[M(\mathrm{d}b-\mathrm{d}Ax)-AQ(A,C)K(\mathrm{d}A^TJr_1+\lambda\mathrm{d}C^Tr_2)+N(\mathrm{d}Cx-\mathrm{d}d)]\\ =&[-x^T\otimes M-(Jr_1)^T\otimes(AQ(A,C)K)\Pi_{mn}]\mathrm{vec}(\mathrm{d}A)+[x^T\otimes N-\lambda(r_2^T\otimes(AQ(A,C)K))\Pi_{pn}]\mathrm{vec}(\mathrm{d}C)+\\ &M\mathrm{d}b-N\mathrm{d}d\\ =&\big[-x^T\otimes M-(Jr_1)^T\otimes(AQ(A,C)K)\Pi_{mn},x^T\otimes N-\lambda(r_2^T\otimes(AQ(A,C)K))\Pi_{pn},M,-N\big]\begin{bmatrix}\mathrm{vecd}A\\ \mathrm{vecd}C\\ \mathrm{d}b\\ \mathrm{d}d\end{bmatrix},\end{aligned}$$

Thus, $\mathrm{d}\psi(A,C,b,d)=[S(A,C,b,d),H(A,C,b,d),M,-N]$, we complete our desired result. □

Next, we will present the explicit expression of normwise, mixed and componentwise condition numbers for residual of ILSQC problem. The proof of the following theorem is similar to the proof of Theorem 4.2, thus it is omitted.

Theorem 5.2 *Let $A\in\mathbb{R}^{m\times n}$, $C\in\mathbb{R}^{p\times n}$, $b\in\mathbb{R}^m$, $d\in\mathbb{R}^p$, then normwise, mixed and componentwise condition numbers for residual vector r of ILSQC problem defined by (5.2), we have*

$$\kappa_{res}(A,C,b,d)=\left\|Q(A,C)\left[\tfrac{1}{\beta}\|x\|_2(M\quad -N)\quad -\tfrac{1}{\alpha}\|t\|_2AQ(A,C)K\right]\begin{bmatrix}C_1I_{m+p}-C_2\frac{rr^T}{\|r\|_2^2} & \frac{\beta}{\alpha}\frac{rx^T}{\|x\|_2\|r\|_2}\\ 0 & I_n\end{bmatrix}\right\|_2\times\frac{\left\|\left(\begin{bmatrix}A\\ C\end{bmatrix},\begin{bmatrix}b\\ d\end{bmatrix}\right)\right\|_F}{\|r\|_2}$$

$$m_{res}(A,C,b,d)=\frac{\||S(A,C,b,d)|\mathrm{vec}(|A|)+|H(A,C,b,d)|\mathrm{vec}(|C|)+|M||b|+|N||d|\|_\infty}{\|r\|_\infty},$$

$$c_{res}(A,C,b,d)=\|\frac{|S(A,C,b,d)|\mathrm{vec}(|A|)+|H(A,C,b,d)|\mathrm{vec}(|B|)+|M||b|+|N||d|}{r}\|_\infty,$$

where $r=\begin{bmatrix}Jr_1\\ \lambda r_2\end{bmatrix}$, $c_1=\sqrt{\frac{\beta^2}{\alpha^2}+\frac{1}{\|x\|_2^2}}$, $c_2=c_1+\frac{1}{\|x\|_2}$.

Now, we want to give the upper bounds of residual vector r for $m_{res}(A,C,b,d)$ and $c_{res}(A,C,b,d)$. The proof is similar to the proof of Corollary 4.3, thus it is omitted.

Corollary 5.3 *Assume that the condition of Theorem 5.2 holds. Then*

$$\begin{aligned}m_{res}(A,C,b,d)\le&m_{res}^{upper}(A,B,b,d)\\ =&\||M||A||x|+|AQ(A,C)K||A^T||(Jr_1)|+|N||C||x|+|\lambda||AQ(A,C)K||C^T||r_2|+\\ &|M||b|+|N||d|\|_\infty/\|r\|_\infty,\\ c_{res}(A,C,b,d)\le&c_{res}^{upper}(A,B,b,d)\\ =&\||M||A||x|+|AQ(A,C)K||A^T||(Jr_1)|+|N||C||x|+|\lambda||AQ(A,C)K||C^T||r_2|+\\ &|M||b|+|N||d|/r\|_\infty.\end{aligned}$$

6. Numerical examples

In this section, we examine the mixed and componentwise condition numbers and their upper bounds that are given in Theorem 4.2 and Corollary 4.3 with the normwise condition number $K^{ILSQC}(A,B,b,d)$. Let

$$A=\begin{bmatrix}5+10^i & 0 & 1\\ -1 & 3 & 1\\ 1 & 0 & 8\\ 0 & 1 & 0\end{bmatrix},\ C=\begin{bmatrix}0 & 1 & 1\\ 0 & 0 & 0\end{bmatrix},\ J=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix},\ b=\begin{bmatrix}1\\2\\1\\0\end{bmatrix},\ d=\begin{bmatrix}1\\1\end{bmatrix},$$

and $\gamma=1.07$. For $i=0:1:5$, we have

$$\operatorname{rank}\begin{pmatrix}R\\ C\end{pmatrix}=\operatorname{rank}\begin{pmatrix}A^TJA\\ C\end{pmatrix}=n=\operatorname{rank}\begin{pmatrix}R^TR\\ C\end{pmatrix}=3,$$

It is easy to check that the matrix A^TJA $(i=0:1:5)$ is positive definite. The condition numbers are computed based on their explicit asseveration in Theorem 4.2. Thus, upon computations in MATLAB 7.9, with precision 2.22×10^{-16}. From Table 1, we determine that: for each i, the mixed $m^{ILSQC}(A,C,b,d)$ and componentwise condition numbers $c^{ILSQC}(A,C,b,d)$ and their upper bounds are smaller than the normwise condition number $K^{ILSQC}(A,C,b,d)$.

i	$m^{ILSQC}(A,C,b,d)$	$c^{ILSQC}(A,C,b,d)$	$m_{upp}^{ILSQC}(A,C,b,d)$	$c_{upp}^{ILSQC}(A,C,b,d)$	$K^{ILSQC}(A,C,b,d)$
0	2.6550	7.0263	4.2371	15.3559	4.1486e+006
1	2.9494	6.3306	4.3822	12.4767	7.5849e+006
2	3.1419	5.9592	4.5400	11.4913	4.8963e+007
3	3.1724	5.9043	4.5675	11.3763	4.7286e+008
4	3.1757	5.8986	4.5705	11.3647	4.7137e+009
5	3.1760	5.8980	4.5708	11.3635	4.7122e+010

Table 1 Comparison of condition numbers in Therorm 4.2 and their upper bounds in Corollary 4.3

As the $(1,1)$-element of A increases, then normwise condition number become larger and larger, whereas comparatively the mixed and componentwise condition numbers have little change. The main reason is that the mixed and componentwise condition numbers notice the structure of the coefficient matrix A $(i=0:1:5)$ with respect to scaling, but the normwise condition number ignores it.

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