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Verified Computation of Eigenpairs in the Generalized Eigenvalue Problem for Nonsquare Matrix Pencils

Shinya MIYAJIMA
Faculty of Science and Engineering, Iwate University, Ueda, Morioka, 020-8551*, Japan*

Abstract Consider an optimization problem arising from the generalized eigenvalue problem $Ax = \lambda Bx$, where $A, B \in \mathbb{C}^{m\times n}$ and $m > n$. Ito et al. showed that the optimization problem can be solved by utilizing right singular vectors of $C := [B, A]$. In this paper, we focus on computing intervals containing the solution. When some singular values of C are multiple or nearly multiple, we can enclose bases of corresponding invariant subspaces of C^HC, where C^H denotes the conjugate transpose of C, but cannot enclose the corresponding right singular vectors. The purpose of this paper is to prove that the solution can be obtained even when we utilize the bases instead of the right singular vectors. Based on the proved result, we propose an algorithm for computing the intervals. Numerical results show property of the algorithm.

Keywords generalized eigenvalue problem; nonsquare pencil; invariant subspace; verified numerical computation

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1. Introduction

Consider the generalized eigenvalue problem $Ax = \lambda Bx$, where $A, B \in \mathbb{C}^{m\times n}$, $m > n$, $\lambda \in \mathbb{C}$ is an eigenvalue and $x \in \mathbb{C}^n \setminus \{0\}$ is an eigenvector corresponding to λ. This problem appears in many fields [1–4] and attracts much attention from theoretical and numerical points of view [5–12]. In practical applications, on the other hand, A and B may contain noise, which may cause n linearly independent eigenvectors to fail to exist even if they are known to exist in the noiseless case. Boutry et al. [5] thus considered an optimization problem that finds the minimum perturbation of the given pair of matrices (A, B) such that the perturbed pair $(\hat{A}, \hat{B})$ has n linearly independent eigenvectors:

$$\begin{cases} \text{Minimize} & \|\hat{A} - A\|_F^2 + \|\hat{B} - B\|_F^2 \\ \text{subject to} & \hat{A}, \hat{B} \in \mathbb{C}^{m\times n}, \quad \{(\lambda_i, x^{(i)})\}_{i=1}^n \subseteq \mathbb{C} \times \mathbb{C}^n, \\ & \hat{A}x^{(i)} = \lambda_i \hat{B}x^{(i)}, \quad i = 1, \ldots, n, \\ & \{x^{(1)}, \ldots, x^{(n)}\}: \text{ linearly independent}, \end{cases} \tag{1.1}$$

where $\|\cdot\|_F$ denotes the Frobenius norm. Note that λ_i and $x^{(i)}$ are not given but decision valuables in the optimization problem. For solving (1.1) numerically, Ito et al. [9] showed that

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E-mail address: miyajima@iwate-u.ac.jp

(1.1) can be reduced to the total least squares problem considered in [13], and presented the following outstanding result:

Theorem 1.1 ([9]) *Let* $[B, A] = U\Sigma V^H$ *be the singular value decomposition of* $[B, A]$*, where* $U \in \mathbb{C}^{m\times m}$ *and* $V \in \mathbb{C}^{2n\times 2n}$ *are unitary,* V^H *denotes the conjugate transpose of* V*,* $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_t)$*,* $t := \min(m, 2n)$ *and* $\sigma_1 \geq \cdots \geq \sigma_t$*. Partition* $V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}$*, where* $V_{11} \in \mathbb{C}^{n\times n}$*. If* $\sigma_n > \sigma_{n+1}$*,* V_{22} *is nonsingular and* $-V_{12}V_{22}^{-1}$ *is diagonalizable, then all the eigenpairs of* $-V_{12}V_{22}^{-1}$ *give* $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ *in (1.1).*

Theorem 1.1 shows that we can obtain $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ by computing all the eigenpairs of $-V_{12}V_{22}^{-1}$ if the assumptions are true. In practical computations, on the other hand, there exists the case where obtaining $-V_{12}V_{22}^{-1}$ explicitly is numerically unstable. This instability is pronounced especially when V_{22} is ill-conditioned. In order to avoid computing $-V_{12}V_{22}^{-1}$ explicitly, they gave Lemma 1.2.

Lemma 1.2 ([9]) *Let* $V \in \mathbb{C}^{2n\times 2n}$ *be unitary. Partition* V *similarly to Theorem 1.1. If* V_{22} *is nonsingular, then so is* V_{11}*, and* $-V_{12}V_{22}^{-1} = (V_{11}^H)^{-1}V_{21}^H$*.*

Lemma 1.2 implies that we can obtain $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ by computing all the eigenpairs of the square pencil $V_{21}^H - \lambda V_{11}^H$ if the assumptions in Theorem 1.1 are true. They thus proposed a fast and robust algorithm for numerically computing $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ by calculating all the eigenpairs of $V_{21}^H - \lambda V_{11}^H$.

The work presented in this paper addresses the problem of verified computation of $\{(\lambda_i, x^{(i)})\}_{i=1}^n$, specifically, computing intervals which are guaranteed to contain $\{(\lambda_i, x^{(i)})\}_{i=1}^n$. To the author's best knowledge, a verification algorithm designed specifically for $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ is not available in literature. If we can obtain intervals containing V_{11} and V_{12}, the verified computation of $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ is possible by executing a known algorithm [14] for all eigenpairs of square pencils. Since columns of $[V_{11}^H, V_{21}^H]^H$ are eigenvectors of the Hermitian matrix $C^HC \in \mathbb{C}^{2n\times 2n}$, where $C := [B, A]$, corresponding to the largest n eigenvalues, computing intervals containing V_{11} and V_{12} seems to be possible by utilizing a known verification algorithm [15] for eigenvectors. On the other hand, verification algorithms for eigenvectors fail when the geometric multiplicity of the corresponding eigenvalue is two or more [16], which means that we cannot enclose some columns of $[V_{11}^H, V_{21}^H]^H$ when some of $\sigma_1, \ldots, \sigma_n$ are multiple. Moreover, the verification algorithms for eigenvectors usually fail even when the eigenvalues are not multiple but closely clustered.

In the multiple or nearly multiple case, the algorithms in [14,17] are applicable. Let $\mu_1, \ldots, \mu_{2n}$ be the eigenvalues of C^HC such that $\mu_1 \geq \cdots \geq \mu_{2n}$. Let also $\{\mu_{i_1^{(j)}}, \ldots, \mu_{i_{p_j}^{(j)}}\}$, $j = 1, \ldots, q$ be sets of eigenvalue clusters, where $i_1^{(1)}, \ldots, i_{p_1}^{(1)}, \ldots, i_1^{(q)}, \ldots, i_{p_q}^{(q)} \in \mathbb{Z}$ satisfy $1 = i_1^{(1)} < \cdots < i_{p_1}^{(1)} < \cdots < i_1^{(q)} < \cdots < i_{p_q}^{(q)} = 2n$ and $p_1 + \cdots + p_q = 2n$. Note that the case of a simple eigenvalue is included in the case $p_j = 1$. For $j = 1, \ldots, q$, let $W_j \in \mathbb{C}^{2n\times p_j}$ and $P_j \in \mathbb{C}^{p_j\times p_j}$ satisfy $C^HCW_j = W_jP_j$ and $\lambda(P_j) \subseteq \{\mu_{i_1^{(j)}}, \ldots, \mu_{i_{p_j}^{(j)}}\}$, where $\lambda(P_j)$ is the spectrum of P_j. By executing the algorithms in [14,17], we can obtain intervals containing W_j and $\lambda(P_j)$. Note that

P_j is not necessarily diagonal, so the columns of W_j are not necessarily the eigenvectors of C^HC. Define $W := [W_1, \dots, W_q] \in \mathbb{C}^{2n\times 2n}$, and partition W similarly to V in Theorem 1.1. Since the columns of W_j are not necessarily the eigenvectors, $V = W$ does not follow in general. Hence, $V_{11} \neq W_{11}$ and $V_{21} \neq W_{21}$ in general.

The purpose of this paper is to prove the following propositions under some assumptions, which are checkable via rounding mode controlled floating point computations:

Proposition 1.3 *All the eigenpairs of $W_{21}^H - \lambda W_{11}^H$ give $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ in (1.1).*

Proposition 1.4 *W_{11} is nonsingular $\Leftrightarrow$ V_{22} is nonsingular.*

Proposition 1.3 means that the verified computation of $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ is possible even when we compute intervals containing W_{11} and W_{21} instead of V_{11} and V_{21}. Proposition 1.4 says we can check the nonsingularity of V_{22} by checking that of W_{11}, which enables us to avoid computing an interval containing $[W_{12}^H, W_{22}^H]^H$ (see Remark 4.1). Based on Propositions 1.3 and 1.4, we propose a verification algorithm for $\{(\lambda_i, x^{(i)})\}_{i=1}^n$. Although this algorithm seems to be a direct application of an already available algorithm, theoretical justification of the direct application is our main contribution.

This paper is organized as follows: Section 2 introduces notation and theories used in this paper. Section 3 proves Propositions 1.3 and 1.4. Section 4 proposes the verification algorithm. Section 5 reports numerical results. Section 6 summarizes the results in this paper and highlights possible extension and future work.

2. Preliminaries

For $M \in \mathbb{C}^{m\times n}$, let M_{ij}, $M_{:j}$, M^T and M^H be the (i,j) element, j-th column, transpose and conjugate transpose of M, respectively, and $|M| := (|M_{ij}|)$. When $m = n$, let $\lambda(M)$ be the spectrum of M. For $M, N \in \mathbb{R}^{m\times n}$, $M \le N$ means $M_{ij} \le N_{ij}$, $\forall i, j$. For $v \in \mathbb{C}^n$, v_i denotes the i-th component of v. For $v, w \in \mathbb{C}^n$ with $\|w\|_\infty < 1$, define $\|v\|_w := \max_i(|v_i|/(1 - |w_i|))$. Let I_n be the $n \times n$ identity matrix and $\mathbf{1} := [1, \dots, 1]^T$. Define $\mathbb{R}_+ := [0, \infty)$, $\mathbb{R}_+^{m\times n} := \{M \in \mathbb{R}^{m\times n} : M \ge 0\}$, $\mathbb{NS}_n := \{X \in \mathbb{C}^{n\times n} : X \text{ is nonsingular}\}$, $\mathbb{H}_n := \{X \in \mathbb{C}^{n\times n} : X^H = X\}$ and $\mathbb{U}_n := \{X \in \mathbb{C}^{n\times n} : X^HX = I_n\}$. For $C \in \mathbb{C}^{m\times n}$ and $R \in \mathbb{R}_+^{m\times n}$, $\langle C, R\rangle$ denotes the interval matrix whose midpoint and radius are C and R, respectively. Let $\mathbb{IR}$ be the set of all real intervals, and $\mathbb{IC}$, $\mathbb{IC}^n$ and $\mathbb{IC}^{m\times n}$ be the sets of all complex interval scalars, n-vectors and $m \times n$ matrices, respectively. For $\boldsymbol{a} \in \mathbb{IC}$, let $\mathrm{mid}(\boldsymbol{a})$ and $\mathrm{rad}(\boldsymbol{a})$ be the midpoint and radius of $\boldsymbol{a}$, respectively, and $|\boldsymbol{a}| := \max_{a\in\boldsymbol{a}} |a|$. We can then define $\mathrm{mid}(\boldsymbol{M})$, $\mathrm{rad}(\boldsymbol{M})$ and $|\boldsymbol{M}|$ for $\boldsymbol{M} \in \mathbb{IC}^{m\times n}$. Expressions containing intervals mean results of interval arithmetic.

Throughout this paper, let $C := [B, A] \in \mathbb{C}^{m\times 2n}$ and $C = U\Sigma V^H$ be the singular value decomposition of C, where $U \in \mathbb{U}_m$, $V \in \mathbb{U}_{2n}$, $\Sigma = \mathrm{diag}(\sigma_1, \dots, \sigma_t)$, $t := \min(m, 2n)$ and $\sigma_1 \ge \dots \ge \sigma_t$. We partition V similarly to Theorem 1.1. Let $\mu_1, \dots, \mu_{2n} \in \mathbb{R}$ be the eigenvalues of C^HC such that $\mu_1 \ge \dots \ge \mu_{2n} \ge 0$. We then have $\mu_i = \sigma_i^2$, $i = 1, \dots, t$. Let $v^{(i)} := V_{:i}$, $i = 1, \dots, 2n$. Then, $v^{(1)}, \dots, v^{(2n)}$ are orthonormal eigenvectors of C^HC corresponding to

$\mu_1, \ldots, \mu_{2n}$. For $p \in \{1, \ldots, 2n\}$, let $i_1, \ldots, i_p \in \mathbb{Z}$ satisfy $1 \le i_1 < \cdots < i_p \le 2n$. Define $\{i_1, \ldots, i_p\}^C := \{1, \ldots, 2n\} \setminus \{i_1, \ldots, i_p\}$ and $\{\mu_{i_1}, \ldots, \mu_{i_p}\}^C := \{\mu_1, \ldots, \mu_{2n}\} \setminus \{\mu_{i_1}, \ldots, \mu_{i_p}\}$.

We cite Lemmas 2.1 to 2.3 for enclosing eigenvalues of an Hermitian matrix. Lemma 2.1 is a modification of the Rump's theorem, whose statement and proof can be found in [18, Theorem 1].

Lemma 2.1 ([19]) *Let $A \in \mathbb{H}_n$, $\tilde{v}^{(i)} \in \mathbb{C}^n \setminus \{0\}$ and $\tilde{\lambda}_i \in \mathbb{R}$, $i = 1, \ldots, n$ with $\tilde{\lambda}_1 \ge \cdots \ge \tilde{\lambda}_n$ be given, λ_i be the eigenvalues of A such that $\lambda_1 \ge \cdots \ge \lambda_n$, $\tilde{V} := [\tilde{v}^{(1)}, \ldots, \tilde{v}^{(n)}]$, $\tilde{D} :=$ $\mathrm{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$, $E := A\tilde{V} - \tilde{V}\tilde{D}$ and $F := I_n - \tilde{V}^H\tilde{V}$. If $\|F\|_\infty < 1$, then $|\lambda_i - \tilde{\lambda}_i| \le \delta$, $i = 1, \ldots, n$, where $\delta := \sqrt{\|E\|_\infty \|E\|_1}/(1 - \|F\|_\infty)$.*

Lemma 2.2 ([20]) *Let A, $\tilde{v}^{(i)}$, $\tilde{\lambda}_i$, λ_i and E be as in Lemma 2.1, and $e^{(i)}$ be the i-th column of E. Then, $\min_j |\lambda_j - \tilde{\lambda}_i| \le \varepsilon_i$, $i = 1, \ldots, n$, where $\varepsilon_i := \|e^{(i)}\|_2/\|\tilde{v}^{(i)}\|_2$.*

Lemma 2.3 ([19]) *Let δ and ε_i be as in Lemmas 2.1 and 2.2, respectively. Then, $\varepsilon_i \le \delta$, $\forall i$.*

We will apply Lemmas 2.4 and 2.5 for verifying $\min_j |\lambda_j - \tilde{\lambda}_i| = |\lambda_i - \tilde{\lambda}_i|$.

Lemma 2.4 ([19]) *Let λ_i and $\tilde{\lambda}_i$, $i = 1, \ldots, n$ be sequences of real numbers such that $\lambda_1 \ge \cdots \ge \lambda_n$ and $\tilde{\lambda}_1 \ge \cdots \ge \tilde{\lambda}_n$, respectively. Assume $|\lambda_i - \tilde{\lambda}_i| \le \delta$ for all i, and*

$$\begin{cases} \tilde{\lambda}_i - \tilde{\lambda}_{i+1} > 2\delta & (i = 1) \\ \tilde{\lambda}_{i-1} - \tilde{\lambda}_i > 2\delta \quad \text{and} \quad \tilde{\lambda}_i - \tilde{\lambda}_{i+1} > 2\delta & (2 \le i \le n-1) \\ \tilde{\lambda}_{i-1} - \tilde{\lambda}_i > 2\delta & (i = n) \end{cases}$$

for some i. Then, $\min_j |\lambda_j - \tilde{\lambda}_i| = |\lambda_i - \tilde{\lambda}_i|$ for some i.

Lemma 2.5 ([19]) *Let λ_i, $\tilde{\lambda}_i$ and δ be as in Lemma 2.4. Assume $\min_j |\lambda_j - \tilde{\lambda}_i| \le \varepsilon_i$ for each i, and some partial sequence $\tilde{\lambda}_{\underline{k}}, \ldots, \tilde{\lambda}_{\overline{k}}$ with $1 \le \underline{k} < \overline{k} \le n$ are clustered such that $\tilde{\lambda}_{\underline{k}-1} - \tilde{\lambda}_{\underline{k}} > 2\delta$, $\tilde{\lambda}_{\overline{k}} - \tilde{\lambda}_{\overline{k}+1} > 2\delta$ and $\tilde{\lambda}_k - \tilde{\lambda}_{k+1} \le 2\delta$, $\forall k = \underline{k}, \ldots, \overline{k}-1$. If $\varepsilon_k + \varepsilon_{k+1} < \tilde{\lambda}_k - \tilde{\lambda}_{k+1}$, $\forall k = \underline{k}, \ldots, \overline{k}-1$, then $\min_j |\lambda_j - \tilde{\lambda}_k| = |\lambda_k - \tilde{\lambda}_k|$, $\forall k = \underline{k}, \ldots, \overline{k}$.*

We refer Lemma 2.6 for enclosing eigenvectors of an Hermitian matrix.

Lemma 2.6 ([15]) *Let A, $\tilde{\lambda}_i$ and λ_i be as in Lemma 2.1, $e^{(i)}$ and ε_i be as in Lemma 2.2, $\rho_i \in \mathbb{R}_+$ satisfy $0 < \rho_i \le \min_{j \ne i} |\lambda_j - \tilde{\lambda}_i|$, and $\xi_i := \|e^{(i)}\|_2/\rho_i$. If $\varepsilon_i < \rho_i$, then there exists an eigenvector $w^{(i)}$ corresponding to λ_i such that $\|w^{(i)} - \tilde{v}^{(i)}\|_2 \le \xi_i$.*

Proof Let $x^{(i)} \in \mathbb{C}^n \setminus \{0\}$ with $\|x^{(i)}\|_2 = 1$ be an eigenvector corresponding to λ_i. Since $A \in \mathbb{H}_n$, we can take $\{x^{(1)}, \ldots, x^{(n)}\}$ as an orthonormal basis. Then, there exist $c_1, \ldots, c_n \in \mathbb{C}$ such that $\tilde{v}^{(i)} = \sum_j c_j x^{(j)}$. Since $x^{(1)}, \ldots, x^{(n)}$ are the orthonormal eigenvectors, we moreover have

$$\begin{aligned} \|e^{(i)}\|_2^2 &= \Big\| A \sum_j c_j x^{(j)} - \tilde{\lambda}_i \sum_j c_j x^{(j)} \Big\|_2^2 = \Big\| \sum_j c_j (\lambda_j - \tilde{\lambda}_i) x^{(j)} \Big\|_2^2 = \sum_j |c_j|^2 |\lambda_j - \tilde{\lambda}_i|^2 \\ &\ge \sum_{j \ne i} |c_j|^2 |\lambda_j - \tilde{\lambda}_i|^2 \ge \min_{j \ne i} |\lambda_j - \tilde{\lambda}_i|^2 \sum_{j \ne i} |c_j|^2. \end{aligned}$$

From this and $0 < \rho_i \le \min_{j \ne i} |\lambda_j - \tilde{\lambda}_i|$, we obtain $\sum_{j \ne i} |c_j|^2 \le \xi_i^2$. This and $\|\tilde{v}^{(i)}\|_2^2 = \sum_j |c_j|^2$

yield $\|\tilde{v}^{(i)}\|_2^2 - \xi_i^2 \le |c_i|^2$. From this and $\varepsilon_i < \rho_i$, we have $\xi_i < \|\tilde{v}^{(i)}\|_2$ and $c_i \ne 0$. Therefore, $c_i x^{(i)}$ is also an eigenvector corresponding to λ_i. We can thus put $w^{(i)} = c_i x^{(i)}$. This and $\sum_{j\ne i} |c_j|^2 \le \xi_i^2$ finally show $\|w^{(i)} - \tilde{v}^{(i)}\|_2^2 = \|c_i x^{(i)} - \sum_j c_j x^{(j)}\|^2 = \sum_{j\ne i} |c_j|^2 \le \xi_i^2$, which completes the proof. □

For enclosing eigenvalues and eigenvectors of a square matrix pencil, we use Lemmas 2.7 and 2.9, respectively.

Lemma 2.7 ([14]) *Let $A, B, \tilde{X}, Y \in \mathbb{C}^{n\times n}$ be given, $\tilde{\Lambda} \in \mathbb{C}^{n\times n}$ be diagonal with $\tilde{\Lambda}_{ii} = \tilde{\lambda}_i$, $i = 1, \dots, n$, $G := Y(A\tilde{X} - B\tilde{X}\tilde{\Lambda})$, $H := I_n - YB\tilde{X}$, $g := |G|\mathbf{1}$ and $h := |H|\mathbf{1}$. If $\|h\|_\infty < 1$, then B, $\tilde{X}$ and Y are nonsingular, and all the eigenvalues of the pencil $A - \lambda B$ are included in $\bigcup_{i=1}^n \langle\tilde{\lambda}_i, r_i\rangle$, where $r := g + \|g\|_h h$.*

Remark 2.8 The eigenvalue inclusion in Lemma 2.7 comes from the Gershgorin theorem. If p of the intervals form a connected domain which is isolated from the other intervals, therefore, this domain contains precisely p eigenvalues.

Lemma 2.9 ([14]) *Let A, B, $\tilde{X}$, $\tilde{\Lambda}$, $\tilde{\lambda}_1, \dots, \tilde{\lambda}_n$, G, h and r be as in Lemma 2.7, $i \in \{1, \dots, n\}$, and $I^{(i)} \in \mathbb{R}^{(n-1)\times n}$ and $J^{(i)} \in \mathbb{R}^{n\times(n-1)}$ be I_n without i-th row and column, respectively. Assume $\|h\|_\infty < 1$ and $\langle\tilde{\lambda}_i, r_i\rangle$ is isolated from the other intervals, and define $\tilde{d} := \tilde{\Lambda}\mathbf{1}$, $f := |\tilde{d} - \tilde{\lambda}_i\mathbf{1}| - r_i\mathbf{1}$, $v := |G|J^{(i)}\mathbf{1}$, $w := |G|_{:i}$, $s := (I^{(i)}(v + \|v\|_h h))./(I^{(i)}f)$, $y := (I^{(i)}(w + \|w\|_h h))./(I^{(i)}f)$ and $u := y + \|y\|_s s$. Then, $\langle\tilde{\lambda}_i, r_i\rangle$ contains precisely one eigenvalue λ^* of the pencil $A - \lambda B$, geometric multiplicity of λ^* is one and there exists an eigenvector x^* corresponding to λ^* satisfying $|x^* - \tilde{X}_{:i}| \le |\tilde{X}|J^{(i)}u$.*

Lemma 2.10 will be used for verifying the nonsingularity of a matrix.

Lemma 2.10 ([21]) *If $\|S\|_\infty < 1$ for $S \in \mathbb{C}^{n\times n}$, then $I_n - S$ is nonsingular.*

In the proof of Lemma 3.1, we will apply Lemma 2.11, which is a modification of [21, Theorem 8.1.9].

Lemma 2.11 ([21]) *Suppose $A \in \mathbb{H}_n$, $Q_1 \in \mathbb{C}^{n\times r}$, $Q_2 \in \mathbb{C}^{n\times(n-r)}$ and $[Q_1, Q_2] \in \mathbb{U}_n$. If $\mathrm{ran}(Q_1)$ is an invariant subspace of A, then $\lambda(A) = \lambda(Q_1^H A Q_1) \cup \lambda(Q_2^H A Q_2)$.*

From Lemma 1.2, we immediately obtain Corollary 2.12.

Corollary 2.12 *V_{11} is nonsingular $\Leftrightarrow$ V_{22} is nonsingular.*

Proof Let $\mathcal{P} := \begin{bmatrix} 0 & I_n \\ I_n & 0 \end{bmatrix}$. Then, $\mathcal{P}V\mathcal{P} = \begin{bmatrix} V_{22} & V_{21} \\ V_{12} & V_{11} \end{bmatrix}$ is also unitary. Lemma 1.2 applied to $V := \mathcal{P}V\mathcal{P}$ shows that V_{22} is nonsingular if V_{11} is nonsingular. This and Lemma 1.2 prove the result. □

3. Proofs of Propositions 1.3 and 1.4

For proving Propositions 1.3 and 1.4, we clarify relations between V and W in Section 1. We first consider the equality $C^H CW = WP$, where $W \in \mathbb{C}^{2n\times p}$, $P \in \mathbb{C}^{p\times p}$ and $p \in$

$\{1,\ldots,2n\}$. We then have $\lambda(P) \subseteq \{\mu_1,\ldots,\mu_{2n}\}$. Let $\mu_{i_1},\ldots,\mu_{i_p}$ be multiple or nearly multiple. If $\lambda(P)=\{\mu_{i_1},\ldots,\mu_{i_p}\}$, we can prove $\mathrm{ran}([v^{(i_1)},\ldots,v^{(i_p)}])=\mathrm{ran}(W)$ (see Lemma 3.2). Even if $\lambda(P)\cap\{\mu_{i_1},\ldots,\mu_{i_p}\}^C=\emptyset$, on the other hand, we cannot assert $\lambda(P)=\{\mu_{i_1},\ldots,\mu_{i_p}\}$. If $W=[v^{(i_1)},\ldots,v^{(i_1)}]$ and $P=\mu_{i_1}I_p$, for example, then $C^HCW=WP$ and $\lambda(P)\cap\{\mu_{i_1},\ldots,\mu_{i_p}\}^C=\emptyset$ follow, but $\lambda(P)=\{\mu_{i_1}\}\neq\{\mu_{i_1},\ldots,\mu_{i_p}\}$. In order to exclude such cases, we present Lemma 3.1.

Lemma 3.1 *Let $W\in\mathbb{C}^{2n\times p}$ and $P\in\mathbb{C}^{p\times p}$ satisfy $C^HCW=WP$. If $\lambda(P)\cap\{\mu_{i_1},\ldots,\mu_{i_p}\}^C=\emptyset$ and* $\mathrm{rank}(W)=p$*, then $\lambda(P)=\{\mu_{i_1},\ldots,\mu_{i_p}\}$.*

Proof Let $W=QR$ be the QR factorization of W, where $Q\in\mathbb{U}_{2n}$, and $R\in\mathbb{C}^{2n\times p}$ is upper triangular. Partition $Q=[Q_1,Q_2]$ and $R=[R_1^H,0]^H$, where $Q_1\in\mathbb{C}^{2n\times p}$ and $R_1\in\mathbb{C}^{p\times p}$. We then have $W=Q_1R_1$. From $\mathrm{rank}(W)=p$, moreover, R_1 is nonsingular. Substituting $W=Q_1R_1$ into $C^HCW=WP$ gives $C^HCQ_1=Q_1R_1PR_1^{-1}$. This and $\mathrm{rank}(Q_1)=p$ show that $\mathrm{ran}(Q_1)$ is an invariant subspace of C^HC. The equalities $C^HCQ_1=Q_1R_1PR_1^{-1}$ and $Q_1^HQ_1=I_p$ give $(CQ_1)^HCQ_1=R_1PR_1^{-1}$. Therefore, Lemma 2.11 applied to $A:=C^HC$ yields $\{\mu_1,\ldots,\mu_{2n}\}=\lambda(P)\cup\lambda((CQ_2)^HCQ_2)$. This and $\lambda(P)\cap\{\mu_{i_1},\ldots,\mu_{i_p}\}^C=\emptyset$ show $\lambda((CQ_2)^HCQ_2)\supseteq\{\mu_{i_1},\ldots,\mu_{i_p}\}^C$. Since $\lambda((CQ_2)^HCQ_2)$ has at most $2n-p$ elements, $\lambda((CQ_2)^HCQ_2)=\{\mu_{i_1},\ldots,\mu_{i_p}\}^C$. Hence, $\{\mu_1,\ldots,\mu_{2n}\}=\lambda(P)\cup\{\mu_{i_1},\ldots,\mu_{i_p}\}^C$, so that $\lambda(P)\supseteq\{\mu_{i_1},\ldots,\mu_{i_p}\}$. Since $\lambda(P)$ has at most p elements, $\lambda(P)=\{\mu_{i_1},\ldots,\mu_{i_p}\}$. □

Under the assumptions in Lemma 3.1, we can develop relations between W in Lemma 3.1 and $[v^{(i_1)},\ldots,v^{(i_p)}]$.

Lemma 3.2 *Let W be as in Lemma 3.1 and $V_i:=[v^{(i_1)},\ldots,v^{(i_p)}]$. Under the assumptions in Lemma 3.1, it holds that* $\mathrm{ran}(V_i)=\mathrm{ran}(W)$.

Proof Let P, Q_1 and R_1 be as in Lemma 3.1 or its proof. Lemma 3.1 gives $\lambda(R_1PR_1^{-1})=\{\mu_{i_1},\ldots,\mu_{i_p}\}$. From $R_1PR_1^{-1}=(CQ_1)^HCQ_1$, moreover, $R_1PR_1^{-1}\in\mathbb{H}_n$. Therefore, there exist $Z\in\mathbb{U}_p$ and $\Omega:=\mathrm{diag}(\mu_{i_1},\ldots,\mu_{i_p})$ such that $R_1PR_1^{-1}=Z\Omega Z^H$. Substituting this into $C^HCQ_1=Q_1R_1PR_1^{-1}$ gives $C^HCQ_1Z=Q_1Z\Omega$. This and $(Q_1Z)^HQ_1Z=I_p$ show $(Q_1Z)_{:1},\ldots,(Q_1Z)_{:p}$ are orthonormal eigenvectors of C^HC corresponding to $\mu_{i_1},\ldots,\mu_{i_p}$. This, $W=Q_1R_1$ and $R_1^{-1}Z\in\mathbb{NS}_p$ prove

$$\mathrm{ran}(V_i)=\mathrm{ran}([v^{(1)},\ldots,v^{(i_1-1)},v^{(i_p+1)},\ldots,v^{(2n)}])^\perp=\mathrm{ran}(Q_1Z)=\mathrm{ran}(WR_1^{-1}Z)=\mathrm{ran}(W).\ \square$$

Lemma 3.3 *Let W and V_i be as in Lemmas 3.1 and 3.2, respectively. Under the assumptions in Lemma 3.1, there exists $L\in\mathbb{NS}_p$ such that $V_i=WL$.*

Proof From Lemma 3.2, there exist $l^{(j)}\in\mathbb{C}^p$, $j=1,\ldots,p$ such that $v^{(i_j)}=Wl^{(j)}$. By putting $L=[l^{(1)},\ldots,l^{(p)}]$, therefore, we have $V_i=WL$. The nonsingularity of L follows from $p=\mathrm{rank}(V_i)=\mathrm{rank}(WL)\le\min(\mathrm{rank}(W),\mathrm{rank}(L))\le\mathrm{rank}(L)\le p$. □

From Lemma 3.3, we obtain Corollary 3.4.

Corollary 3.4 *Let p_j and $i_1^{(j)},\ldots,i_{p_j}^{(j)}$ for $j=1,\ldots,q$ be as in Section 1. Suppose $W_j\in\mathbb{C}^{2n\times p_j}$*

and $P_j \in \mathbb{C}^{p_j \times p_j}$ *satisfy* $C^H C W_j = W_j P_j$, $\mathrm{rank}(W_j) = p_j$ *and* $\lambda(P_j) \cap \{\mu_{i_1^{(j)}}, \ldots, \mu_{i_{p_j}^{(j)}}\}^C = \emptyset$ *for* $j = 1, \ldots, q$. *Define* $W := [W_1, \ldots, W_q]$. *Then, there exist* $L_j \in \mathbb{NS}_{p_j}$, $j = 1, \ldots, q$ *such that* $V = W\mathrm{diag}(L_1, \ldots, L_q)$.

Proof For $j = 1, \ldots, q$, let $V_j := [v^{(i_1^{(j)})}, \ldots, v^{(i_{p_j}^{(j)})}]$. From Lemma 3.3, there exists $L_j \in \mathbb{NS}_{p_j}$ such that $V_j = W_j L_j$ for $j = 1, \ldots, q$. Thus, $[W_1 L_1, \ldots, W_q L_q] = W\mathrm{diag}(L_1, \ldots, L_q)$ and $[V_1, \ldots, V_q] = V$ prove the result. □

Remark 3.5 Since V and $\mathrm{diag}(L_1, \ldots, L_q)$ are nonsingular, so is W.

From Corollary 3.4, we can prove Proposition 1.4 and equivalence of the generalized eigenvalue problems $V_{21}^H x = \lambda V_{11}^H x$ and $W_{21}^H x = \lambda W_{11}^H x$.

Lemma 3.6 *Let* p_j, $j = 1, \ldots, q$ *and* W *be as in Corollary 3.4. Partition* W *similarly to* V *in Theorem 1.1. With all the assumptions in Corollary 3.4, suppose there exists* $r \in \{1, \ldots, q\}$ *such that* $p_1 + \cdots + p_r = p_{r+1} + \cdots + p_q = n$. *Then,*

(a) V_{11} *is nonsingular* $\Leftrightarrow$ W_{11} *is nonsingular;*

(b) V_{22} *is nonsingular* $\Leftrightarrow$ W_{22} *is nonsingular;*

(c) V_{22} *is nonsingular* $\Leftrightarrow$ W_{11} *is nonsingular;*

(d) *If* V_{11} *is nonsingular,* $(V_{11}^H)^{-1} V_{21}^H = (W_{11}^H)^{-1} W_{21}^H$;

(e) *For* $x \in \mathbb{C}^n$ *and* $\lambda \in \mathbb{C}$, $V_{21}^H x = \lambda V_{11}^H x \Leftrightarrow W_{21}^H x = \lambda W_{11}^H x$.

Proof Let L_j be as in Corollary 3.4 and $L := \mathrm{diag}(L_1, \ldots, L_q)$. From $p_1 + \cdots + p_r = p_{r+1} + \cdots + p_q = n$, L can be written as $L = \mathrm{diag}(L_\alpha, L_\beta)$, where $L_\alpha := \mathrm{diag}(L_1, \ldots, L_r) \in \mathbb{NS}_n$ and $L_\beta := \mathrm{diag}(L_{r+1}, \ldots, L_q) \in \mathbb{NS}_n$. This and $V = WL$ show $V_{11} = W_{11} L_\alpha$, $V_{12} = W_{12} L_\beta$, $V_{21} = W_{21} L_\alpha$ and $V_{22} = W_{22} L_\beta$, which prove (a), (b), (d) and (e). From (a) and Corollary 2.12, we finally obtain (c). □

We finally prove Proposition 1.3.

Theorem 3.7 *Let* W_{11} *and* W_{21} *be as in Lemma 3.6. With all the assumptions in Lemma 3.6, suppose* $\mu_n > \mu_{n+1}$, W_{11} *is nonsingular, and* $(W_{11}^H)^{-1} W_{21}^H$ *is diagonalizable. Then, all the eigenpairs of* $W_{21}^H - \lambda W_{11}^H$ *give* $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ *in (1.1).*

Proof From $\mu_n = \sigma_n^2$, $\mu_{n+1} = \sigma_{n+1}^2$ and $\mu_n > \mu_{n+1}$, we have $\sigma_n > \sigma_{n+1}$. The nonsingularity of W_{11} and Lemma 3.6 (a) and (c) give $V_{11}, V_{22} \in \mathbb{NS}_n$. Since $(W_{11}^H)^{-1} W_{21}^H$ is diagonalizable, Lemma 3.6 (d) shows that so is $(V_{11}^H)^{-1} V_{21}^H$. These discussions, Theorem 1.1, Lemmas 1.2 and 3.6 (e) prove the result. □

4. Proposed algorithm

Assume as a result of a numerical spectral decomposition of $C^H C$, we have $\tilde{D} \in \mathbb{R}^{2n \times 2n}$ and $\tilde{V} \in \mathbb{C}^{2n \times 2n}$ with $\tilde{D} = \mathrm{diag}(\tilde{\mu}_1, \ldots, \tilde{\mu}_{2n})$ such that $C^H C \tilde{V} \approx \tilde{V} \tilde{D}$, $\tilde{V}^H \tilde{V} \approx I_{2n}$ and $\tilde{\mu}_1 \geq \cdots \geq \tilde{\mu}_{2n}$. We compute $\eta_i \in \mathbb{R}_+$ satisfying $|\mu_i - \tilde{\mu}_i| \leq \eta_i$, i.e., $\mu_i \in \langle \tilde{\mu}_i, \eta_i \rangle \in \mathbb{IR}$, $i = 1, \ldots, 2n$, which are required for checking some conditions.

Let $E := C^HC\tilde{V} - \tilde{V}\tilde{D}$, $F := I_n - \tilde{V}^H\tilde{V}$, and $\tilde{v}^{(i)}$ and $e^{(i)}$ be the i-th columns of $\tilde{V}$ and E, respectively, for $i = 1, \ldots, 2n$. We can then expect $E \approx 0$, $F \approx 0$ and $e^{(i)} \approx 0$. Suppose $\|F\|_\infty < 1$, and define δ and ε_i similarly to Lemmas 2.1 and 2.2, respectively. Then, Lemmas 2.1 and 2.2 give $|\mu_i - \tilde{\mu}_i| \leq \delta$ and $\min_j |\mu_j - \tilde{\mu}_i| \leq \varepsilon_i$, $\forall i$, respectively. Taking Lemma 2.3 into account, we compute the above η_i such that

$$\eta_i = \begin{cases} \varepsilon_i, & \text{if } \min_j |\mu_j - \tilde{\mu}_i| = |\mu_i - \tilde{\mu}_i| \text{ is verified} \\ \delta, & \text{otherwise} \end{cases}.$$

For verifying $\min_j |\mu_j - \tilde{\mu}_i| = |\mu_i - \tilde{\mu}_i|$, we can apply Lemmas 2.4 and 2.5.

If $\langle \tilde{\mu}_i, \eta_i \rangle$ is isolated from the other intervals, we can compute an interval containing an eigenvector corresponding to μ_i based on Lemma 2.6. In fact, if we put

$$\rho_i = \begin{cases} \tilde{\mu}_1 - \tilde{\mu}_2 - \eta_2, & i = 1; \\ \min(\tilde{\mu}_{i-1} - \tilde{\mu}_i - \eta_{i-1}, \tilde{\mu}_i - \tilde{\mu}_{i+1} - \eta_{i+1}), & 2 \leq i \leq 2n-1; \\ \tilde{\mu}_{2n-1} - \tilde{\mu}_{2n} - \eta_{2n-1}, & i = 2n, \end{cases}$$

then the isolation yields $0 < \rho_i \leq \min_{j \neq i} |\mu_j - \tilde{\mu}_i|$. If $\varepsilon_i < \rho_i$, then there exists an eigenvector $w^{(i)}$ corresponding to μ_i such that $\|w^{(i)} - \tilde{v}^{(i)}\|_2 \leq \xi_i$, where $\xi_i := \|e^{(i)}\|_2/\rho_i$. Thus, $w^{(i)}$ can be enclosed by $w_j^{(i)} \in \langle \tilde{v}_j^{(i)}, |w_j^{(i)} - \tilde{v}_j^{(i)}| \rangle \subseteq \langle \tilde{v}_j^{(i)}, \|w^{(i)} - \tilde{v}^{(i)}\|_2 \rangle \subseteq \langle \tilde{v}_j^{(i)}, \xi_i \rangle$, $j = 1, \ldots, 2n$. Since $\|w^{(i)}\|_2 \neq 1$ in general, $v^{(i)} \neq w^{(i)}$ in general. Because μ_i is a simple eigenvalue, however, there exists $l_i \in \mathbb{C} \setminus \{0\}$ such that $v^{(i)} = l_i w^{(i)}$, which is a special case of $V_j = W_j L_j$ in the proof of Corollary 3.4.

If $\langle \tilde{\mu}_i, \eta_i \rangle$ is not isolated, let $\bigcup_{k=1}^{p} \langle \tilde{\mu}_{i_k}, \eta_{i_k} \rangle$, where $i \in \{i_1, \ldots, i_p\}$, be a connected domain which is isolated from the other intervals, and $\tilde{V}_i := [\tilde{v}^{(i_1)}, \ldots, \tilde{v}^{(i_p)}]$. In this case, we can apply [14, Algorithm 2] for computing $\langle \tilde{V}_i, \Xi_i \rangle \in \mathbb{IC}^{2n \times p}$ and $\langle \tilde{\mu}, \omega \rangle \in \mathbb{IR}$ such that $\langle \tilde{V}_i, \Xi_i \rangle \ni W$ and $\langle \tilde{\mu}, \omega \rangle \supseteq \lambda(P)$, where $W \in \mathbb{C}^{2n \times p}$ and $P \in \mathbb{C}^{p \times p}$ satisfy $C^HCW = WP$, with $\tilde{V}_i$ and $\tilde{\mu}_{i_1}, \ldots, \tilde{\mu}_{i_p}$ being inputs. If $\langle \tilde{\mu}, \omega \rangle \cap \bigcup_{j \in \{i_1, \ldots, i_p\}^C} \langle \tilde{\mu}_j, \eta_j \rangle = \emptyset$, then $\lambda(P) \cap \{\mu_{i_1}, \ldots, \mu_{i_p}\}^C = \emptyset$.

We can verify the condition $\mathrm{rank}(W) = p$ in Lemma 3.1 using $\langle \tilde{V}_i, \Xi_i \rangle$. In fact, a center-radius interval arithmetic evaluation [22] yield

$$I_p - W^HW \in I_p - \langle \tilde{V}_i^H, \Xi_i^T \rangle \langle \tilde{V}_i, \Xi_i \rangle \subseteq \langle I_p - \tilde{V}_i^H\tilde{V}_i, |\tilde{V}_i|^T\Xi_i + \Xi_i^T(|\tilde{V}_i| + \Xi_i) \rangle,$$

which gives

$$\begin{aligned} \|I_p - W^HW\|_\infty &\leq \|I_p - \tilde{V}_i^H\tilde{V}_i\|_\infty + \||\tilde{V}_i|^T\Xi_i + \Xi_i^T(|\tilde{V}_i| + \Xi_i)\|_\infty \\ &\leq \|I_p - \tilde{V}_i^H\tilde{V}_i\|_\infty + \|\tilde{V}_i\|_1\|\Xi_i\|_\infty + \|\Xi_i\|_1\||\tilde{V}_i| + \Xi_i\|_\infty =: \zeta. \end{aligned}$$

If $\zeta < 1$, then Lemma 2.10 shows that $I_p - (I_p - W^HW) = W^HW$ is nonsingular, which implies $\mathrm{rank}(W) = p$. Observe that $I_p - \tilde{V}_i^H\tilde{V}_i \approx 0$ follows from $\tilde{V}^H\tilde{V} \approx I_{2n}$.

We can verify the condition $\mu_n > \mu_{n+1}$ in Theorem 3.7 by checking $\tilde{\mu}_n - \eta_n > \tilde{\mu}_{n+1} + \eta_{n+1}$. In practical execution, the index sets $\{i_1^{(j)}, \ldots, i_{p_j}^{(j)}\}$, $j = 1, \ldots, q$ in Corollary 3.4 means that $\bigcup_{k=1}^{p_j} \langle \tilde{\mu}_{i_k^{(j)}}, \eta_{i_k^{(j)}} \rangle$ is a connected domain which is isolated from the other intervals. Therefore, if $\tilde{\mu}_n - \eta_n > \tilde{\mu}_{n+1} + \eta_{n+1}$, then $\langle \tilde{\mu}_n, \eta_n \rangle$ is isolated from $\langle \tilde{\mu}_{n+1}, \eta_{n+1} \rangle$, so that n and $n+1$ are allocated to different index sets. This implies the existence of $r \in \{1, \ldots, q\}$ such that

$p_1 + \cdots + p_r = p_{r+1} + \cdots + p_q = n$. Hence, extra work is unnecessary for checking the existence of r in Lemma 3.6.

Suppose we have intervals $\boldsymbol{W}_{11}$ and $\boldsymbol{W}_{21}$ containing W_{11} and W_{21}, respectively. Then, we can obtain n intervals containing all the eigenvalues of $W_{21}^H - \lambda W_{11}^H$ using Lemma 2.7. Let $\mathrm{mid}(\boldsymbol{W}_{21})^H \tilde{X} \approx \mathrm{mid}(\boldsymbol{W}_{11})^H \tilde{X}\tilde{\Lambda}$ be a numerical generalized eigendecomposition where $\tilde{\Lambda} = \mathrm{diag}(\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)$, $Y \in \mathbb{C}^{n\times n}$ be an approximation to $(\mathrm{mid}(\boldsymbol{W}_{11})^H \tilde{X})^{-1}$, $G := Y(W_{21}^H \tilde{X} - W_{11}^H \tilde{X}\tilde{\Lambda})$ and $H := I_n - Y W_{11}^H \tilde{X}$. We have $G \in Y(\boldsymbol{W}_{21}^H \tilde{X} - \boldsymbol{W}_{11}^H \tilde{X}\tilde{\Lambda})$ and $H \in I_n - Y\boldsymbol{W}_{11}^H \tilde{X}$, so that $|G| \le |Y(\boldsymbol{W}_{21}^H \tilde{X} - \boldsymbol{W}_{11}^H \tilde{X}\tilde{\Lambda})| =: G_S$ and $|H| \le |I_n - Y\boldsymbol{W}_{11}^H \tilde{X}| =: H_S$. Let $g := |G|\mathbf{1}$, $h := |H|\mathbf{1}$, $g_S := G_S\mathbf{1}$ and $h_S := H_S\mathbf{1}$. These two inequalities show $g \le g_S$ and $h \le h_S$. Therefore, if $\|h_S\|_\infty < 1$, then $\|h\|_\infty < 1$, so that we can define $r := g + \|g\|_h h$ and $r_S := g_S + \|g_S\|_{h_S} h_S$. From $g \le g_S$ and $h \le h_S$, moreover, we obtain $r \le r_S$, which shows $\bigcup_{i=1}^n \langle\tilde{\lambda}_i, r_i\rangle \subseteq \bigcup_{i=1}^n \langle\tilde{\lambda}_i, (r_S)_i\rangle$. From this and Lemma 2.7, all the eigenvalues of $W_{21}^H - \lambda W_{11}^H$ are contained in $\bigcup_{i=1}^n \langle\tilde{\lambda}_i, (r_S)_i\rangle$.

Note that the nonsingularity of W_{11} is verified within this process. From $\|h\|_\infty \le \|h_S\|_\infty$, in fact, $\|h_S\|_\infty < 1$ yields $\|h\|_\infty < 1$. This and Lemma 2.7 imply the nonsingularity of W_{11}. Therefore, extra work is unnecessary for verifying the nonsingularity.

Remark 4.1 Lemma 3.6 (b) gives another way for verifying the nonsingularity of V_{22}: checking that of W_{22}. If we adopt this way, on the other hand, we need to compute an interval containing $[W_{12}^H, W_{22}^H]^H$. If we verify the nonsingularity of V_{22} by checking that of W_{11}, however, such an interval is unnecessary. In other words, Proposition 1.4 enables us to avoid computing the interval containing $[W_{12}^H, W_{22}^H]^H$.

If $\langle\tilde{\lambda}_i, (r_S)_i\rangle \cap \bigcup_{j\ne i}\langle\tilde{\lambda}_j, (r_S)_j\rangle = \emptyset$, $\forall i = 1, \ldots, n$, then $W_{21}^H - \lambda W_{11}^H$ has n distinct eigenvalues (see Remark 2.8). The diagonalizability of $(W_{11}^H)^{-1} W_{21}^H$ can be verified by this way. In this case, we can compute intervals containing eigenvectors corresponding to the eigenvalues contained in $\langle\tilde{\lambda}_i, (r_S)_i\rangle$ for all i by Lemma 2.9. Specifically, we can compute an upper bound for $|\tilde{X}|J^{(i)}u$ in Lemma 2.9 reusing G_S, h_S and r_S.

Based on discussion above, we propose Algorithm 4.2.

Algorithm 4.2 This algorithm computes $\boldsymbol{\lambda}_i \in \mathbb{IC}$ and $\boldsymbol{x}^{(i)} \in \mathbb{IC}^n$, $i = 1, \ldots, n$ such that $\boldsymbol{\lambda}_i \ni \lambda_i$ and $\boldsymbol{x}^{(i)} \ni x^{(i)}$ for λ_i and $x^{(i)}$ in (1.1).

Step 1. Compute η_i for $i = 1, \ldots, n+1$. If $\tilde{\mu}_n - \eta_n > \tilde{\mu}_{n+1} + \eta_{n+1}$ cannot be verified, terminate with failure.

Step 2. Initialize $q \in \mathbb{Z}$ and the list $\mathcal{L}$ as $q = 1$ and $\mathcal{L} = \{1, \ldots, n\}$, respectively.

Step 3. If $\mathcal{L} = \emptyset$, then go to Step 7. Otherwise, let i be the smallest integer in $\mathcal{L}$.

Step 4. If $\langle\tilde{\mu}_i, \eta_i\rangle$ is isolated from the other intervals, then delete i from $\mathcal{L}$ and go to Step 5. Otherwise, let $\bigcup_{k=1}^p \langle\tilde{\mu}_{i_k}, \eta_{i_k}\rangle$, where $i \in \{i_1, \ldots, i_p\} \subseteq \mathcal{L}$, be a connected domain which is isolated from the other intervals, delete $i_1, \ldots, i_p$ from $\mathcal{L}$ and go to Step 6.

Step 5. Compute $\boldsymbol{W}_q \in \mathbb{IC}^{2n}$ which contains an eigenvector corresponding to μ_i utilizing Lemma 2.6. Update $q = q + 1$ and go back to Step 3.

Step 6. Let $P \in \mathbb{C}^{p\times p}$ and $W \in \mathbb{C}^{2n\times p}$ satisfy $C^H C W = WP$. Compute $\boldsymbol{\mu}_q \in \mathbb{IR}$ and $\boldsymbol{W}_q \in \mathbb{IC}^{2n\times p}$ containing $\lambda(P)$ and W, respectively, by executing [14, Algorithm 2] with $\tilde{V}_i$ and

$\tilde{\mu}_{i_1}, \ldots, \tilde{\mu}_{i_p}$ being inputs. If $\boldsymbol{\mu}_q \cap \bigcup_{j \in \{i_1,\ldots,i_p\}^C} \langle \tilde{\mu}_j, \eta_j \rangle = \emptyset$ or $\mathrm{rank}(W) = p$ cannot be verified, then terminate with failure. Otherwise, update $q = q + 1$ and go back to Step 3.

Step 7. Let $\boldsymbol{W} := [\boldsymbol{W}_1, \ldots, \boldsymbol{W}_{q-1}] \in \mathbb{IC}^{2n\times n}$. Partition $\boldsymbol{W} = [\boldsymbol{W}_{11}^H, \boldsymbol{W}_{21}^H]^H$, where $\boldsymbol{W}_{11} \in \mathbb{IC}^{n\times n}$.

Step 8. Compute $\boldsymbol{\lambda}_i \in \mathbb{IC}$, $i = 1, \ldots, n$, which contains each eigenvalue of $W_{21}^H - \lambda W_{11}^H$, based on Lemma 2.7. If $\boldsymbol{\lambda}_i \cap \bigcup_{j\neq i} \boldsymbol{\lambda}_j = \emptyset$, $\forall i$ cannot be verified, then terminate with failure.

Step 9. For all i, compute $\boldsymbol{x}^{(i)} \in \mathbb{IC}^n$, which contains an eigenvector corresponding to the eigenvalue included in $\boldsymbol{\lambda}_i$, based on Lemma 2.9. Terminate.

Let p_j, $j = 1, \ldots, q$ be as in Section 1, and r be as in Lemma 3.6. Steps 5 and 6 involve $\mathcal{O}(p_j n^2)$ operations for each $j \in \{1, \ldots, r\}$. Since $p_1 + \cdots + p_r = n$, Steps 2 to 7 require $\mathcal{O}(\sum_{j=1}^r p_j n^2) = \mathcal{O}(n^3)$ operations. Thus, a large p_j does not enlarge the overall cost of Algorithm 4.2. Costs of the other parts are $\mathcal{O}(mn^2)$, since the matrix multiplication $C^H C$ involves $\mathcal{O}(mn^2)$ operations. Therefore, Algorithm 4.2 involves $\mathcal{O}(mn^2)$ operations.

Remark 4.3 When p_j for $j \in \{1, \ldots, r\}$ is large, the corresponding $\mathrm{rad}(\boldsymbol{\mu}_q)$ becomes large. See Example 5.3, where the large radius is illustrated. Observe that the large $\mathrm{rad}(\boldsymbol{\mu}_q)$ does not enlarge $\mathrm{rad}(\boldsymbol{\lambda}_i)$ and $\mathrm{rad}(\boldsymbol{x}^{(i)})$ for $i = 1, \ldots, n$. This is because $\boldsymbol{\mu}_q$ is used only for checking $\boldsymbol{\mu}_q \cap \bigcup_{j \in \{i_1,\ldots,i_p\}^C} \langle \tilde{\mu}_j, \eta_j \rangle = \emptyset$.

5. Numerical results

We used a computer with Intel Core 1.51GHz CPU, 16.0GB RAM and MATLAB R2012a with Intel Math Kernel Library and IEEE 754 double precision. We denote Algorithm 4.2 by `M`. We executed the numerical spectral decomposition and generalized eigendecomposition by the MATLAB function `eig`. See `http://web.cc.iwate-u.ac.jp/~miyajima/NSGEP.zip` for details of the implementation, where the INTLAB [23] code of `M` (denoted by `M.m`) is uploaded.

Suppose as a result of execution of `M`, we have $\langle \tilde{\lambda}_i, r_i \rangle \in \mathbb{IC}$ and $\langle \tilde{x}^{(i)}, s^{(i)} \rangle \in \mathbb{IC}^n$ containing λ_i and $x^{(i)}$ in (1.1), respectively, for $i = 1, \ldots, n$. To assess quality of enclosure, define the maximum radius (MR) as $\max(\max_i r_i, \max_i \|s^{(i)}\|_\infty)$. As mentioned in Section 1, the verification algorithm designed specifically for $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ seems to be unavailable in literature. To try to hint on how large the MR is, we executed [9, Algorithm 1] via symbolic computation using the MATLAB Symbolic Math Toolbox. Considering this result as the "exact" eigenpair $(\hat{\lambda}_i, \hat{x}^{(i)})$, define the maximum difference (MD) as $\max(\max_i |\tilde{\lambda}_i - \hat{\lambda}_i|, \max_i \|\tilde{x}^{(i)} - \hat{x}^{(i)}\|_\infty)$. We also report the MD and compare it with the MR. For observing performance of `M`, we report CPU time (sec) of [9, Algorithm 1] executed in floating point computation (denoted by $\mathrm{T}_{\mathtt{I}}$) adding with that of `M` (denoted by $\mathrm{T}_{\mathtt{M}}$). Let p_j, $j = 1, \ldots, q$ be as in Section 1, and r be as in Lemma 3.6. In the examples below, we also report r, q, and p_j for $j = 1, \ldots, q$. Note that $p_j = 1$ for $j \in \{1, \ldots, q\}$ implies that the corresponding eigenvalue of $C^H C$ is isolated.

Example 5.1 We created matrices $A_0, B_0 \in \mathbb{C}^{m\times n}$ so that $A_0 - \lambda B_0$ has n eigenpairs and added noise to them by the following procedure employed in [5, 9]:

- Choose random matrices $\tilde{A}, \tilde{B} \in \mathbb{C}^{n\times n}$ and $\tilde{Q} \in \mathbb{C}^{m\times n}$ whose entries' real and imaginary parts are drawn independently from the Gaussian distribution with zero mean and standard deviation equal to one.
- Compute the QR decomposition of $\tilde{Q}$ to define Q_0 as its Q part.
- Define A_0 and B_0 by $A_0 := Q_0\tilde{A}$ and $B_0 := Q_0\tilde{B}$, respectively.
- Define A and B by $A := A_0 + N_A$ and $B := B_0 + N_B$, respectively, where N_A and N_B are matrices of random noise.

Real and imaginary parts of the entries of N_A and N_B are drawn independently from the Gaussian distribution with zero mean and standard deviation equal to ς. Then, $A_0 - \lambda B_0$ has the same eigenpairs as the square matrix pencil $\tilde{A} - \lambda\tilde{B}$. Table 1 displays the MD, MR, $\mathrm{T}_{\mathtt{I}}$ and $\mathrm{T}_{\mathtt{M}}$ for various m and n when $\varsigma = 0.25$. Table 2 reports the similar quantities for various ς when $m = 100$ and $n = 5$. We obtained $r = n$, $q = 2n$, and $p_j = 1$ for $j = 1, \dots, q$ in all the problems. Hence, the number of the isolated eigenvalues of $C^HC \in \mathbb{C}^{2n\times 2n}$ is $2n$, i.e., all the eigenvalues are isolated. We see from Table 1 that the MR increased as n increased, whereas it stayed about the same even when m increased. Table 2 shows magnitude of ς did not affect the MR.

m	n	MD	MR	$\mathrm{T}_{\mathtt{I}}$	$\mathrm{T}_{\mathtt{M}}$
100	5	5.4e–14	5.0e–11	6.0e–3	7.5e–2
200	5	7.1e–15	2.1e–11	2.4e–3	2.2e–2
300	5	7.5e–15	2.5e–11	5.7e–3	3.0e–2
100	10	8.4e–15	2.5e–10	1.0e–3	2.3e–2
200	10	2.3e–13	5.1e–9	3.6e–3	2.0e–2
300	10	1.3e–14	4.5e–10	9.1e–3	2.4e–2
100	15	1.4e–14	1.4e–9	2.1e–3	2.4e–2
200	15	5.7e–14	6.4e–9	5.8e–3	2.5e–2
300	15	7.8e–14	1.8e–9	7.9e–3	1.8e–2

Table 1 The MD, MR, $\mathrm{T}_{\mathtt{I}}$ and $\mathrm{T}_{\mathtt{M}}$ for various m and n in Example 5.1

ς	MD	MR	$\mathrm{T}_{\mathtt{I}}$	$\mathrm{T}_{\mathtt{M}}$
0.50	2.3e–15	7.2e–12	1.1e–3	1.9e–2
0.75	2.1e–14	2.1e–11	7.0e–4	1.9e–2
1.00	1.7e–14	4.9e–11	5.9e–4	2.4e–2
1.25	3.5e–15	3.8e–11	7.3e–4	2.1e–2
1.50	1.9e–14	4.0e–11	6.7e–4	1.9e–2

Table 2 The MD, MR, $\mathrm{T}_{\mathtt{I}}$ and $\mathrm{T}_{\mathtt{M}}$ for various ς in Example 5.1

Example 5.2 We consider the two problems in [9, Section 3.3], where

$$A = S\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T_1, \ B = ST_1, \ S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ -1 & 1 \end{bmatrix}, \ T_1 = \begin{bmatrix} 1.0001 & 1 \\ 1 & 0.9999 \end{bmatrix}, \tag{5.1}$$

$$A = S \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} T_2, \; B = ST_2, \; T_2 = \begin{bmatrix} 1.0001 & -1 \\ 1 & 0.9999 \end{bmatrix}. \tag{5.2}$$

In (5.1), B is almost rank-deficient, whereas B in (5.2) is well-conditioned. In the both problem, $A - \lambda B$ has eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -1$. We obtained $r = n = 2$, $q = 3$, $p_1 = p_2 = 1$, and $p_3 = 2$ in the both problem. Hence, the number of the isolated eigenvalues of $C^H C \in \mathbb{C}^{4\times 4}$ is 2. Table 3 displays the similar quantities for (5.1) and (5.2). The MR was large in (5.1).

problem	MD	MR	$\mathrm{T}_{\mathtt{I}}$	$\mathrm{T}_{\mathtt{M}}$
(5.1)	1.6e–8	6.3e–3	1.1e–3	4.5e–2
(5.2)	1.8e–40	4.6e–11	5.0e–4	1.8e–2

Table 3 The MD, MR, $\mathrm{T}_{\mathtt{I}}$ and $\mathrm{T}_{\mathtt{M}}$ in Example 5.2

Example 5.3 Consider the case where $C^H C$ has multiple or nearly multiple eigenvalues. In the first problem, we generated $A, B \in \mathbb{C}^{80\times 40}$ by the following MATLAB code:

```
[U,~,V] = svd(randn(80,80) + 1i*randn(80,80));  S = eye(80);
S(1,1) = 40;  S(2,2) = 40;  for i = 3:40, S(i,i) = 42 - i;  end;
C = U*S*V';  A = C(:,41:80);  B = C(:,1:40);
```

Then, $\mu_1 \approx \mu_2 \approx 40^2$, $\mu_i \approx (42-i)^2$ for $i = 3, \dots, 40$, and $\mu_i \approx 1$ for $i = 41, \dots, 80$. We obtained $r = 39$, $q = 40$, $p_1 = 2$, $p_i = 1$ for $i = 2, \dots, 39$, and $p_{40} = 40$. Hence, the number of the isolated eigenvalues of $C^H C \in \mathbb{C}^{80\times 80}$ is 38. In the second problem, we generated $A, B \in \mathbb{C}^{80\times 40}$ such that

```
S = eye(80);  for i = 1:40, S(i,i) = 40;  end;  C = U*S*V';
A = C(:,41:80);  B = C(:,1:40);
```

by reusing `U` and `V` given above. Then, $\mu_i \approx 40^2$ for $i = 1, \dots, 40$, and $\mu_i \approx 1$ for $i = 41, \dots, 80$. We obtained $r = 1$, $q = 2$, and $p_1 = p_2 = 40$. Hence, the number of the isolated eigenvalues is 0. Table 4 reports the similar quantities and $\mathrm{rad}(\boldsymbol{\mu}_q)$, where $\boldsymbol{\mu}_q$ is as in Step 6 of Algorithm 4.2, for the two problems. Note that $\mathrm{rad}(\boldsymbol{\mu}_q)$ in the first and second problems correspond to the cases where $p_1 = 2$ and $p_1 = 40$, respectively. This result shows `M` worked well even in the multiple or nearly multiple case. We can moreover observe that $\mathrm{rad}(\boldsymbol{\mu}_q)$ in the second problem ($p_1 = 40$) is larger than that in the first problem ($p_1 = 2$) (see Remark 4.3).

problem	MD	MR	$\mathrm{T}_{\mathtt{I}}$	$\mathrm{T}_{\mathtt{M}}$	$\mathrm{rad}(\boldsymbol{\mu}_q)$
first	3.2e–12	2.9e–7	6.4e–3	3.8e–2	4.3e–11
second	3.1e–13	2.8e–8	8.0e–3	4.6e–2	1.5e–9

Table 4 The MD, MR, $\mathrm{T}_{\mathtt{I}}$, $\mathrm{T}_{\mathtt{M}}$ and $\mathrm{rad}(\boldsymbol{\mu}_q)$ in Example 5.3

6. Conclusion

In this paper, we proved Propositions 1.3 and 1.4, proposed Algorithm 4.2, and reported the numerical results. It was mentioned in [9, Proposition 3] that all the eigenpairs of $A - \lambda B$ are contained in $\{(\lambda_i, x^{(i)})\}_{i=1}^n$ of (1.1) even if the number of eigenpairs of $A - \lambda B$ is less than n. This assertion gives all the eigenpairs of $A - \lambda B$ are contained in $\{(\boldsymbol{\lambda}_i, \boldsymbol{x}^{(i)})\}_{i=1}^n$, where $\boldsymbol{\lambda}_i$ and $\boldsymbol{x}^{(i)}$ are as in Algorithm 4.2. Note that several pairs in $\{(\boldsymbol{\lambda}_i, \boldsymbol{x}^{(i)})\}_{i=1}^n$ do not contain the eigenpairs of $A - \lambda B$ when the number is less than n. If $A\boldsymbol{x}^{(j)} - \boldsymbol{\lambda}_j B\boldsymbol{x}^{(j)} \not\ni 0$, then we can assert $(\boldsymbol{\lambda}_j, \boldsymbol{x}^{(j)})$ does not contain the eigenpair. We can thus exclude such pairs from $\{(\boldsymbol{\lambda}_i, \boldsymbol{x}^{(i)})\}_{i=1}^n$. Computing intervals containing all the eigenpairs of $A - \lambda B$ is possible in this manner.

It is clear that Algorithm 4.2 fails when $W_{21}^H - \lambda W_{11}^H$ has semi-simple eigenvalues. In this case, we cannot verify the diagonalizability of $(W_{11}^H)^{-1} W_{21}^H$. Hence, our first future work will be to develop a verification algorithm which is applicable even in this case. Partition U and Σ in Theorem 1.1 such that $U = [U_1, U_2]$ and $\Sigma = \mathrm{diag}(\Sigma_1, \Sigma_2)$, where $U_1 \in \mathbb{C}^{m\times n}$ and $\Sigma_1 \in \mathbb{C}^{n\times n}$, respectively. Then, $\hat{A}$ and $\hat{B}$ in (1.1) can be represented as $\hat{A} = U_1 \Sigma_1 V_{21}^H$ and $\hat{B} = U_1 \Sigma_1 V_{11}^H$, respectively [9, Eq. (11)]. As mentioned in Section 1, we cannot obtain intervals containing some columns of U_1, V_{11} and V_{21} when some singular values of C are multiple or nearly multiple. Thus, our second future work will be to compute intervals containing $\hat{A}$ and $\hat{B}$ in this case.

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