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Automorphism Groups of Some Graphs for the Ring of Gaussian Integers Modulo p^s

Hengbin ZHANG*, **Jizhu NAN**
School of Mathematical Sciences, Dalian University of Technology, Liaoning 116024*, P. R. China*

Abstract In this paper, the automorphism group is completely determined, of the unitary Cayley graph, the unit graph and the total graph, over the ring of Gaussian integers modulo a prime power.

Keywords automorphism; unit graph; unitary Cayley graph; Gaussian integers

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1. Introduction

Given a ring R, by $D(R)$ and $U(R)$ we denote the set of zero-divisors and the group of units, respectively. Then the unitary Cayley graph G_R, the unit graph $G(R)$ and the total graph $T(\Gamma(R))$ of the ring R are defined to be simple graphs with the same vertex set R and with the edge $\{a, b\}$, where $a - b \in U(R)$, $a + b \in U(R)$ and $a + b \in D(R)$, respectively. Obviously, $T(\Gamma(R))$ is the complement of $G(R)$, provided R is a finite ring.

For a graph G, a bijection σ on vertex set is called an automorphism of G if σ preserves adjacency. Note that the set of all automorphisms of G forms a group under usual composition of functions. Using the algebraic structure to determine the automorphisms of a family of graph has attracted considerable attention during the past decades [1–3]. In 1995, Dejter and Giudici defined the unitary Cayley graph in [4]. They proved that $G_{\mathbb{Z}_n}$ is a bipartite graph when n is even, where $\mathbb{Z}_n$ is the additive cyclic group of integers mod n. Grimaldi defined the unit graph $G(\mathbb{Z}_n)$ in [5]. The total graph was introduced and investigated by Anderson and Badawi in [6]. They also studied the three induced subgraphs $\mathrm{Nil}(\Gamma(R))$, $Z(\Gamma(R))$, and $\mathrm{Reg}(\Gamma(R))$ of $T(\Gamma(R))$, with vertices $\mathrm{Nil}(R)$, $Z(R)$, and $\mathrm{Reg}(R)$, respectively. Here, R is a commutative ring, $\mathrm{Nil}(R)$ is the ideal of nilpotent elements, $Z(R)$ is the set of zero-divisors, and $\mathrm{Reg}(R)$ is the set of regular elements. For some other recent papers on these graphs [7–9].

In this paper, we shall focus on the unit graph, the unitary Cayley graph and the total graph, over the ring $\mathbb{Z}_{p^s}[i]$ of Gaussian integers mod p^s. Recall that the ring $\mathbb{Z}_n[i]$ of Gaussian integers modulo n is the set $\{a + bi \mid a, b \in \mathbb{Z}_n\}$ with ordinary addition and multiplication of complex numbers, and Euclidian norm $N(a + ib) = a^2 + b^2$, where $i^2 = -1$. Let $\mathbb{Z}_{p^s}[i]$ be the ring of Gaussian integers modulo p^s, where p is prime and s is a positive integer.

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* Corresponding author
E-mail address: zhhb210@163.com (Hengbin ZHANG); jznan@163.com (Jizhu NAN)

This paper is organized as follows. In Section 2, we give some preliminaries, notation and lemmas. In Section 3, we show that $G_{\mathbb{Z}_{2^s}[i]}$ is a complete bipartite graph. Then, we get the automorphism groups of $G_{\mathbb{Z}_{2^s}[i]}$, $G(\mathbb{Z}_{2^s}[i])$ and $T(\Gamma(\mathbb{Z}_{2^s}[i]))$. In Section 4, we show that $G_{\mathbb{Z}_{p^s}[i]}$ is a complete multipartite graph, then it is easy to have the automorphism groups of $G_{\mathbb{Z}_{2^s}[i]}$, where $p \equiv 3 \pmod 4$. We use regular graph of $\mathbb{Z}_{p^s}[i]$ to determine the automorphism groups of $G(\mathbb{Z}_{p^s}[i])$ and $T(\Gamma(\mathbb{Z}_{p^s}[i]))$. In Section 5, after defining some automorphisms, we show the automorphism groups of $G_{\mathbb{Z}_{p^s}[i]}$, $G(\mathbb{Z}_{p^s}[i])$ and $T(\Gamma(\mathbb{Z}_{p^s}[i]))$, where $p \equiv 1 \pmod 4$.

2. Preliminaries

We use $D(R)$ and $U(R)$ to denote the set of zero-divisors and the group of units of a ring R, respectively. For a set T, T^* denotes the non-zero elements of T, $|T|$ denotes the size of T, $T\backslash S$ denotes the set of elements that belong to T and not to set S. We will use $V(G)$ to denote the vertex set of a graph G. Let $x, y \in V(G)$. If x and y are adjacent vertices, then they are called the neighbors of each other. We write $N_G(x)$ for the set of neighbors of x in G.

Lemma 2.1 ([10, Theorem 2]) *Let p be a prime and s be a positive integer.*

(i) Let $p = 2$ and $a + bi \in \mathbb{Z}_{p^s}[i]$. Then $a + bi \in U(\mathbb{Z}_{p^s}[i])$ if and only if $a \not\equiv b \pmod 2$.

(ii) Let $p = 3 \pmod 4$ and $a + bi \in \mathbb{Z}_{p^s}[i]$. Then $a + bi \in U(\mathbb{Z}_{p^s}[i])$ if and only if one of a and b is prime to p.

(iii) Let $p = 1 \pmod 4$, $p = \pi\overline{\pi}$ for some π in $\mathbb{Z}[i]$ and $a \in \mathbb{Z}[i]/(\pi^s)$, where $\overline{\pi}$ is the complex conjugate of π. Then $a \in U(\mathbb{Z}[i]/(\pi^s))$ if and only if a is prime to p.

If G_2 is a permutation group on $\{1, 2, \ldots, n\}$, then the wreath product $G_1 \wr G_2$ is generated by the direct product of n copies of G_1, together with the elements of G_2 acting on these n copies of G_1.

Lemma 2.2 ([11, P.139, P.188]) *(i) A graph and its complement have the same automorphism group.*

(ii) For $n \geq 2$, let $K_{n,n}$ be the complete bipartite graph of degree n. Then $\mathrm{Aut}(K_{n,n}) = S_n \wr S_2$.

(iii) Let the connected components of G consist of n_1 copies of G_1, n_2 copies of $G_2, \ldots, n_r$ copies of G_r, where $G_1, G_2, \ldots, G_r$ are pairwise non-isomorphic. Then $\mathrm{Aut}(G) = (\mathrm{Aut}(G_1) \wr S_{n_1}) \times (\mathrm{Aut}(G_2) \wr S_{n_2}) \times \cdots \times (\mathrm{Aut}(G_r) \wr S_{n_r})$.

Lemma 2.3 ([7, Theorem 2.6]) *Let R be a finite ring. Then the following statements hold.*

(i) If R is a local ring of even order, then $\mathrm{Aut}(G_R) \cong \mathrm{Aut}(G(R))$.

(ii) If R is a ring of odd order, then $\mathrm{Aut}(G_R) \ncong \mathrm{Aut}(G(R))$.

3. Automorphisms of some graphs for $\mathbb{Z}_{2^s}[i]$

In this section, we determine the automorphism groups of the unit graph, the unitary Cayley graph and the total graph of $\mathbb{Z}_{2^s}[i]$. We first prove some lemmas about these graphs. From the definitions of the unit graph and the unitary Cayley graph, it is easy to have the following lemma.

Lemma 3.1 *Let $a+bi \in \mathbb{Z}_{2^s}[i]$, where s is a positive integer. Then,*

(i) $N_{G_{\mathbb{Z}_{2^s}[i]}}(a+bi) = (a+bi) + U(\mathbb{Z}_{2^s}[i])$;

(ii) $N_{G(\mathbb{Z}_{2^s}[i])}(a+bi) = -(a+bi) + U(\mathbb{Z}_{2^s}[i])$.

Lemma 3.2 *Let s be a positive integer. Then $G_{\mathbb{Z}_{2^s}[i]}$ and $G(\mathbb{Z}_{2^s}[i])$ are the union of some independent sets. In particular,*

$$V(G_{\mathbb{Z}_{2^s}[i]}) = V(G(\mathbb{Z}_{2^s}[i])) = \bigcup_{\alpha\in\{0,1\}} (\alpha + D(\mathbb{Z}_{2^s}[i])).$$

Proof From Lemma 2.1 (i), $a+bi \in D(\mathbb{Z}_{2^s}[i])$ if and only if $a \equiv b \pmod 2$. Suppose that $\alpha = a+bi$, $\beta = c+di \in D(\mathbb{Z}_{2^s}[i])$ and $\alpha \neq \beta$, then $a \equiv b \pmod 2$ and $c \equiv d \pmod 2$. So $a-c \equiv b-d \pmod 2$ and $\alpha - \beta \in D(\mathbb{Z}_{2^s}[i])$. It means that α is not connected to β in $G_{\mathbb{Z}_{2^s}[i]}$. Furthermore, the set $D(\mathbb{Z}_{2^s}[i])$ is an independent set in $G_{\mathbb{Z}_{2^s}[i]}$. It is easy to check that $1 + D(\mathbb{Z}_{2^s}[i]) = U(\mathbb{Z}_{2^s}[i])$. Similarly, the set $1 + D(\mathbb{Z}_{2^s}[i])$ is an independent set in $G_{\mathbb{Z}_{2^s}[i]}$. The proof for the case $G(\mathbb{Z}_{2^s}[i])$ is similar. □

Theorem 3.3 *Let s be a positive integer. Then*

$$\mathrm{Aut}(G_{\mathbb{Z}_{2^s}[i]}) \cong \mathrm{Aut}(G(\mathbb{Z}_{2^s}[i])) \cong \mathrm{Aut}(T(\Gamma(\mathbb{Z}_{2^s}[i]))) \cong S_{2^{2s-1}} \wr S_2.$$

Proof From Lemmas 2.2 (i) and 2.3 (i), we know that $\mathrm{Aut}(G(\mathbb{Z}_{2^s}[i])) \cong \mathrm{Aut}(T(\Gamma(\mathbb{Z}_{2^s}[i])))$ and $\mathrm{Aut}(G_{\mathbb{Z}_{2^s}[i]}) \cong \mathrm{Aut}(G(\mathbb{Z}_{2^s}[i]))$. We only need to show that $\mathrm{Aut}(G_{\mathbb{Z}_{2^s}[i]}) \cong S_{2^{2s-1}} \wr S_2$. From Lemma 2.1 (i), it is immediate that $|D(\mathbb{Z}_{2^s}[i])| = |1 + D(\mathbb{Z}_{2^s}[i])| = |U(\mathbb{Z}_{2^s}[i])| = 2^{2s-1}$. By Lemma 2.2 (ii), what is left is to show that $G_{\mathbb{Z}_{2^s}[i]}$ is a complete bipartite graph of degree 2^{2s-1}. Suppose that $\alpha = a+bi \in 1 + D(\mathbb{Z}_{2^s}[i])$, $\beta = c+di \in D(\mathbb{Z}_{2^s}[i])$, then $a \not\equiv b \pmod 2$ and $c \equiv d \pmod 2$ by Lemma 2.1 (i). So $a-c \not\equiv b-d \pmod 2$ and $\alpha-\beta \in 1 + D(\mathbb{Z}_{2^s}[i]) = U(\mathbb{Z}_{2^s}[i])$. It means that α is connected to β in $G_{\mathbb{Z}_{2^s}[i]}$. Furthermore, every vertex in the set $D(\mathbb{Z}_{2^s}[i])$ is connected to all vertices in the set $1 + D(\mathbb{Z}_{2^s}[i])$. Then by Lemma 3.2, $G_{\mathbb{Z}_{2^s}[i]}$ is a complete bipartite graph of degree 2^{2s-1}, which completes the proof. □

4. Automorphisms of some graphs for $\mathbb{Z}_{p^s}[i]$, $p \equiv 3 \pmod 4$

In this section, we determine the automorphism groups of the unit graph, the unitary Cayley graph and the total graph of $\mathbb{Z}_{p^s}[i]$, where $p \equiv 3 \pmod 4$. Similarly, from the definitions of the unit graph and the unitary Cayley graph, it is easy to have the following lemma.

Lemma 4.1 *Let $a+bi \in \mathbb{Z}_{p^s}[i]$, where $p \equiv 3 \pmod 4$ and s is a positive integer. Then,*

(i) $N_{G_{\mathbb{Z}_{p^s}[i]}}(a+bi) = (a+bi) + U(\mathbb{Z}_{p^s}[i])$;

(ii) If $a+bi \in D(\mathbb{Z}_{p^s}[i])$, then $N_{G(\mathbb{Z}_{p^s}[i])}(a+bi) = -(a+bi) + U(\mathbb{Z}_{p^s}[i])$;

(iii) If $a+bi \in U(\mathbb{Z}_{p^s}[i])$, then $N_{G(\mathbb{Z}_{p^s}[i])}(a+bi) = (-(a+bi) + U(\mathbb{Z}_{p^s}[i])) \backslash \{a+bi\}$.

Lemma 4.2 *Let $p \equiv 3 \pmod 4$ and s be a positive integer. Then $G_{\mathbb{Z}_{p^s}[i]}$ and $G(\mathbb{Z}_{p^s}[i])$ are the* *union of some independent sets. In particular,*

$$V(G_{\mathbb{Z}_{p^s}[i]}) = V(G(\mathbb{Z}_{p^s}[i])) = \bigcup_{a,b=0}^{p-1}((a+bi) + D(\mathbb{Z}_{p^s}[i])).$$

Proof Let $p \equiv 3 \pmod 4$ and s be a positive integer. By Lemma 2.1 (ii), $a+bi \in D(\mathbb{Z}_{p^s}[i])$ if and only if a and b are not prime to p. Suppose that $\alpha = a+bi$, $\beta = c+di \in D(\mathbb{Z}_{2^s}[i])$ and $\alpha \neq \beta$, then $p|a$, $p|b$, $p|c$ and $p|d$. So $p|(a-c)$, $p|(b-d)$ and $(\alpha-\beta) \in D(\mathbb{Z}_{2^s}[i])$. It means that α is not connected to β in $G_{\mathbb{Z}_{p^s}[i]}$. Furthermore, the set $D(\mathbb{Z}_{p^s}[i])$ is an independent set in $G_{\mathbb{Z}_{p^s}[i]}$. It is easy to check that $\bigcup_{a,b=0}^{p-1}((a+bi)+D(\mathbb{Z}_{p^s}[i]))\backslash D(\mathbb{Z}_{p^s}[i]) = U(\mathbb{Z}_{p^s}[i])$. Similarly, for $a, b \in \{0,1,\ldots,p-1\}$, the set $(a+bi)+D(\mathbb{Z}_{p^s}[i])$ is an independent set in $G_{\mathbb{Z}_{p^s}[i]}$. The proof for the case $G(\mathbb{Z}_{p^s}[i])$ is similar. $\square$

Recall that the total graph of ring R is a graph with all elements of R as vertices, and two distinct vertices α,β are adjacent if and only if $\alpha+\beta \in D(R)$. It is denoted by $T(\Gamma(R))$. Let regular graph of R, $\mathrm{Reg}(\Gamma(R))$, be the induced subgraph of $T(\Gamma(R))$ on the regular elements of R. For a finite ring, the regular elements are the unit elements. So $\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))$ is an induced subgraph of $T(\Gamma(\mathbb{Z}_{p^s}[i]))$ on the unit elements of $\mathbb{Z}_{p^s}[i]$. We first determine the automorphism group of $\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))$.

Theorem 4.3 *Let $p \equiv 3 \pmod 4$ and s be a positive integer. Then,*

$$\mathrm{Aut}(\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))) \cong (S_{p^{2s-2}} \wr S_2) \wr S_{\frac{p^2-1}{2}}.$$

Proof Let us denote by R_p the set $\{a+bi \in \mathbb{Z}_{p^s}[i] | 0 \le a,b \le p-1\}$. From Lemma 2.1 (ii), $a+bi \in D(\mathbb{Z}_{p^s}[i])$ if and only if a and b are not prime to p. Then there exists only one zero divisor 0 in R_p.

Suppose that $0 \neq \alpha \in R_p$, then there exists a unique $0 \neq \beta \in R_p$ such that $\alpha+\beta = p+pi$. We next show that the subgraph of $\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))$ induced by $\alpha + D(\mathbb{Z}_{p^s}[i]) \bigcup \beta + D(\mathbb{Z}_{p^s}[i])$ is a complete bipartite connected components of $\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))$. Since $p \equiv 3 \pmod 4$, we know that $(p,2\alpha)=1$ and $(p,2\beta)=1$. Therefore, $\alpha + D(\mathbb{Z}_{p^s}[i])$ and $\beta + D(\mathbb{Z}_{p^s}[i])$ are the independent sets in $\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))$. And by $\alpha+\beta = p+pi$, it is obvious that $\alpha + D(\mathbb{Z}_{p^s}[i]) \bigcup \beta + D(\mathbb{Z}_{p^s}[i])$ is a complete bipartite subgraph of $\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))$. For $\gamma \in R_p\backslash\{\alpha,\beta\}$, it is clear that $\alpha+\gamma \neq p+pi$ and $\beta+\gamma \neq p+pi$. Thus, $(p,\alpha+\gamma)=1$ and $(p,\beta+\gamma)=1$. Hence, for any $a+bi \in D(\mathbb{Z}_{p^s}[i])$, $(p,\alpha+\gamma+a+bi)=1$ and $(p,\beta+\gamma+a+bi)=1$, this means that $\alpha+\gamma+a+bi \in U(\mathbb{Z}_{p^s}[i])$ and $\beta+\gamma+a+bi \in U(\mathbb{Z}_{p^s}[i])$. Therefore, all vertices in $\alpha + D(\mathbb{Z}_{p^s}[i]) \bigcup \beta + D(\mathbb{Z}_{p^s}[i])$ are not adjacent to $\bigcup_{\gamma \in R_p\backslash\{\alpha,\beta\}}(\gamma + D(\mathbb{Z}_{p^s}[i]))$. From Lemma 2.1 (ii), $|D(\mathbb{Z}_{p^s}[i])| = p^{2s-2}$. Consequently, the subgraph of $\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))$ induced by $\alpha + D(\mathbb{Z}_{p^s}[i]) \bigcup \beta + D(\mathbb{Z}_{p^s}[i])$ is a complete bipartite connected components of $\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))$, which gives $\mathrm{Aut}(\mathrm{Reg}(\Gamma(\alpha + D(\mathbb{Z}_{p^s}[i]) \bigcup \beta + D(\mathbb{Z}_{p^s}[i])))) \cong S_{p^{2s-2}} \wr S_2$.

Since $(p,2)=1$, the equation $X+Y = p+pi$ has $\frac{p^2-1}{2}$ distinct pairs of solutions in R_p. Thus, $\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))$ consist of $\frac{p^2-1}{2}$ copies of $\alpha + D(\mathbb{Z}_{p^s}[i]) \bigcup \beta + D(\mathbb{Z}_{p^s}[i])$. By Lemma 2.2 (iii), we get $\mathrm{Aut}(\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))) \cong (S_{p^{2s-2}} \wr S_2) \wr S_{\frac{p^2-1}{2}}$. $\square$

Now we determine the automorphism groups of the unit graph, the unitary Cayley graph and the total graph of $\mathbb{Z}_{p^s}[i]$, where $p \equiv 3 \pmod 4$.

Theorem 4.4 *Let $p \equiv 3 \pmod 4$ and s be a positive integer. Then*

$$\mathrm{Aut}(G_{\mathbb{Z}_{p^s}[i]}) \cong S_{p^{2s-2}} \wr S_{p^2}$$

and

$$\mathrm{Aut}(G(\mathbb{Z}_{p^s}[i])) \cong \mathrm{Aut}(T(\Gamma(\mathbb{Z}_{p^s}[i]))) \cong ((S_{p^{2s-2}} \wr S_2) \wr S_{\frac{p^2-1}{2}}) \times S_{p^{2s-2}}.$$

Proof The proof for $\mathrm{Aut}(G_{\mathbb{Z}_{p^s}[i]}) \cong S_{p^{2s-2}} \wr S_{p^2}$ is similar to Theorem 3.3. In fact, $G_{\mathbb{Z}_{p^s}[i]}$ is a complete p^2-partite graph $K_{p^{2s-2},p^{2s-2},\ldots,p^{2s-2}}$.

By Lemma 2.2 (i), we get $\mathrm{Aut}(G(\mathbb{Z}_{p^s}[i])) \cong \mathrm{Aut}(T(\Gamma(\mathbb{Z}_{p^s}[i])))$. We only need to show that $\mathrm{Aut}(T(\Gamma(\mathbb{Z}_{p^s}[i]))) \cong ((S_{p^{2s-2}} \wr S_2) \wr S_{\frac{p^2-1}{2}}) \times S_{p^{2s-2}}$. From Lemma 4.1 (ii) and (iii), we know that the unit elements and zero divisors have different degrees in graph $T(\Gamma(\mathbb{Z}_{p^s}[i]))$. It is obvious that $D(\mathbb{Z}_{p^s}[i])$ and $U(\mathbb{Z}_{p^s}[i])$ are two connected components of $T(\Gamma(\mathbb{Z}_{p^s}[i]))$ and $D(\mathbb{Z}_{p^s}[i])$ is closed under addition. Hence, the connected component of $T(\Gamma(\mathbb{Z}_{p^s}[i]))$ induced by the zero divisors is a complete subgraph. By Theorem 4.3, $\mathrm{Aut}(\mathrm{Reg}(\Gamma(\mathbb{Z}_{p^s}[i]))) \cong (S_{p^{2s-2}} \wr S_2) \wr S_{\frac{p^2-1}{2}}$. Therefore, $\mathrm{Aut}(T(\Gamma(\mathbb{Z}_{p^s}[i]))) \cong ((S_{p^{2s-2}} \wr S_2) \wr S_{\frac{p^2-1}{2}}) \times S_{p^{2s-2}}$, by Lemma 2.2 (iii). $\square$

5. Automorphisms of some graphs for $\mathbb{Z}_{p^s}[i]$, $p \equiv 1 \pmod 4$

Let $p \equiv 1 \pmod 4$. Then $p = \pi\overline{\pi}$ for some π in $\mathbb{Z}[i]$, where $\overline{\pi}$ is the complex conjugate of π. In [10], we know that $\mathbb{Z}[i]/(\pi^s) \cong \mathbb{Z}_{p^s}$. Then by Chinese remainder theorem,

$$\mathbb{Z}_{p^s}[i] \cong \mathbb{Z}[i]/(p^s) \cong \mathbb{Z}[i]/(\pi^s) \times \mathbb{Z}[i]/(\overline{\pi}^s) \cong \mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}.$$

In this section, we use $\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}$ instead of $\mathbb{Z}_{p^s}[i]$. It is well known that $U(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}) = U(\mathbb{Z}_{p^s}) \times U(\mathbb{Z}_{p^s})$. Then by Lemma 2.1 (iii) and the definitions of the unit graph, the unitary Cayley graph, it is easy to have the following lemma.

Lemma 5.1 *Let $(a, b) \in \mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}$, where $p \equiv 1 \pmod 4$ and s is a positive integer. Then,*

(i) $N_{G_{\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}}}(a, b) = (a, b) + U(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})$;

(ii) If $(a, b) \in D(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})$, then $N_{G(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})}(a, b) = -(a, b) + U(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})$;

(iii) If $(a, b) \in U(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})$, then $N_{G(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})}(a, b) = (-(a, b) + U(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}))\backslash\{(a, b)\}$.

Lemma 5.2 *Let $p \equiv 1 \pmod 4$ and s be a positive integer. Then $G_{\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}}$ and $G(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})$ are the union of some independent sets. In particular,*

$$V(G_{\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}}) = V(G(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})) = \bigcup_{a,b=0}^{p-1} ((a, b) + D(\mathbb{Z}_{p^s}) \times D(\mathbb{Z}_{p^s})).$$

Proof The proof is similar to Lemma 4.2. $\square$

In order to get the automorphism groups of $G_{\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}}$, $G(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})$ and $T(\Gamma(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}))$, we need to define the following mappings. Let

$$f : \mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s} \to \mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}$$
$$(a, b) \mapsto (b, a).$$

Then $\{f, f^2 = e\}$ is a cycle group with order 2, denoted by S_2. Let Z_p be a subset of $\mathbb{Z}_{p^s}$, set $Z_p = \{a \mid 0 \le a \le p-1\}$. Let S_p be the symmetric group over the set Z_p and $g \in S_p$, define

$$\begin{aligned} h_{e,g} : Z_p \times Z_p &\to Z_p \times Z_p \\ (a,b) &\mapsto (a, g(b)). \end{aligned}$$

Set $H_p = \{h_{e,g} \mid g \in S_p\}$. Similarly, we have

$$\begin{aligned} h_{g,e} : Z_p \times Z_p &\to Z_p \times Z_p \\ (a,b) &\mapsto (g(a), b). \end{aligned}$$

Note that $h_{g,e} = f h_{e,g} f$.

We will denote by $\mathrm{Aut}(G_{Z_p \times Z_p})$ the automorphism group of subgraph of $G_{\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s}}$ induced by $Z_p \times Z_p$. It is easy to check that the restriction of S_2 to $Z_p \times Z_p$ and H_p are subgroups of $\mathrm{Aut}(G_{Z_p \times Z_p})$. Let $\langle S_2 \cup H_p \rangle$ denote the subgroup of $\mathrm{Aut}(G_{Z_p \times Z_p})$ generated by $S_2 \cup H_p$.

Let $S_2 \wr S_{\frac{p-1}{2}}$ be the symmetric group over a partition $Z_p^* = \cup_{a+b=p}\{a, b\}$, where $Z_p^* = Z_p \backslash \{0\}$ and $g \in S_2 \wr S_{\frac{p-1}{2}}$, define

$$\begin{aligned} k_{e,g} : Z_p \times Z_p &\to Z_p \times Z_p \\ (a,b) &\mapsto (a, g(b)), \quad b \neq 0, \\ (a,b) &\mapsto (a, b), \qquad\ b = 0. \end{aligned}$$

Set $K_p = \{k_{e,g} \mid g \in S_2 \wr S_{\frac{p-1}{2}}\}$. Similar to $h_{g,e}$, we have

$$\begin{aligned} k_{g,e} : Z_p \times Z_p &\to Z_p \times Z_p \\ (a,b) &\mapsto (g(a), b), \quad a \neq 0, \\ (a,b) &\mapsto (a, b), \qquad\ a = 0. \end{aligned}$$

Note that $k_{g,e} = f k_{e,g} f$. We will denote by $\mathrm{Aut}(G(Z_p \times Z_p))$ the automorphism group of subgraph of $G(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})$ induced by $Z_p \times Z_p$. It is easy to check that the restriction of S_2 to $Z_p \times Z_p$ and K_p are subgroups of $\mathrm{Aut}(G(Z_p \times Z_p))$. Let $\langle S_2 \cup K_p \rangle$ denote the subgroup of $\mathrm{Aut}(G(Z_p \times Z_p))$ generated by $S_2 \cup K_p$.

Theorem 5.3 *Let $p \equiv 1 \pmod 4$. Then*

$$\mathrm{Aut}(G_{Z_p \times Z_p}) = \langle S_2 \cup H_p \rangle$$

and

$$\mathrm{Aut}(G(Z_p \times Z_p)) = \langle S_2 \cup K_p \rangle.$$

Proof It is obvious that $\mathrm{Aut}(G_{Z_p \times Z_p}) \supseteq \langle S_2 \cup H_p \rangle$. Let $\sigma \in \mathrm{Aut}(G_{Z_p \times Z_p})$. We next show that σ can be generated by finite composite of elements in $S_2 \cup H_p$. Suppose that $\sigma(0,0) = (a,b)$. Then there exist $g_1, g_2 \in S_p$ such that $g_1(a) = 0$ and $g_2(b) = 0$. Thus, $h_{g_2,e} h_{e,g_1} \sigma(0,0) = h_{g_2,e} h_{e,g_1}(a,b) = (0,0)$.

Set $\sigma_1 = h_{g_2,e} h_{e,g_1} \sigma$. Since automorphisms preserve adjacency and $(0,1) \notin N_{G_{Z_p \times Z_p}}(0,0)$, we know that $\sigma_1(0,1) \notin N_{G_{Z_p \times Z_p}}(0,0)$. Then $\sigma_1(0,1) \in \{(a,b) \in Z_p \times Z_p \mid a = 0,\ b \neq 0 \text{ or } a \neq$

$0,\ b=0\}$. Without loss of generality we can assume $\sigma_1(0,1)=(a_1,0)$. Then there exists $g_3\in S_p$ such that $g_3(0)=0$ and $g_3(a_1)=1$. Thus, we get $fh_{g_3,e}\sigma_1(0,0)=fh_{g_3,e}(0,0)=(0,0)$ and $fh_{g_3,e}\sigma_1(0,1)=fh_{g_3,e}(a_1,0)=f(1,0)=(0,1)$.

Set $\sigma_2=fh_{g_3,e}\sigma_1$. Since $\sigma_2(N_{G_{Z_p\times Z_p}}(0,0))=N_{G_{Z_p\times Z_p}}(0,0)$ and $\sigma_2(N_{G_{Z_p\times Z_p}}(0,1))=N_{G_{Z_p\times Z_p}}(0,1)$, we know that $\sigma_2(Z_p\times Z_p\backslash N_{G_{Z_p\times Z_p}}(0,0))=Z_p\times Z_p\backslash N_{G_{Z_p\times Z_p}}(0,0)$ and $\sigma_2(Z_p\times Z_p\backslash N_{G_{Z_p\times Z_p}}(0,1))=Z_p\times Z_p\backslash N_{G_{Z_p\times Z_p}}(0,1)$. In fact,

$$(Z_p\times Z_p\backslash N_{G_{Z_p\times Z_p}}(0,0))\cap(Z_p\times Z_p\backslash N_{G_{Z_p\times Z_p}}(0,1))=\{(0,b)\mid 0\le b\le p-1\}.$$

Then there exists $g_4\in S_p$ such that $h_{e,g_4}\sigma_2(0,b)=(0,b)$, where $0\le b\le p-1$.

Set $\sigma_3=h_{e,g_4}\sigma_2$. Similarly, there exists $g_5\in S_p$ such that $h_{g_5,e}\sigma_3(a,0)=(a,0)$ and $h_{g_5,e}\sigma_3(0,b)=(0,b)$, where $0\le a,b\le p-1$.

Set $\sigma_4=h_{g_5,e}\sigma_3$. Since automorphisms preserve adjacency and

$$(Z_p\times Z_p\backslash N_{G_{Z_p\times Z_p}}(0,b))\cap(Z_p\times Z_p\backslash N_{G_{Z_p\times Z_p}}(a,0))=\{(0,0),(a,b)\},$$

we can get $\sigma_4(a,b)=(a,b)$, where $0\le a,b\le p-1$. Therefore, σ_4 is the identity element e of $\mathrm{Aut}(G_{Z_p\times Z_p})$. This gives $e=h_{g_5,e}h_{e,g_4}fh_{g_3,e}h_{g_2,e}h_{e,g_1}\sigma$. Hence $\sigma=h^{-1}_{e,g_1}h^{-1}_{g_2,e}h^{-1}_{g_3,e}fh^{-1}_{e,g_4}h^{-1}_{g_5,e}$, which gives σ can be generated by finite composite of elements in $S_2\cup H_p$.

For any $\sigma\in\mathrm{Aut}(G(Z_p\times Z_p))$, by Lemma 5.1, we know that $N_{G(Z_p\times Z_p)}(0,0)=Z_p^*\times Z_p^*$ and $\sigma(Z_p^*\times Z_p^*)=Z_p^*\times Z_p^*$. Since automorphism preserves adjacency, $N_{G(Z_p\times Z_p)}(\sigma(0,0))=\sigma(N_{G(Z_p\times Z_p)}(0,0))=\sigma(Z_p^*\times Z_p^*)=Z_p^*\times Z_p^*=N_{G(Z_p\times Z_p)}(0,0)$. Then, $\sigma(0,0)=(0,0)$. Similar to the proof of $\mathrm{Aut}(G_{Z_p\times Z_p})$, $\mathrm{Aut}(G(Z_p\times Z_p))\cong\langle S_2\cup K_p\rangle$, which completes the proof. □

Since every non-zero element in $\mathbb{Z}_{p^s}$ can be written uniquely as $t_0+t_1p+\cdots+t_{s-1}p^{s-1}$, where $t_i\in\{0,1,\dots,p-1\}$, $i\in\{0,1,\dots,s-1\}$, and $U(\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s})=U(\mathbb{Z}_{p^s})\times U(\mathbb{Z}_{p^s})$, it is easy to get the following lemma.

Lemma 5.4 *Let $p\equiv 1 \pmod 4$ and s be a positive integer. Then for $\alpha,\beta\in\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s}$, the following conditions are equivalent.*

(i) $N_{G_{\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s}}}(\alpha)=N_{G_{\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s}}}(\beta)$.

(ii) $\alpha,\beta\in(a,b)+D(\mathbb{Z}_{p^s})\times D(\mathbb{Z}_{p^s})$ *for some* $a,b\in\{0,1,\dots,p-1\}$.

(iii) $N_{G(\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s})}(\alpha)=N_{G(\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s})}(\beta)$.

Now we show the automorphism groups of the unit graph, the unitary Cayley graph and the total graph of $\mathbb{Z}_{p^s}[i]$, where $p\equiv 1 \pmod 4$.

Theorem 5.5 *Let $p\equiv 1 \pmod 4$ and s be a positive integer. Then,*

$$\mathrm{Aut}(G_{\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s}})\cong(S_{p^{2s-2}})^{p^2}\rtimes\langle S_2\cup H_p\rangle,$$

and

$$\begin{aligned}\mathrm{Aut}(G(\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s}))&\cong\mathrm{Aut}(T(\Gamma(\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s})))\\&\cong(S_{p^{2s-2}})^{p^2}\rtimes\langle S_2\cup K_p\rangle.\end{aligned}$$

Proof Recall that $|D(\mathbb{Z}_{p^s})\times D(\mathbb{Z}_{p^s})|=p^{2s-2}$. Let $(S_{p^{2s-2}})^{p^2}$ be a product of symmetric groups over $\bigcup_{a,b=0}^{p-1}((a,b)+D(\mathbb{Z}_{p^s})\times D(\mathbb{Z}_{p^s}))$. We claim that $\mathrm{Aut}(G_{\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s}})/(S_{p^{2s-2}})^{p^2}\cong\mathrm{Aut}(G_{Z_p\times Z_p})$.

Let

$$\varphi : \mathrm{Aut}(G_{\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s}}) \to \mathrm{Aut}(G_{Z_p\times Z_p})$$
$$\sigma \mapsto \sigma|_{Z_p\times Z_p},$$

where $\sigma|_{Z_p\times Z_p}$ is the restriction of σ to $Z_p \times Z_p$. By Lemma 5.4, it is easily seen that φ is an epimorphism and $\ker(\varphi) = (S_{p^{2s-2}})^{p^2}$. Therefore, $\mathrm{Aut}(G_{\mathbb{Z}_{p^s}\times\mathbb{Z}_{p^s}}) \cong (S_{p^{2s-2}})^{p^2} \rtimes \langle S_2 \cup H_p\rangle$ by Theorem 5.3.

The proof for the case $G(\mathbb{Z}_{p^s} \times \mathbb{Z}_{p^s})$ is similar. $\square$

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