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On Split δ-Jordan Lie Triple Systems

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Abstract The aim of this article is to study the structures of arbitrary split δ-Jordan Lie triple systems, which are a generalization of split Lie triple systems. By developing techniques of connections of roots for this kind of triple systems, we show that any of such δ-Jordan Lie triple systems T with a symmetric root system is of the form $T = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$ with U a subspace of T_0 and any $I_{[\alpha]}$ a well described ideal of T, satisfying $\{I_{[\alpha]}, T, I_{[\beta]}\} = \{I_{[\alpha]}, I_{[\beta]}, T\} = \{T, I_{[\alpha]}, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$.

Keywords split δ-Jordan Lie triple system; Lie triple system; root system

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1. Introduction

The concept of Lie triple systems (LTSs) was introduced by Nathan Jacobson in 1949 to study subspaces of associative algebras closed under triple commutators $[[u, v], w]$. The role played by LTSs in the theory of symmetric spaces is parallel to that of Lie algebras in the theory of Lie groups: the tangent space at every point of a symmetric space has the structure of a Lie triple system (LTS). Because of close relation to Lie algebras and theoretical physics, LTSs are widely studied recently [1–3]. The notion of δ-Jordan Lie triple systems (δ-JLTSs) was introduced by Susumu Okubo in 1997 (see [4]). The case of $\delta = 1$ implies δ-JLTSs are LTSs and the other case of $\delta = -1$ gives Jordan Lie triple systems. So a question arises whether some known results on LTSs can be extended to the framework of δ-JLTSs. δ-JLTSs are the natural generalization of LTSs and have important applications. Recently, deformations, nijenhuis operators, abelian extensions and T^*-extensions of δ-JLTSs are studied [5].

In the present paper, we are interested in studying the structures of arbitrary δ-JLTSs by focussing on the split ones. The class of the split ones is specially related to addition quantum numbers, graded contractions, and deformations. Recently, in [6–12], the structures of arbitrary split Lie algebras, arbitrary split Leibniz algebras, arbitrary split LTSs, arbitrary split Leibniz

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triple systems and arbitrary graded Leibniz triple systems have been determined by the techniques of connections of roots. Our work is essentially motivated by the work on split LTSs [6].

Throughout this paper, δ-JLTSs T are considered of arbitrary dimension and over an arbitrary base field $\mathbb{K}$. It is worth to mention that, unless otherwise stated, there is not any restriction on $\dim T_\alpha$ or $\{k \in \mathbb{K}\colon k\alpha \in \Lambda^1$, for a fixed $\alpha \in \Lambda^1\}$, where T_α denotes the root space associated to the root α, and Λ^1 the set of nonzero roots of T. This paper proceeds as follows. In Section 2, we establish the preliminaries on split δ-JLTSs theory. In Section 3, we show that such an arbitrary δ-JLTSs with a symmetric root system is of the form $T = U + \sum_{[\alpha]\in\Lambda^1/\sim} I_{[\alpha]}$ with U a subspace of T_0 and any $I_{[\alpha]}$ a well described ideal of T, satisfying $\{I_{[\alpha]}, T, I_{[\beta]}\} = \{I_{[\alpha]}, I_{[\beta]}, T\} = \{T, I_{[\alpha]}, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$.

2. Preliminaries

First we recall the definitions of δ-Jordan Lie algebra and δ-Jordan Lie triple system.

Definition 2.1 ([4]) *A δ-Jordan Lie algebra L is a vector space over a field $\mathbb{K}$ endowed with a bilinear map $[\cdot,\cdot] : L \times L \to L$ satisfying*

(1) $[x, y] = -\delta[y, x]$, $\delta = \pm 1$,

(2) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$, $\forall x, y, z \in L$.

Remark 2.2 ([4]) A δ-Jordan Lie algebra L is called a Lie algebra if $\delta = 1$, and a δ-Jordan Lie algebra L is called a Jordan Lie algebra if $\delta = -1$.

Definition 2.3 ([4]) *A δ-JLTS is a vector space T endowed with a trilinear operation $\{\cdot,\cdot,\cdot\} : T \times T \times T \to T$ satisfying*

(1) $\{x, y, z\} = -\delta\{y, x, z\}$, $\delta = \pm 1$,

(2) $\{x, y, z\} + \{y, z, x\} + \{z, x, y\} = 0$,

(3) $\{x, y, \{a, b, c\}\} = \{\{x, y, a\}, b, c\} + \{a, \{x, y, b\}, c\} + \delta\{a, b, \{x, y, c\}\}$,

for $x, y, z, a, b, c \in T$.

When $\delta = 1$, a δ-JLTS is a LTS. So LTSs are special examples of δ-JLTSs.

Example 2.4 If L is a δ-Jordan Lie algebra with product $[\cdot,\cdot]$, then L becomes a δ-JLTS by putting $\{x, y, z\} = [[x, y], z]$.

Definition 2.5 *Let I be a subspace of a δ-JLTS T. Then I is called a subsystem of T, if $\{I, I, I\} \subseteq I$; I is called an ideal of T, if $\{I, T, T\} \subseteq I$.*

Definition 2.6 ([4]) *The standard embedding of a δ-JLTS T is the $\mathbb{Z}_2$-graded δ-Jordan Lie algebra $L = L^0 \oplus L^1$, L^0 being the $\mathbb{K}$-span of $\{L(x, y).x, y \in T\}$, where $L(x, y)$ denotes the left multiplication operator in T, $L(x, y)(z) := \{x, y, z\}$; $L^1 := T$ and where the product is given by*

$$[(L(x, y), z), (L(u, v), w)] := (L(\{u, v, y\}, x) - L(\{u, v, x\}, y) + L(z, w), \{x, y, w\} - \delta\{u, v, z\}).$$

Let us observe that L^0 with the product induced by the one in $L = L^0 \oplus L^1$ becomes a δ-Jordan

Lie algebra.

Definition 2.7 *Let T be a δ-JLTS, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a maximal abelian subalgebra (MASA) of L^0. For a linear functional $\alpha \in (H^0)^*$, we define the root space of T (with respect to H^0) associated to α as the subspace $T_\alpha := \{t_\alpha \in T : [h, t_\alpha] = \alpha(h)t_\alpha$ for any $h \in H^0\}$. The elements $\alpha \in (H^0)^*$ satisfying $T_\alpha \neq 0$ are called roots of T with respect to H^0 and we denote $\Lambda^1 := \{\alpha \in (H^0)^* \setminus \{0\} : T_\alpha \neq 0\}$.*

Let us observe that $T_0 = \{t_0 \in T : [h, t_0] = 0$ for any h $\in H^0\}$. In the following, we shall denote by Λ^0 the set of all nonzero $\alpha \in (H^0)^*$ such that $L^0_\alpha := \{v^0_\alpha \in L^0 : [h, v^0_\alpha] = \alpha(h)v^0_\alpha$ for any h $\in H^0\} \neq 0$.

Lemma 2.8 *Let T be a δ-JLTS, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be an MASA of L^0. For $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}$ and $\xi, q \in \Lambda^0 \cup \{0\}$, the following assertions hold.*

(1) If $[T_\alpha, T_\beta] \neq 0$, then $\delta(\alpha + \beta) \in \Lambda^0 \cup \{0\}$ and $[T_\alpha, T_\beta] \subseteq L^0_{\delta(\alpha+\beta)}$.

(2) If $[L^0_\xi, T_\alpha] \neq 0$, then $\delta(\xi + \alpha) \in \Lambda^1 \cup \{0\}$ and $[L^0_\xi, T_\alpha] \subseteq T_{\delta(\xi+\alpha)}$.

(3) If $[T_\alpha, L^0_\xi] \neq 0$, then $\delta(\alpha + \xi) \in \Lambda^1 \cup \{0\}$ and $[T_\alpha, L^0_\xi] \subseteq T_{\delta(\alpha+\xi)}$.

(4) If $[L^0_\xi, L^0_q] \neq 0$, then $\delta(\xi + q) \in \Lambda^0 \cup \{0\}$ and $[L^0_\xi, L^0_q] \subseteq L^0_{\delta(\xi+q)}$.

(5) If $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, then $\alpha+\beta+\delta\gamma \in \Lambda^1 \cup \{0\}$ and $\{T_\alpha, T_\beta, T_\gamma\} \subseteq T_{\delta^2\alpha+\delta^2\beta+\delta\gamma} = T_{\alpha+\beta+\delta\gamma}$.

Proof (1) For any $x \in T_\alpha$, $y \in T_\beta$ and $h \in H^0$, by Definition 2.1 (2), one has $[h, [x, y]] = \delta[x, [h, y]] + \delta[[h, x], y] = \delta[x, \beta(h)y] + \delta[\alpha(h)x, y] = \delta(\alpha + \beta)(h)[x, y]$.

(2) For any $x \in L^0_\xi$, $y \in T_\alpha$ and $h \in H^0$, by Definition 2.1 (2), one has $[h, [x, y]] = \delta[x, [h, y]] + \delta[[h, x], y] = \delta[x, \alpha(h)y] + \delta[\xi(h)x, y] = \delta(\xi + \alpha)(h)[x, y]$.

(3) For any $x \in T_\alpha$, $y \in L^0_\xi$, and $h \in H^0$, by Definition 2.1 (2), one has $[h, [x, y]] = \delta[x, [h, y]] + \delta[[h, x], y] = \delta[x, \xi(h)y] + \delta[\alpha(h)x, y] = \delta(\alpha + \xi)(h)[x, y]$.

(4) For any $x \in L^0_\xi$, $y \in L^0_q$ and $h \in H^0$, by Definition 2.1 (2), one has $[h, [x, y]] = \delta[x, [h, y]] + \delta[[h, x], y] = \delta[x, q(h)y] + \delta[\xi(h)x, y] = \delta(\xi + q)(h)[x, y]$.

(5) It is a consequence of Lemma 2.8 (1) and (2). □

Definition 2.9 *Let T be a δ-JLTS, $L = L^0 \oplus L^1$ be its standard embedding, and H^0 be a MASA of L^0. We shall call that T is a split δ-JLTS (with respect to H^0) if $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$. We say that Λ^1 is the root system of T.*

We also note that the facts $H^0 \subset L^0 = [T, T]$ and $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$ imply

$$H^0 = [T_0, T_0] + \sum_{\alpha\in\Lambda^1} [T_\alpha, T_{-\alpha}]. \tag{2.1}$$

Finally, as $[T_0, T_0] \subset L^0_0 = H^0$, we have

$$\{T_0, T_0, T_0\} = 0. \tag{2.2}$$

Similarly, we also have

$$\{T_\alpha, T_{-\alpha}, T_0\} = 0. \tag{2.3}$$

Definition 2.10 *A root system Λ^1 of a split δ-JLTS T is called symmetric if it satisfies that $\alpha \in \Lambda^1$ implies $-\alpha \in \Lambda^1$.*

A similar concept applies to the set Λ^0 of nonzero roots of L^0.

In the following, T denotes a split δ-JLTS with a symmetric root system Λ^1, and $T = T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)$ the corresponding root decomposition. We begin the study of split δ-JLTS by developing the concept of connections of roots.

Definition 2.11 *Let α and β be two nonzero roots. We shall say that α and β are connected if there exists a family $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\} \subset \Lambda^1 \cup \{0\}$ of roots of T such that*

(1) $\{\alpha_1, \delta^2\alpha_1+\delta^2\alpha_2+\delta\alpha_3, \delta^4\alpha_1+\delta^4\alpha_2+\delta^3\alpha_3+\delta^2\alpha_4+\delta\alpha_5, \dots, \delta^{2n}\alpha_1+\cdots+\delta^2\alpha_{2n}+\delta\alpha_{2n+1}\} \subset \Lambda^1$;

(2) $\{\delta\alpha_1+\delta\alpha_2, \delta^3\alpha_1+\delta^3\alpha_2+\delta^2\alpha_3+\delta\alpha_4, \dots, \delta^{2n-1}\alpha_1+\cdots+\delta\alpha_{2n}\} \subset \Lambda^0$;

(3) $\alpha_1 = \alpha$ and $\delta^{2n}\alpha_1+\cdots+\delta^2\alpha_{2n}+\delta\alpha_{2n+1} \in \pm\beta$.

We shall also say that $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β.

Let $\Lambda^1_\alpha := \{\beta \in \Lambda^1 : \alpha \text{ and } \beta \text{ are connected}\}$. We can easily get that $\{\alpha\}$ is a connection from α to itself and to $-\alpha$. Therefore, $\pm\alpha \in \Lambda^1_\alpha$.

Definition 2.12 *A subset Ω^1 of a root system Λ^1, associated to a split δ-JLTS T, is called a root subsystem if it is symmetric, and for $\alpha, \beta, \gamma \in \Omega^1 \cup \{0\}$ such that $\delta(\alpha+\beta) \in \Lambda^0$ and $\alpha+\beta+\delta\gamma \in \Lambda^1$ then $\alpha+\beta+\delta\gamma \in \Omega^1$.*

Let Ω^1 be a root subsystem of Λ^1. We define

$$T_{0,\Omega^1} := \mathrm{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha+\beta+\delta\gamma = 0;\ \alpha, \beta, \gamma \in \Omega^1 \cup \{0\}\} \subset T_0$$

and $V_{\Omega^1} := \oplus_{\alpha\in\Omega^1} T_\alpha$. Taking into account the fact that $\{T_0, T_0, T_0\} = 0$, it is straightforward to verify that $T_{\Omega^1} := T_{0,\Omega^1} \oplus V_{\Omega^1}$ is a subsystem of T. We will say that T_{Ω^1} is a subsystem associated to the root subsystem Ω^1.

Proposition 2.13 *If Λ^0 is symmetric, then the relation $\sim$ in Λ^1, defined by $\alpha \sim \beta$ if and only if $\beta \in \Lambda^1_\alpha$, is of equivalence.*

Proof $\{\alpha\}$ is a connection from α to itself and therefore $\alpha \sim \alpha$.

If $\alpha \sim \beta$ and $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β, then

$$\{\delta^{2n}\alpha_1+\cdots+\delta\alpha_{2n+1}, -\delta\alpha_{2n+1}, -\delta\alpha_{2n}, \dots, -\delta\alpha_2\}$$

is a connection from β to α in case $\delta^{2n}\alpha_1+\cdots+\delta^2\alpha_{2n}+\delta\alpha_{2n+1} = \beta$, and

$$\{-\delta^{2n}\alpha_1-\cdots-\delta\alpha_{2n+1}, \delta\alpha_{2n+1}, \delta\alpha_{2n}, \dots, \delta\alpha_2\}$$

in case $\delta^{2n}\alpha_1+\cdots+\delta^2\alpha_{2n}+\delta\alpha_{2n+1} = -\beta$. Therefore $\beta \sim \alpha$.

Finally, suppose $\alpha \sim \beta$ and $\beta \sim \gamma$, $\{\alpha_1, \alpha_2, \dots, \alpha_{2n}, \alpha_{2n+1}\}$ is a connection from α to β and $\{\beta_1, \dots, \beta_{2m+1}\}$ is a connection from β to γ. If $m \neq 0$, then

$$\{\alpha_1, \dots, \alpha_{2n+1}, \beta_2, \dots, \beta_{2m+1}\}$$

is a connection from α to γ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = \beta$, and

$$\{\alpha_1, \ldots, \alpha_{2n+1}, -\beta_2, \ldots, -\beta_{2m+1}\}$$

in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = -\beta$. If $m = 0$, then $\gamma \in \pm\beta$ and so

$$\{\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \alpha_{2n+1}\}$$

is a connection from α to γ. Therefore, $\alpha \sim \gamma$ and $\sim$ is of equivalence. □

Proposition 2.14 *Let α be a nonzero root and suppose Λ^0 is symmetric. Then Λ^1_α is a root subsystem.*

Proof If $\beta \in \Lambda^1_\alpha$, then there exists a connection $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \alpha_{2n+1}\}$ from α to β. It is clear that $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \alpha_{2n+1}\}$ also connects α to $-\beta$ and therefore $-\beta \in \Lambda^1_\alpha$. Let $\beta_1, \beta_2, \beta_3 \in \Lambda^1_\alpha \cup \{0\}$ be such that $\delta(\beta_1 + \beta_2) \in \Lambda^0$ and $\beta_1 + \beta_2 + \delta\beta_3 \in \Lambda^1$. If $\beta_1 = 0$, as $\delta(\beta_1 + \beta_2) \in \Lambda^0$ then $\beta_2 \neq 0$ and there exists a connection $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \alpha_{2n+1}\}$ from α to β_2. We have $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, 0, \beta_3\}$ is a connection from α to $\beta_2 + \delta\beta_3$ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = \beta_2$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, 0, -\beta_3\}$ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = -\beta_2$. So $\beta_1 + \beta_2 + \delta\beta_3 = \beta_2 + \delta\beta_3 \in \Lambda^1_\alpha$. Suppose $\beta_1 \neq 0$, then there exists a connection $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n}, \alpha_{2n+1}\}$ from α to β_1. Hence, $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, \beta_2, \beta_3\}$ is a connection from α to $\beta_1 + \beta_2 + \delta\beta_3$ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = \beta_1$ and $\{\alpha_1, \alpha_2, \ldots, \alpha_{2n+1}, -\beta_2, -\beta_3\}$ in case $\delta^{2n}\alpha_1 + \cdots + \delta^2\alpha_{2n} + \delta\alpha_{2n+1} = -\beta_1$. Therefore, $\beta_1 + \beta_2 + \delta\beta_3 \in \Lambda^1_\alpha$. □

3. Decompositions

In this section, we will show that for a fixed $\alpha_0 \in \Lambda^1$, the subsystem $T_{\Lambda^1_{\alpha_0}}$ associated to the root subsystem $\Lambda^1_{\alpha_0}$ is an ideal of T.

Lemma 3.1 *The following assertions hold.*

(1) If $\alpha, \beta \in \Lambda^1$ with $[T_\alpha, T_\beta] \neq 0$, then α is connected with β.

(2) If $\alpha, \beta \in \Lambda^1$, $\alpha \in \Lambda^0$ and $[L^0_\alpha, T_\beta] \neq 0$, then α is connected with β.

(3) If $\alpha, \beta \in \Lambda^1$, $\alpha \in \Lambda^0$ and $[T_\beta, L^0_\alpha] \neq 0$, then α is connected with β.

(4) If $\alpha, \beta \in \Lambda^1$, $\alpha, \beta \in \Lambda^0$ and $[L^0_\alpha, L^0_\beta] \neq 0$, then α is connected with β.

(5) If $\alpha, \overline{\beta} \in \Lambda^1$ such that α is not connected with $\overline{\beta}$, then $[T_\alpha, T_{\overline{\beta}}] = 0$, $[L^0_\alpha, T_{\overline{\beta}}] = 0$ and $[T_{\overline{\beta}}, L^0_\alpha] = 0$ if furthermore $\alpha \in \Lambda^0$. If $\alpha, \overline{\beta} \in \Lambda^1$ such that α is not connected with $\overline{\beta}$, then $[L^0_\alpha, L^0_{\overline{\beta}}] = 0$ if furthermore $\alpha, \overline{\beta} \in \Lambda^0$.

Proof (1) Suppose $[T_\alpha, T_\beta] \neq 0$, by Lemma 2.8 (1), one gets $\delta(\alpha + \beta) \in \Lambda^0 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^0$, one gets $\{\alpha, \beta, -\delta\alpha\}$ is a connection from α to β.

(2) Suppose $[L^0_\alpha, T_\beta] \neq 0$, by Lemma 2.8 (2), one gets $\delta(\alpha + \beta) \in \Lambda^1 \cup \{0\}$. If $\alpha + \beta = 0$, then $\beta = -\alpha$ and so α is connected with β. Suppose $\alpha + \beta \neq 0$. Since $\alpha + \beta \in \Lambda^1$, we obtain $\{\alpha, 0, -\delta\alpha - \delta\beta\}$ is a connection from α to β.

(3) Suppose $[T_\beta, L^0_\alpha] \neq 0$, by Lemma 2.8 (3), one gets $\delta(\beta + \alpha) \in \Lambda^1 \cup \{0\}$. If $\beta + \alpha = 0$,

then $\beta=-\alpha$ and it is clear that α is connected with β. Suppose $\beta+\alpha\neq 0$. Since $\beta+\alpha\in\Lambda^1$, one gets $\{\beta,-\delta\alpha-\delta\beta,0\}$ is a connection from β to α. By the symmetry, we get α is connected with β.

(4) Suppose $[L^0_\alpha,L^0_\beta]\neq 0$, by Lemma 2.8 (4), one has $\delta(\alpha+\beta)\in\Lambda^0\cup\{0\}$. If $\alpha+\beta=0$, then $\beta=-\alpha$ and so α is connected with β. Suppose $\alpha+\beta\neq 0$. Since $\alpha+\beta\in\Lambda^0$, one gets $\{\alpha,\beta,-\delta\alpha\}$ is a connection from α to β.

(5) It is a consequence of Lemma 3.1 (1), (2), (3) and (4). □

Lemma 3.2 *If $\alpha,\overline{\beta}\in\Lambda^1$ are not connected, then $\{T_\alpha,T_{-\alpha},T_{\overline{\beta}}\}=0$.*

Proof If $[T_\alpha,T_{-\alpha}]=0$, it is clear. One may suppose that $[T_\alpha,T_{-\alpha}]\neq 0$ and $\{T_\alpha,T_{-\alpha},T_{\overline{\beta}}\}\neq 0$. So either $\{T_{-\alpha},T_{\overline{\beta}},T_\alpha\}\neq 0$ or $\{T_{\overline{\beta}},T_\alpha,T_{-\alpha}\}\neq 0$, contradicting Lemma 3.1 (5). Hence, $\{T_\alpha,T_{-\alpha},T_{\overline{\beta}}\}=0$. □

Lemma 3.3 *Fix $\alpha_0\in\Lambda^1$ and suppose Λ^0 is symmetric. For $\alpha\in\Lambda^1_{\alpha_0}$ and $\beta,\gamma\in\Lambda^1\cup\{0\}$, then the following assertions hold.*

(1) If $\{T_\alpha,T_\beta,T_\gamma\}\neq 0$ then β, γ, $\alpha+\beta+\delta\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$.

(2) If $\{T_\gamma,T_\alpha,T_\beta\}\neq 0$ then γ, β, $\gamma+\alpha+\delta\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$.

(3) If $\{T_\beta,T_\gamma,T_\alpha\}\neq 0$ then β, γ, $\beta+\gamma+\delta\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Proof (1) It is easy to see that $[T_\alpha,T_\beta]\neq 0$, for $\alpha\in\Lambda^1_{\alpha_0}$ and $\beta\in\Lambda^1\cup\{0\}$. By Lemma 3.1 (1), one gets $\alpha\sim\beta$ in the case $\beta\neq 0$. From here, $\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$. In order to complete the proof, we will show γ, $\alpha+\beta+\delta\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$. We distinguish two cases.

Case 1. Suppose $\alpha+\beta+\delta\gamma=0$. It is clear that $\alpha+\beta+\delta\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$. The fact that $\{T_0,T_0,T_0\}=0$ and $\{T_\alpha,T_{-\alpha},T_0\}=0$ for $\alpha\in\Lambda^1$ gives us $\gamma\neq 0$. By Lemma 2.8 (1), one gets $\delta(\alpha+\beta)\in\Lambda^0$. As $\alpha+\beta=-\delta\gamma$, $\{\alpha,\beta,0\}$ would be a connection from α to γ and we conclude $\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Case 2. Suppose $\alpha+\beta+\delta\gamma\neq 0$. We treat separately two cases.

Suppose $\alpha+\beta\neq 0$. By Lemma 2.8 (1), one gets $\delta(\alpha+\beta)\in\Lambda^0$ and so $\{\alpha,\beta,\gamma\}$ is a connection from α to $\alpha+\beta+\delta\gamma$. Hence $\alpha+\beta+\delta\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$. In the case $\gamma\neq 0$, $\{\alpha,\beta,-\delta\alpha-\delta\beta-\gamma\}$ is a connection from α to γ. So $\gamma\in\Lambda^1_{\alpha_0}$. Hence $\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Suppose $\alpha+\beta=0$. Then necessarily $\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$. Indeed, if $\gamma\neq 0$ and α is not connected with γ, by Lemma 3.2, $\{T_\alpha,T_\beta,T_\gamma\}=\{T_\alpha,T_{-\alpha},T_\gamma\}=0$, a contradiction. We also have $\alpha+\beta+\delta\gamma=\delta\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$.

(2) The fact that $[T_\gamma,T_\alpha]\neq 0$ implies by Lemma 3.1 (1) that $\alpha\sim\gamma$ in the case $\gamma\neq 0$. From here, $\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$. In order to complete the proof, we will show β, $\gamma+\alpha+\delta\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$. We distinguish two cases.

Case 1. Suppose $\gamma+\alpha+\delta\beta=0$. It is clear that $\gamma+\alpha+\delta\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$. The fact that $\{T_0,T_0,T_0\}=0$ and $\{T_\alpha,T_{-\alpha},T_0\}=0$ for $\alpha\in\Lambda^1$ gives us $\beta\neq 0$. By Lemma 2.8 (1), one has $\gamma+\alpha\in\Lambda^0$. As $\gamma+\alpha=-\delta\beta$, $\{\alpha,\gamma,0\}$ would be a connection from α to β and we conclude $\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Case 2. Suppose $\gamma+\alpha+\delta\beta\neq 0$. We treat separately two cases.

Suppose $\gamma+\alpha\neq 0$. By Lemma 2.8 (1), one gets $\gamma+\alpha\in\Lambda^0$ and so $\{\alpha,\gamma,\beta\}$ is a connection from α to $\gamma+\alpha+\delta\beta$. Hence $\gamma+\alpha+\delta\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$. In the case $\beta\neq 0$, we have $\{\alpha,\gamma,-\delta\alpha-\delta\gamma-\beta\}$ is a connection from α to $\delta\beta$. So $\beta\in\Lambda^1_{\alpha_0}$. Hence $\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Suppose $\gamma+\alpha=0$. Then necessarily $\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$. Indeed, if $\beta\neq 0$ and α is not connected with β, by Lemma 3.2, $\{T_\gamma,T_\alpha,T_\beta\}=\{T_{-\alpha},T_\alpha,T_\beta\}=0$, a contradiction. We also have $\gamma+\alpha+\delta\beta=\delta\beta\in\Lambda^1_{\alpha_0}\cup\{0\}$.

(3) By the definition of δ-JLTS, one has $\{T_\beta,T_\gamma,T_\alpha\}\subset\{T_\alpha,T_\beta,T_\gamma\}+\{T_\gamma,T_\alpha,T_\beta\}$. So either $\{T_\alpha,T_\beta,T_\gamma\}\neq 0$ or $\{T_\gamma,T_\alpha,T_\beta\}\neq 0$. By Lemma 3.3 (1) and (2), one gets $\beta,\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$. Next we will show that $\beta+\gamma+\delta\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$. We treat separately three cases.

Case 1. Suppose $\beta\neq 0$. Then $\beta\in\Lambda^1_{\alpha_0}$. By Lemma 3.3 (1), one has $\beta+\gamma+\delta\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Case 2. Suppose $\beta=0$ and $\gamma\neq 0$. Then $\gamma\in\Lambda^1_{\alpha_0}$. By Lemma 3.3 (2), one has $\beta+\gamma+\delta\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Case 3. Suppose $\beta=0$ and $\gamma=0$. Then $\beta+\gamma+\delta\alpha=\delta\alpha\in\Lambda^1_{\alpha_0}$. We also have $\beta+\gamma+\delta\alpha\in\Lambda^1_{\alpha_0}\cup\{0\}$. □

Lemma 3.4 *Fix $\alpha_0\in\Lambda^1$ and suppose Λ^0 is symmetric. For $\alpha,\beta,\gamma\in\Lambda^1_{\alpha_0}\cup\{0\}$ with $\alpha+\beta+\delta\gamma=0$ and $\tau,\epsilon\in\Lambda^1\cup\{0\}$, the following assertions hold.*

(1) If $\{\{T_\alpha,T_\beta,T_\gamma\},T_\tau,T_\epsilon\}\neq 0$, then τ, ϵ, $\tau+\delta\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$.

(2) If $\{T_\epsilon,\{T_\alpha,T_\beta,T_\gamma\},T_\tau\}\neq 0$, then τ, ϵ, $\epsilon+\delta\tau\in\Lambda^1_{\alpha_0}\cup\{0\}$.

(3) If $\{T_\tau,T_\epsilon,\{T_\alpha,T_\beta,T_\gamma\}\}\neq 0$, then τ, ϵ, $\tau+\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Proof (1) From the fact that $\alpha+\beta+\delta\gamma=0$, $\{T_0,T_0,T_0\}=0$ and $\{T_\alpha,T_{-\alpha},T_0\}=0$ whenever $\alpha\in\Lambda^1$, one may suppose that at least two distinct elements in $\{\alpha,\beta,\gamma\}$ are nonzero and one may consider the case $\{T_\alpha,T_\beta,T_\gamma\}\neq 0$, $\alpha+\beta\neq 0$ and $\gamma\neq 0$. Since

$$\begin{aligned}0\neq&\{\{T_\alpha,T_\beta,T_\gamma\},T_\tau,T_\epsilon\}\subset\{T_\alpha,T_\beta,\{T_\gamma,T_\tau,T_\epsilon\}\}-\\&\{T_\gamma,\{T_\alpha,T_\beta,T_\tau\},T_\epsilon\}-\delta\{T_\gamma,T_\tau,\{T_\alpha,T_\beta,T_\epsilon\}\},\end{aligned}$$

any of the above three summands is nonzero. In order to complete the proof, we firstly will show τ, $\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$. We distinguish three cases.

Case 1. Suppose $\{T_\alpha,T_\beta,\{T_\gamma,T_\tau,T_\epsilon\}\}\neq 0$. As $\gamma\neq 0$ and $\{T_\gamma,T_\tau,T_\epsilon\}\neq 0$, Lemma 3.3 (1) shows that τ,ϵ are connected with γ in the case of being nonzero roots and so $\tau,\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Case 2. Suppose $\{T_\gamma,\{T_\alpha,T_\beta,T_\tau\},T_\epsilon\}\neq 0$. As $\alpha+\beta\neq 0$ and $\gamma\neq 0$. So either $\alpha\neq 0$ or $\beta\neq 0$. By Lemma 3.3 (1) and (2), one has $\tau,\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Case 3. Suppose $\{T_\gamma,T_\tau,\{T_\alpha,T_\beta,T_\epsilon\}\}\neq 0$. As $\alpha+\beta\neq 0$ and $\gamma\neq 0$. So either $\alpha\neq 0$ or $\beta\neq 0$. By Lemma 3.3 (1) and (2), one has $\tau,\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$.

Finally, we will show $\tau+\delta\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$. From the fact that $\alpha+\beta+\delta\gamma=0$, $\{T_0,T_0,T_0\}=0$ and $\{\{T_\alpha,T_\beta,T_\gamma\},T_\tau,T_\epsilon\}\neq 0$, let us suppose that at least one element in $\{\tau,\epsilon\}$ is nonzero. So either $\tau\in\Lambda^1_{\alpha_0}$ or $\epsilon\in\Lambda^1_{\alpha_0}$. Then $\{\{T_\alpha,T_\beta,T_\gamma\},T_\tau,T_\epsilon\}\subset\{T_0,T_\tau,T_\epsilon\}$. By Lemma 3.3 (2) and (3), one has $\tau+\delta\epsilon\in\Lambda^1_{\alpha_0}\cup\{0\}$.

(2) From the fact that $\alpha+\beta+\delta\gamma=0$, $\{T_0,T_0,T_0\}=0$ and $\{T_\alpha,T_{-\alpha},T_0\}=0$ whenever $\alpha\in\Lambda^1$, one may suppose that at least two distinct elements in $\{\alpha,\beta,\gamma\}$ are nonzero and one

may consider the case $\{T_\alpha, T_\beta, T_\gamma\} \neq 0$, $\alpha+\beta \neq 0$ and $\gamma \neq 0$. Since

$$0 \neq \{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\} \subset \{T_\alpha, T_\beta, \{T_\epsilon, T_\gamma, T_\tau\}\} - \delta\{T_\epsilon, T_\gamma, \{T_\alpha, T_\beta, T_\tau\}\} - \{\{T_\alpha, T_\beta, T_\epsilon\}, T_\gamma, T_\tau\},$$

any of the above three summands is nonzero. In order to complete the proof, we firstly will show τ, $\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. We distinguish three cases.

Case 1. Suppose $\{T_\alpha, T_\beta, \{T_\epsilon, T_\gamma, T_\tau\}\} \neq 0$. As $\gamma \neq 0$ and $\{T_\epsilon, T_\gamma, T_\tau\} \neq 0$, Lemma 3.3 (2) shows that ϵ, τ are connected with γ in the case of being nonzero roots and so $\epsilon, \tau \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 2. Suppose $\{T_\epsilon, T_\gamma, \{T_\alpha, T_\beta, T_\tau\}\} \neq 0$. As $\alpha+\beta \neq 0$ and $\gamma \neq 0$. So either $\alpha \neq 0$ or $\beta \neq 0$. By Lemma 3.3 (1) and (2), one has $\epsilon, \tau \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Case 3. Suppose $\{\{T_\alpha, T_\beta, T_\epsilon\}, T_\gamma, T_\tau\} \neq 0$. As $\alpha+\beta \neq 0$ and $\gamma \neq 0$. So either $\alpha \neq 0$ or $\beta \neq 0$. By Lemma 3.3 (1) and (2), one has $\epsilon, \tau \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Finally, we will show $\epsilon + \delta\tau \in \Lambda^1_{\alpha_0} \cup \{0\}$. From the fact that $\alpha+\beta+\delta\gamma = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\} \neq 0$, let us suppose that at least one element in $\{\epsilon, \tau\}$ is nonzero. So either $\epsilon \in \Lambda^1_{\alpha_0}$ or $\tau \in \Lambda^1_{\alpha_0}$. Then $\{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\} \subset \{T_\epsilon, T_0, T_\tau\}$. By Lemma 3.3 (1) and (3), one has $\epsilon + \delta\tau \in \Lambda^1_{\alpha_0} \cup \{0\}$.

(3) By the definition of δ-JLTS, one has

$$0 \neq \{T_\tau, T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}\} \subset \{\{T_\alpha, T_\beta, T_\gamma\}, T_\tau, T_\epsilon\} + \{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\}.$$

Suppose $\{\{T_\alpha, T_\beta, T_\gamma\}, T_\tau, T_\epsilon\} \neq 0$, by Lemma 3.4 (1), one has τ, ϵ, $\tau + \delta\epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$. Suppose $\{T_\epsilon, \{T_\alpha, T_\beta, T_\gamma\}, T_\tau\} \neq 0$, by Lemma 3.4 (2), one has τ, ϵ, $\epsilon + \delta\tau \in \Lambda^1_{\alpha_0} \cup \{0\}$. Therefore, in these two cases, we get τ, ϵ, $\tau + \epsilon \in \Lambda^1_{\alpha_0} \cup \{0\}$.

Lemma 3.5 *Fix $\alpha_0 \in \Lambda^1$ and suppose Λ^0 is symmetric. If $\alpha_1, \alpha_2, \alpha_3 \in \Lambda^1_{\alpha_0} \cup \{0\}$ with $\alpha_1 + \alpha_2 + \delta\alpha_3 = 0$ and $\overline{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, then the following assertions hold.*

(1) $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\overline{\epsilon}}] = 0$.

(2) In case $\overline{\epsilon} \in \Lambda^0$, then $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\overline{\epsilon}}] = 0$.

(3) $[[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_0], T_{\overline{\epsilon}}] = 0$.

Proof (1) From the fact $\alpha_1 + \alpha_2 + \delta\alpha_3 = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$, one gets if $\alpha_3 = 0$ then it is clear that $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\overline{\epsilon}}] = 0$. Let us consider the case $\alpha_3 \neq 0$. By the definition of δ-Jordan Lie algebra, we have

$$[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\overline{\epsilon}}] \subset \delta[[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, T_{\overline{\epsilon}}]] + [T_{\alpha_3}, [[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}]]. \tag{3.4}$$

Let us consider the first summand in (3.4). As $\alpha_3 \neq 0$, one has $\alpha_3 \in \Lambda^1_{\alpha_0}$. For $\overline{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$ and Lemma 3.1 (5), one easily gets $[T_{\alpha_3}, T_{\overline{\epsilon}}] = 0$. Therefore, $[[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, T_{\overline{\epsilon}}]] = 0$.

Let us now consider the second summand in (3.4), it suffices to verify that

$$[T_{\alpha_3}, [[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}]] = 0.$$

To do so, we first assert that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\overline{\epsilon}}] = 0$. Indeed, by the definition of δ-Jordan Lie algebra, we have

$$[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] \subset \delta[T_{\alpha_1}, [T_{\alpha_2}, T_{\bar{\epsilon}}]] - [T_{\alpha_2}, [T_{\alpha_1}, T_{\bar{\epsilon}}]], \tag{3.5}$$

where $\alpha_1, \alpha_2 \in \Lambda^1_{\alpha_0} \cup \{0\}$, $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$. In the following, we distinguish three cases.

Case 1. $\alpha_1 \neq 0$ and $\alpha_2 \neq 0$. As $\alpha_1 \in \Lambda^1_{\alpha_0}$ and $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1 (1), one gets $[T_{\alpha_1}, T_{\bar{\epsilon}}] = 0$. As $\alpha_2 \in \Lambda^1_{\alpha_0}$ and $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1 (1), one gets $[T_{\alpha_2}, T_{\bar{\epsilon}}] = 0$. Therefore by (3.5), one can show that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] = 0$.

Case 2. $\alpha_1 \neq 0$ and $\alpha_2 = 0$. As $\alpha_1 \in \Lambda^1_{\alpha_0}$ and $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1 (1), one gets $[T_{\alpha_1}, T_{\bar{\epsilon}}] = 0$. That is $[T_{\alpha_2}, [T_{\alpha_1}, T_{\bar{\epsilon}}]] = 0$. As $\alpha_2 = 0$, $[T_{\alpha_2}, T_{\bar{\epsilon}}] = [T_0, T_{\bar{\epsilon}}] \subset L^0_{\delta\bar{\epsilon}}$. By Lemma 3.1 (5), one gets $[T_{\alpha_1}, [T_{\alpha_2}, T_{\bar{\epsilon}}]] = 0$. Therefore, by (3.5), one can show that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] = 0$.

Case 3. $\alpha_1 = 0$ and $\alpha_2 \neq 0$. As $\alpha_2 \in \Lambda^1_{\alpha_0}$ and $\bar{\epsilon} \in \Lambda^1 \setminus \Lambda^1_{\alpha_0}$, by Lemma 3.1 (1), one gets $[T_{\alpha_2}, T_{\bar{\epsilon}}] = 0$. That is $[T_{\alpha_1}, [T_{\alpha_2}, T_{\bar{\epsilon}}]] = 0$. As $\alpha_1 = 0$, $[T_{\alpha_1}, T_{\bar{\epsilon}}] = [T_0, T_{\bar{\epsilon}}] \subset L^0_{\delta\bar{\epsilon}}$. By Lemma 3.1 (5), we get $[T_{\alpha_2}, [T_{\alpha_1}, T_{\bar{\epsilon}}]] = 0$. Therefore, by (3.5), one can show that $[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] = 0$.

So $[T_{\alpha_3}, [[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}]]$=0 is a consequence of $[[T_{\alpha_1}, T_{\alpha_2}], T_{\bar{\epsilon}}] = 0$. By (3.4), one gets

$$[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\bar{\epsilon}}] = 0.$$

The proof is completed. □

(2) From the fact $\alpha_1 + \alpha_2 + \delta\alpha_3 = 0$, $\{T_0, T_0, T_0\} = 0$ and $\{T_\alpha, T_{-\alpha}, T_0\} = 0$ for $\alpha \in \Lambda^1$, one gets if $\alpha_3 = 0$ then it is clear that $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\bar{\epsilon}}] = 0$. Let us consider the case $\alpha_3 \neq 0$. Note that

$$[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\bar{\epsilon}}] \subset \delta[[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, L^0_{\bar{\epsilon}}]] - [T_{\alpha_3}, [[T_{\alpha_1}, T_{\alpha_2}], L^0_{\bar{\epsilon}}]]. \tag{3.6}$$

Let us consider the first summand in (3.6). As $\alpha_3 \neq 0$, one gets $[[T_{\alpha_1}, T_{\alpha_2}], [T_{\alpha_3}, L^0_{\bar{\epsilon}}]] = 0$ by Lemma 3.1 (5). Let us now consider the second summand in (3.6). As either $\alpha_1 \neq 0$ or $\alpha_2 \neq 0$, by the definition of δ-Jordan Lie algebra, the fact $[T_0, L^0_{\bar{\epsilon}}] \subset T_{\delta\bar{\epsilon}}$ and Lemma 3.1 (5), we obtain that $[T_{\alpha_3}, [[T_{\alpha_1}, T_{\alpha_2}], L^0_{\bar{\epsilon}}]] = 0$. So, the second summand in (3.6) is also zero and then $[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, L^0_{\bar{\epsilon}}] = 0$.

(3) It is a consequence of Lemma 3.5 (1), (2) and

$$[[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_0], T_{\bar{\epsilon}}] \subset \delta[\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, [T_0, T_{\bar{\epsilon}}]] - [T_0, [\{T_{\alpha_1}, T_{\alpha_2}, T_{\alpha_3}\}, T_{\bar{\epsilon}}]].$$

Definition 3.6 *A δ-JLTS T is said to be simple, if $\{T, T, T\} \neq 0$ and its only ideals are $\{0\}$ and T.*

Theorem 3.7 *Suppose Λ^0 is symmetric, the following assertions hold.*

(1) For any $\alpha_0 \in \Lambda^1$, the subsystem

$$T_{\Lambda^1_{\alpha_0}} = T_{0,\Lambda^1_{\alpha_0}} \oplus V_{\Lambda^1_{\alpha_0}}$$

of T associated to the root subsystem $\Lambda^1_{\alpha_0}$ is an ideal of T.

(2) If T is simple, then there exists a connection from α to β for any α, $\beta \in \Lambda^1$.

Proof (1) Recall that

$$T_{0,\Lambda^1_{\alpha_0}} := \mathrm{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha + \beta + \delta\gamma = 0;\ \alpha, \beta, \gamma \in \Lambda^1_{\alpha_0} \cup \{0\}\} \subset T_0$$

and $V_{\Lambda^1_{\alpha_0}} := \oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma$. In order to complete the proof, it suffices to show that

$$\{T_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}.$$

We first check that $\{T_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. It is easy to see that

$$\{T_{\Lambda^1_{\alpha_0}}, T, T\} = \{T_{0,\Lambda^1_{\alpha_0}} \oplus V_{\Lambda^1_{\alpha_0}}, T, T\} = \{T_{0,\Lambda^1_{\alpha_0}}, T, T\} + \{V_{\Lambda^1_{\alpha_0}}, T, T\}.$$

Next, we will show that $\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. Note that

$$\begin{aligned}\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} =& \{T_{0,\Lambda^1_{\alpha_0}}, T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha), T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)\}\\ =& \{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_0\} + \{T_{0,\Lambda^1_{\alpha_0}}, T_0, \oplus_{\alpha\in\Lambda^1} T_\alpha\} +\\ & \{T_{0,\Lambda^1_{\alpha_0}}, \oplus_{\alpha\in\Lambda^1} T_\alpha, T_0\} + \{T_{0,\Lambda^1_{\alpha_0}}, \oplus_{\alpha\in\Lambda^1} T_\alpha, \oplus_{\beta\in\Lambda^1} T_\beta\}.\end{aligned}$$

Here, it is clear that $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_0\} \subset \{T_0, T_0, T_0\} = 0$. Taking into account $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_\alpha\}$, for $\alpha \in \Lambda^1$, Lemma 3.4 (1) and the fact that either $\alpha \in \Lambda^1_{\alpha_0}$ or $\alpha \notin \Lambda^1_{\alpha_0}$, give us that $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_\alpha\} \subset V_{\Lambda^1_{\alpha_0}}$ or $\{T_{0,\Lambda^1_{\alpha_0}}, T_0, T_\alpha\} = 0$. Similarly, one gets that $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_0\} \subset V_{\Lambda^1_{\alpha_0}}$ or $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_0\} = 0$. Next, we will consider $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\}$, where $\alpha, \beta \in \Lambda^1$. We treat five cases.

Case 1. If $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\alpha + \delta\beta = 0$, then one has $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} \subset T_{0,\Lambda^1_{\alpha_0}}$.

Case 2. If $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\alpha + \delta\beta \neq 0$, since $\Lambda^1_{\alpha_0}$ is a root subsystem, one gets

$$\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} \subset V_{\Lambda^1_{\alpha_0}}.$$

Case 3. If $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta \notin \Lambda^1_{\alpha_0}$, by Lemma 3.4 (1), one has $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} = 0$.

Case 4. If $\beta \in \Lambda^1_{\alpha_0}$ and $\alpha \notin \Lambda^1_{\alpha_0}$, by Lemma 3.4 (1), one has $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} = 0$.

Case 5. If $\beta \notin \Lambda^1_{\alpha_0}$ and $\alpha \notin \Lambda^1_{\alpha_0}$, by Lemma 3.4 (1), one has $\{T_{0,\Lambda^1_{\alpha_0}}, T_\alpha, T_\beta\} = 0$.

Therefore, $\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$.

Next, we will show that $\{V_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. It is obvious that

$$\begin{aligned}\{V_{\Lambda^1_{\alpha_0}}, T, T\} =& \{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha), T_0 \oplus (\oplus_{\alpha\in\Lambda^1} T_\alpha)\}\\ =& \{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_0, T_0\} + \{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, T_0, \oplus_{\alpha\in\Lambda^1} T_\alpha\} +\\ & \{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha\in\Lambda^1} T_\alpha, T_0\} + \{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha\in\Lambda^1} T_\alpha, \oplus_{\beta\in\Lambda^1} T_\beta\}.\end{aligned}$$

Here, it is clear that $\{T_\gamma, T_0, T_0\} \subset V_{\Lambda^1_{\alpha_0}}$, for $\gamma \in \Lambda^1_{\alpha_0}$. Next, we will consider $\{T_\gamma, T_0, T_\alpha\}$, for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$. We treat three cases.

Case 1. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \notin \Lambda^1_{\alpha_0}$, by Lemma 3.3 (1), one has $\{T_\gamma, T_0, T_\alpha\} = 0$.

Case 2. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$ and $\gamma + \delta\alpha \neq 0$, by $\Lambda^1_{\alpha_0}$ is a root subsystem, one has

$$\{T_\gamma, T_0, T_\alpha\} \subset V_{\Lambda^1_{\alpha_0}}.$$

Case 3. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$ and $\gamma + \delta\alpha = 0$, it is clear that $\{T_\gamma, T_0, T_\alpha\} \subset T_{0,\Lambda^1_{\alpha_0}}$.

Hence, $\{T_\gamma, T_0, T_\alpha\} \subset T_{\Lambda^1_{\alpha_0}}$, for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$. Similarly, it is easy to get $\{T_\gamma, T_\alpha, T_0\} \subset T_{\Lambda^1_{\alpha_0}}$, for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$. At last, we will consider $\{\oplus_{\gamma\in\Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha\in\Lambda^1} T_\alpha, \oplus_{\beta\in\Lambda^1} T_\beta\}$, for $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1$ and $\beta \in \Lambda^1$. We treat five cases.

Case 1. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\gamma + \alpha + \delta\beta = 0$, one gets $\{T_\gamma, T_\alpha, T_\beta\} \subset T_{0,\Lambda^1_{\alpha_0}}$.

Case 2. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$, $\beta \in \Lambda^1_{\alpha_0}$ and $\gamma + \alpha + \delta\beta \neq 0$, one gets

$$\{\oplus_{\gamma \in \Lambda^1_{\alpha_0}} T_\gamma, \oplus_{\alpha \in \Lambda^1} T_\alpha, \oplus_{\beta \in \Lambda^1} T_\beta\} \subset V_{\Lambda^1_{\alpha_0}}.$$

Case 3. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \in \Lambda^1_{\alpha_0}$ and $\beta \notin \Lambda^1_{\alpha_0}$, by Lemma 3.3 (1) and (2), one gets

$$\{T_\gamma, T_\alpha, T_\beta\} = 0.$$

Case 4. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \notin \Lambda^1_{\alpha_0}$ and $\beta \in \Lambda^1_{\alpha_0}$, by Lemma 3.3 (1) and (3), one gets

$$\{T_\gamma, T_\alpha, T_\beta\} = 0.$$

Case 5. If $\gamma \in \Lambda^1_{\alpha_0}$, $\alpha \notin \Lambda^1_{\alpha_0}$ and $\beta \notin \Lambda^1_{\alpha_0}$, by Lemma 3.3 (1), one gets $\{T_\gamma, T_\alpha, T_\beta\} = 0$. So, $\{V_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. Therefore, $\{T_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$ is a consequence of $\{T_{0,\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$ and $\{V_{\Lambda^1_{\alpha_0}}, T, T\} \subset T_{\Lambda^1_{\alpha_0}}$. Consequently, this proves that $T_{\Lambda^1_{\alpha_0}}$ is an ideal of T.

(2) The simplicity of T implies $T_{\Lambda^1_{\alpha_0}} = T$. Hence $\Lambda^1_{\alpha_0} = \Lambda^1$.

Theorem 3.8 *Suppose Λ^0 is symmetric. Then for a vector space complement U of*

$$\mathrm{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha + \beta + \delta\gamma = 0, \textit{where } \alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}\} \textit{ in } T_0,$$

we have

$$T = U + \sum_{[\alpha] \in \Lambda^1/\sim} I_{[\alpha]},$$

where any $I_{[\alpha]}$ is one of the ideals $T_{\Lambda^1_{\alpha_0}}$ of T described in Theorem 3.7. Moreover

$$\{I_{[\alpha]}, T, I_{[\beta]}\} = \{I_{[\alpha]}, I_{[\beta]}, T\} = \{T, I_{[\alpha]}, I_{[\beta]}\} = 0 \textit{ if } [\alpha] \neq [\beta].$$

Proof Let us denote $\xi_0 :=\mathrm{span}_{\mathbb{K}}\{\{T_\alpha, T_\beta, T_\gamma\} : \alpha + \beta + \delta\gamma = 0$, where $\alpha, \beta, \gamma \in \Lambda^1 \cup \{0\}\}$ in T_0. By Proposition 2.13, we can consider the quotient set $\Lambda^1/\sim := \{[\alpha] : \alpha \in \Lambda^1\}$. By denoting $I_{[\alpha]} := T_{\Lambda^1_\alpha}$, $T_{0,[\alpha]} := T_{0,\Lambda^1_\alpha}$ and $V_{[\alpha]} := V_{\Lambda^1_\alpha}$, one gets $I_{[\alpha]} := T_{0,[\alpha]} \oplus V_{[\alpha]}$. From

$$T = T_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha) = (U + \xi_0) \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha),$$

it follows

$$\oplus_{\alpha \in \Lambda^1} T_\alpha = \oplus_{[\alpha] \in \Lambda^1/\sim} V_{[\alpha]}, \quad \xi_0 = \sum_{[\alpha] \in \Lambda^1/\sim} T_{0,[\alpha]},$$

indent which implies

$$T = U + \xi_0 \oplus (\oplus_{\alpha \in \Lambda^1} T_\alpha) = U + \sum_{[\alpha] \in \Lambda^1/\sim} I_{[\alpha]},$$

where each $I_{[\alpha]}$ is an ideal of T by Theorem 3.7.

Next, it is sufficient to show that $\{I_{[\alpha]}, T, I_{[\beta]}\} = 0$ if $[\alpha] \neq [\beta]$. Note that,

$$\begin{aligned}\{I_{[\alpha]}, T, I_{[\beta]}\} =& \{T_{0,[\alpha]} \oplus V_{[\alpha]}, T_0 \oplus (\oplus_{\gamma \in \Lambda^1} T_\gamma), T_{0,[\beta]} \oplus V_{[\beta]}\} \\ =& \{T_{0,[\alpha]}, T_0, T_{0,[\beta]}\} + \{T_{0,[\alpha]}, T_0, V_{[\beta]}\} + \{T_{0,[\alpha]}, \oplus_{\gamma \in \Lambda^1} T_\gamma, T_{0,[\beta]}\} + \\ & \{T_{0,[\alpha]}, \oplus_{\gamma \in \Lambda^1} T_\gamma, V_{[\beta]}\} + \{V_{[\alpha]}, T_0, T_{0,[\beta]}\} + \{V_{[\alpha]}, T_0, V_{[\beta]}\} + \\ & \{V_{[\alpha]}, \oplus_{\gamma \in \Lambda^1} T_\gamma, T_{0,[\beta]}\} + \{V_{[\alpha]}, \oplus_{\gamma \in \Lambda^1} T_\gamma, V_{[\beta]}\}.\end{aligned}$$

Here, it is clear that $\{T_{0,[\alpha]},T_0,T_{0,[\beta]}\}\subset\{T_0,T_0,T_0\}=0$. If $[\alpha]\neq[\beta]$, by Lemmas 3.3 and 3.4, it is easy to see $\{T_{0,[\alpha]},T_0,V_{[\beta]}\}=0$, $\{T_{0,[\alpha]},\oplus_{\gamma\in\Lambda^1}T_\gamma,V_{[\beta]}\}=0$, $\{V_{[\alpha]},T_0,T_{0,[\beta]}\}=0$, $\{V_{[\alpha]},T_0,V_{[\beta]}\}=0$, $\{V_{[\alpha]},\oplus_{\gamma\in\Lambda^1}T_\gamma,T_{0,[\beta]}\}=0$, $\{V_{[\alpha]},\oplus_{\gamma\in\Lambda^1}T_\gamma,V_{[\beta]}\}=0$.

Next, we will show $\{T_{0,[\alpha]},\oplus_{\gamma\in\Lambda^1}T_\gamma,T_{0,[\beta]}\}=0$. Indeed, for $\{T_{\alpha_1},T_{\alpha_2},T_{\alpha_3}\}\in T_{0,[\alpha]}$ with $\alpha_1,\alpha_2,\alpha_3\in\Lambda^1_\alpha\cup\{0\}$, $\alpha_1+\alpha_2+\delta\alpha_3=0$, and for $\{T_{\beta_1},T_{\beta_2},T_{\beta_3}\}\in T_{0,[\beta]}$ with $\beta_1,\beta_2,\beta_3\in\Lambda^1_\beta\cup\{0\}$, $\beta_1+\beta_2+\delta\beta_3=0$, by the definition of δ-JLTS, one gets

$$\begin{aligned}&\{\{T_{\alpha_1},T_{\alpha_2},T_{\alpha_3}\},\oplus_{\gamma\in\Lambda^1}T_\gamma,\{T_{\beta_1},T_{\beta_2},T_{\beta_3}\}\}\\&\quad\subset\delta\{T_{\beta_1},T_{\beta_2},\{\{T_{\alpha_1},T_{\alpha_2},T_{\alpha_3}\},\oplus_{\gamma\in\Lambda^1}T_\gamma,T_{\beta_3}\}\}+\\&\qquad\{\{\{T_{\alpha_1},T_{\alpha_2},T_{\alpha_3}\},\oplus_{\gamma\in\Lambda^1}T_\gamma,T_{\beta_1}\},T_{\beta_2},T_{\beta_3}\}+\\&\qquad\{T_{\beta_1},\{\{T_{\alpha_1},T_{\alpha_2},T_{\alpha_3}\},\oplus_{\gamma\in\Lambda^1}T_\gamma,T_{\beta_2}\},T_{\beta_3}\}.\end{aligned}$$

By Lemma 3.4, it is easy to see that

$$\begin{aligned}&\{T_{\beta_1},T_{\beta_2},\{\{T_{\alpha_1},T_{\alpha_2},T_{\alpha_3}\},\oplus_{\gamma\in\Lambda^1}T_\gamma,T_{\beta_3}\}\}=0,\\&\{\{\{T_{\alpha_1},T_{\alpha_2},T_{\alpha_3}\},\oplus_{\gamma\in\Lambda^1}T_\gamma,T_{\beta_1}\},T_{\beta_2},T_{\beta_3}\}=0,\\&\{T_{\beta_1},\{\{T_{\alpha_1},T_{\alpha_2},T_{\alpha_3}\},\oplus_{\gamma\in\Lambda^1}T_\gamma,T_{\beta_2}\},T_{\beta_3}\}=0,\end{aligned}$$

for $\alpha_1,\alpha_2,\alpha_3\in\Lambda^1_\alpha\cup\{0\}$, $\alpha_1+\alpha_2+\delta\alpha_3=0$, $\beta_1,\beta_2,\beta_3\in\Lambda^1_\beta\cup\{0\}$, $\beta_1+\beta_2+\delta\beta_3=0$, $[\alpha]\neq[\beta]$. So $\{I_{[\alpha]},T,I_{[\beta]}\}=0$ if $[\alpha]\neq[\beta]$.

A similar argument gives us $\{I_{[\alpha]},I_{[\beta]},T\}=\{T,I_{[\alpha]},I_{[\beta]}\}=0$ if $[\alpha]\neq[\beta]$.

Definition 3.9 *The annihilator of a δ-JLTS T is the set* $\mathrm{Ann}(T)=\{x\in T:\{x,T,T\}=0\}$.

Corollary 3.10 *Suppose Λ^0 is symmetric. If $\mathrm{Ann}(T)=0$, and $\{T,T,T\}=T$, then T is the direct sum of the ideals given in Theorem 3.8, $T=\oplus_{[\alpha]\in\Lambda^1/\sim}I_{[\alpha]}$.*

Proof From $\{T,T,T\}=T$ and Theorem 3.8, we have

$$\Big\{U+\sum_{[\alpha]\in\Lambda^1/\sim}I_{[\alpha]},U+\sum_{[\alpha]\in\Lambda^1/\sim}I_{[\alpha]},U+\sum_{[\alpha]\in\Lambda^1/\sim}I_{[\alpha]}\Big\}=U+\sum_{[\alpha]\in\Lambda^1/\sim}I_{[\alpha]}.$$

Taking into account $U\subset T_0$, Lemma 3.3 and the fact that $\{I_{[\alpha]},T,I_{[\beta]}\}=\{I_{[\alpha]},I_{[\beta]},T\}=\{T,I_{[\alpha]},I_{[\beta]}\}=0$ if $[\alpha]\neq[\beta]$ (see Theorem 3.8) give us that $U=0$. That is,

$$T=\sum_{[\alpha]\in\Lambda^1/\sim}I_{[\alpha]}.$$

To finish, it is sufficient to show the direct character of the sum. For $x\in I_{[\alpha]}\cap\sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\not\sim\alpha}}I_{[\beta]}$, using again the equation $\{I_{[\alpha]},T,I_{[\beta]}\}=0$ for $[\alpha]\neq[\beta]$, we obtain

$$\{x,T,I_{[\alpha]}\}=\Big\{x,T,\sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\not\sim\alpha}}I_{[\beta]}\Big\}=0.$$

So $\{x,T,T\}=\{x,T,I_{[\alpha]}+\sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\not\sim\alpha}}I_{[\beta]}\}=\{x,T,I_{[\alpha]}\}+\{x,T,\sum_{\substack{[\beta]\in\Lambda^1/\sim\\ \beta\not\sim\alpha}}I_{[\beta]}\}=0+0=0$. That is, $x\in\mathrm{Ann}(T)=0$. Thus $x=0$, as desired.

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