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Algebraic Properties of Toeplitz Operators on Cutoff Harmonic Bergman Space

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Abstract In this paper, we first investigate the finite-rank product problems of several Toeplitz operators with quasihomogeneous symbols on the cutoff harmonic Bergman space b_n^2. Next, we characterize finite rank commutators and semi-commutators of two Toeplitz operators with quasihomogeneous symbols on b_n^2.

Keywords Toeplitz operator; Cutoff Harmonic Bergman space; quasihomogeneous; finite rank

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1. Introduction

Let D be the open unit disk in the complex plane C and $\mathrm{d}A = \frac{r}{\pi}\mathrm{d}r\mathrm{d}\theta$ be the normalized area measure on D. $L^2(D, \mathrm{d}A)$ is the Hilbert space of Lebesgue square integrable functions on D with the inner product

$$\langle f, g\rangle = \int_D f(z)\overline{g(z)}\mathrm{d}A(z).$$

The Bergman space $L_a^2(D)$ is the closed subspace of all analytic functions in $L^2(D, \mathrm{d}A)$. Harmonic Bergman space $L_h^2(D)$ is the closed subspace of $L^2(D, \mathrm{d}A)$ consisting of the harmonic functions on D. It is clear that

$$L_h^2(D) = L_a^2(D) + \overline{zL_a^2(D)}.$$

For a fixed positive integer n, $\{\overline{z}, \overline{z}^2, \ldots, \overline{z}^n\} \subset \overline{zL_a^2}$, we define

$$W_n = \{\overline{z}, \overline{z}^2, \ldots, \overline{z}^n\}^{\bigvee},$$

in which $\{\cdot\}^{\bigvee}$ denotes the linear closed space spanned by $\{\cdot\}$. Denote by b_n^2 the cutoff harmonic Bergman space, we have

$$b_n^2 = L_a^2 \bigoplus W_n. \tag{1.1}$$

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It is well known that $K_z(w)=\frac{1}{(1-\bar{z}w)^2}$ is the reproducing kernel for L_a^2, $R_z(w)=K_z(w)+\overline{K_z(w)}-1$ is the reproducing kernel for L_h^2. From the relationship of (1.1), we know that b_n^2 is a reproducing Hilbert space, its kernel is denoted by $R_z^{(n)}$, and given by

$$R_z^{(n)}=K_z(w)+\sum_{i=1}^{n}(i+1)(z\overline{w})^i.$$

Let P be the orthogonal projection from $L^2(D)$ onto $L_a^2(D)$, Q be the orthogonal projection from $L^2(D)$ onto $L_h^2(D)$, P_n denote the orthogonal projection from $L^2(D)$ onto W_n, and $Q_n=P\bigoplus P_n$ be the projection from $L^2(D)$ onto b_n^2. It is clear that Q_n converges to Q by strong operator topology. Since $L_a^2(D)$, $L_h^2(D)$ and b_n^2 are reproducing Hilbert space, we have

$$Pf(z)=\langle f,K_z\rangle,\ Qf(z)=\langle f,R_z\rangle,\ Q_nf(z)=\langle f,R_z^{(n)}\rangle,\quad \forall f\in L^2(D).$$

For $\varphi\in L^\infty(D)$, the Toeplitz operator $T_\varphi: L^2(D)\to b_n^2$ with symbol φ is defined by

$$T_\varphi f(z)=Q_n(\varphi f)=\int_D f(w)\varphi(w)\overline{R_z^{(n)}(w)}\mathrm{d}A(w).$$

In 1964, Brown and Halmos [1] proved that if $T_fT_g=0$ on the Hardy space $H^2(T)$, then either f or g must be identically zero. In [2], Ahern and Čučković showed that the result is analogous to that in [1] for two Toeplitz operators with harmonic symbols on the Bergman space of unit disk. Moreover in [3] they proved that if $T_fT_g=0$, where f is arbitrary bounded and g is radial, then either $f\equiv 0$ or $g\equiv 0$. Those zero product results have been generalized to finite rank product result in [4]. Čučković and Louhichi [5] studied finite rank product of several quasihomogeneous Toeplitz operators on the Bergman space of the unit disk. Furthermore, Yang and Lu [6] studied finite rank product of quasihomogeneous Toeplitz operators on the harmonic Bergman space of unit disk.

For two Toeplitz operators T_φ and T_ψ the commutator and semi-commutator are defined by

$$[T_\varphi,T_\psi]=T_\varphi T_\psi-T_\psi T_\varphi,\quad (T_\varphi,T_\psi]=T_{\varphi\psi}-T_\varphi T_\psi.$$

For the problem of finite-rank commutator or semi-commutator, Axler [7] and Ding [8] have characterized it on the Hardy space completely. On the Bergman space the problem seems to be far from solution. Guo, Sun and Zheng [9] completely characterized the finite rank commutator and semi-commutator of two Toeplitz operators with bounded harmonic symbols on the Bergman space of the unit disk. Čučković and Louhichi [5] investigated the finite rank semi-commutators and commutators of Toeplitz operators with quasihomogeneous symbols on the Bergman space of the unit disk. Yang and Lu [6] have studied the finite rank commutators and semi-commutators of quasihomogeneous Toeplitz operators on the harmonic Bergman space. Ding [10] solved the finite rank commutators of Toeplitz operator with bounded harmonic symbols on the cutoff harmonic Bergman space.

Motivated by Čučković and Louhichi [5], Ding [10] and Choe [11], we will discuss the finite rank (semi)commutators of quasihomogeneous Toeplitz operators on the cutoff harmonic Bergman space.

2. Preliminaries

Before we state our results, we need to introduce the Mellin transform.

Definition 2.1 *Let $f \in L^1([0,1], r\mathrm{d}r)$. The Mellin transform $\hat{f}$ of a function f is defined by*

$$\widehat{f}(z) = \int_0^1 f(r) r^{z-1} \mathrm{d}r.$$

It is clear that $\hat{f}$ is well defined on the right half-plane $\{z : \operatorname{Re} z \geq 2\}$ and analytic on $\{z : \operatorname{Re} z > 2\}$. It is important and helpful to know that the Mellin transform $\hat{f}$ is uniquely determined by its value on an arithmetic sequence of integers. In fact, we have the following classical theorem [12, p.102].

Theorem 2.2 *Suppose that f is a bounded analytic function on $\{z : \operatorname{Re} z > 0\}$ which vanishes at the pairwise distinct points $z_1, z_2, \ldots$, where*

(1) $\inf\{|z_n|\} > 0$,

(2) $\sum_{n\geq 1} \operatorname{Re}(\frac{1}{z_n}) = \infty$.

Then f vanishes identically on $\{z : \operatorname{Re} z > 0\}$.

Remark 2.3 We shall often use this theorem to show that if $f \in L^1([0,1], r\mathrm{d}r)$ and if there exists a sequence $(n_k)_{k\geq 0} \subset \mathbb{N}$ such that

$$\widehat{f}(n_k) = 0 \quad \text{and} \quad \sum_{k\geq 0} \frac{1}{n_k} = \infty,$$

then $\hat{f}(z) = 0$ for all $z \in \{z : \operatorname{Re}(z) > 2\}$ and so $f = 0$.

Let $p \in \mathbb{Z}$. A function $\varphi \in L^1(D, \mathrm{d}A)$ is called a quasihomogeneous function of degree p if φ is of the form $e^{ip\theta} f$, where f is a radial function, i.e.,

$$\varphi(re^{i\theta}) = e^{ip\theta} f(r).$$

The main reason for many researchers to study Toeplitz operators with quasimonogeneous symbols is that any function f in $L^2(D, \mathrm{d}A)$ has the polar decomposition

$$f(re^{i\theta}) = \sum_{k\in\mathbb{Z}} e^{ik\theta} f_k(r),$$

where f_k are radial functions in $L^2([0,1], r\mathrm{d}r)$.

Lemma 2.4 *Let $p \geq 0$ and φ be a bounded radial function. Then for each $k \in N$,*

$$T_{e^{ip\theta}\varphi}(z^k) = 2(p+k+1)\widehat{\varphi}(2k+p+2) z^{p+k}, \quad \forall\, k \geq 0.$$

$$T_{e^{-ip\theta}\varphi}(z^k) = \begin{cases} 2(k-p+1)\widehat{\varphi}(2k-p+2) z^{k-p}, & k \geq p, \\ 2(p-k+1)\widehat{\varphi}(p+2)\overline{z}^{p-k}, & p-n \leq k < p, \\ 0, & k > p-n. \end{cases}$$

If $p \geq n$,

$$T_{e^{ip\theta}\varphi}(\overline{z}^k) = 2(p-k+1)\widehat{\varphi}(p+2) z^{p-k}, \quad \forall\, 1 \leq k \leq n$$

$$T_{e^{-ip\theta}\varphi}(\overline{z}^k) = 0, \quad \forall\, 1 \le k \le n.$$

If $p < n$,

$$T_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)\widehat{\varphi}(p+2)z^{p-k}, & k \le p, \\ 2(k-p+1)\widehat{\varphi}(2k-p+2)\overline{z}^{k-p}, & p < k \le n. \end{cases}$$

$$T_{e^{-ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(k+p+1)\widehat{\varphi}(2k+p+2)\overline{z}^{k+p}, & 1 \le k < n-p, \\ 0, & n-p < k \le n. \end{cases}$$

3. Finite rank product of n Toeplitz operators

We will discuss the finite rank product of Toeplitz operators with quasihomogeneous symbols in this section.

Theorem 3.1 *Let $p_1, \ldots, p_m \in \mathbb{Z}^+ \bigcup\{0\}$ and $\varphi_1, \ldots, \varphi_m$ be bounded radial functions. If $T_{e^{ip_m\theta}\varphi_m} \cdots T_{e^{ip_1\theta}\varphi_1}$ is of finite rank M, then $\varphi_i = 0$ for some $i \in \{1, 2, \ldots, m\}$.*

Proof We denote by S the product of Toeplitz operators $T_{e^{ip_m\theta}\varphi_m} \cdots T_{e^{ip_1\theta}\varphi_1}$.

For any $\overline{z}^k \in b_n^2$ $(1 \le k \le n)$, it is clear that $\{S(\overline{z}^k) | 1 \le k \le n\}$ has finite rank, and its rank is less than n.

On the other hand, $\{S(z^k) | k \ge 0\}$ must have finite rank, which, by [5, Theorem 2], implies that $\varphi_i = 0$ for some $i \in 1, 2, \ldots, m$. The proof of this Theorem is completed. □

Theorem 3.2 *Let $p_1, \ldots, p_m \in \mathbb{Z}^+ \bigcup\{0\}$ and $\varphi_1, \ldots, \varphi_m$ be bounded radial functions. If $T_{e^{-ip_m\theta}\varphi_m} \cdots T_{e^{-ip_1\theta}\varphi_1}$ has finite rank, then $\varphi_i = 0$ for some $i \in \{1, 2, \ldots, m\}$.*

Proof Let S denote the product of Toeplitz operators $T_{e^{-ip_m\theta}\varphi_m} \cdots T_{e^{-ip_1\theta}\varphi_1}$. Since S has finite rank on b_n^2, it follows $\{S(\overline{z}^k) : 1 \le k \le n\}$ and $\{S(z^k) : k \ge 0\}$ are have finite rank.

For $\{S(\overline{z}^k) : 1 \le k \le n\}$, it is clear that it has finite rank for any φ_i.

On the other hand, by Lemma 2.4 for $k \ge \sum_{j=1}^m p_j$,

$$\begin{aligned} S(z^k) =& T_{e^{-ip_m\theta}\varphi_m} \cdots T_{e^{-ip_1\theta}\varphi_1}(z^k) \\ =& 2(k-p_1+1)\widehat{\varphi_1}(2k-p_1+2)2(k-p_1-p_2+1)\widehat{\varphi_2}(2k-2p_1-p_2+2)\cdots \\ & 2(k-p_1-\cdots-p_m+1)\widehat{\varphi_m}(2k-2p_1+\cdots-2p_{m-1}-p_m+2)z^{k-p_1-\cdots-p_m}. \end{aligned}$$

By [6, Theorem 3.3], we can get $\varphi_i = 0$ from the fact that $\{S(z^k) : k \ge 0\}$ have finite rank. The proof of this Theorem is completed. □

Corollary 3.3 *Let $p_1, \ldots, p_m \in \mathbb{Z}$ and $\varphi_1, \ldots, \varphi_m$ be bounded radial functions. If $T_{e^{ip_m\theta}\varphi_m} \cdots T_{e^{ip_1\theta}\varphi_1}$ is of finite rank M, then $\varphi_i = 0$ for some $i \in \{1, 2, \ldots, m\}$.*

4. Finite rank commutator

In this section, we investigate the commutator $[T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi}]$ and $[T_{e^{ip\theta}\varphi}, T_{e^{-is\theta}\psi}]$, $p, s \ge 0$.

Theorem 4.1 *Let p, s be non-negative integers and at least one of them is nonzero. Let φ and*

ψ be two integrable radial functions on D such that $T_{e^{ip\theta}\varphi}$ and $T_{e^{is\theta}\psi}$ are bounded operators. If the commutators $[T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi}]$ have finite ranks M and $p+s<n$ respectively, then M is at most equal to $s+p$, otherwise if $p+s\geq n$, M is at most equal to n.

Proof Let S denote the commutator $[T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi}]$. Since S has finite rank on $b_n^2(D)$, we know that $\{S(z^k)\}_{k\geq 0}$ and $\{S(\overline{z}^k)\}_{1\leq k\leq n}$ must have finite rank.

Firstly, if $\{S(z^k)\}_{k\geq 0}$ has the finite rank N, we have

$$S(z^k)=0,\quad \forall\, k\geq N_1\geq N. \tag{4.1}$$

By Lemma 2.4, Eq. (4.1) is equivalent to

$$2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s+p+2)=2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p+s+2),$$

by the proof of Theorem 6 in [5], we known that the rank of $\{S(z^k)\}_{k\geq 0}$ is equal to zero, i.e., $N=0$.

On the other hand, we suppose the rank of $\{S(\overline{z}^k)\}_{1\leq k\leq n}$ is equal to M, by Lemma 2.4, we can obtain the following results.

If $p>n$,

$$\begin{aligned}&T_{e^{is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k)\\&\quad=2(p-k+1)2(p+s-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2pp-2k+s+2)z^{p+s-k},\quad 1\leq k\leq n.\end{aligned}$$

If $p\leq n$, $p+s\geq n$, then

$$T_{e^{is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k)=\begin{cases}2(p-k+1)2(p+s-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k+s+2)z^{p+s-k}, & 1\leq k\leq p,\\ 2(k-p+1)2(p+s-k+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(s+2)z^{p+k-s}, & p<k\leq n.\end{cases}$$

If $p\leq n$, $p+s<n$, then

$$T_{e^{is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k)=\begin{cases}2(p-k+1)2(p+s-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k+s+2)z^{p+s-k}, & 1\leq k\leq p,\\ 2(k-p+1)2(p+s-k+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(s+2)z^{p+k-s}, & p<k\leq p+s,\\ 2(k-p+1)2(k-p-s+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p-s+2)\overline{z}^{k-s-p}, & p+s<k\leq n.\end{cases}$$

If $s>n$, then

$$T_{e^{ip\theta}\varphi}T_{e^{is\theta}\psi}(\overline{z}^k)=2(s-k+1)2(p+s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k+p+2)z^{p+s-k},\quad 1\leq k\leq n.$$

If $s\leq n$, $p+s\geq n$, then

$$T_{e^{ip\theta}\varphi}T_{e^{is\theta}\psi}(\overline{z}^k)=\begin{cases}2(s-k+1)2(p+s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k+p+2)z^{p+s-k}, & 1\leq k\leq s,\\ 2(k-s+1)2(p+s-k+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(p+2)z^{p+s-k}, & s<k\leq n.\end{cases}$$

If $s\leq n$, $p+s<n$, then

$$T_{e^{ip\theta}\varphi}T_{e^{is\theta}\psi}(\overline{z}^k)=\begin{cases}2(s-k+1)2(p+s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k+p+2)z^{p+s-k}, & 1\leq k\leq s,\\ 2(k-s+1)2(p+s-k+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(p+2)z^{p+s-k}, & s<k\leq p+s,\\ 2(k-s+1)2(k-s-p+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s-p+2)\overline{z}^{k-s-p}, & p+s<k\leq n.\end{cases}$$

From above calculation, we can express $S(\overline{z}^k)$ as follows.

Case 1. $p+s<n$

$$S(\overline{z}^k)=\begin{cases}\lambda(k,s,p)z^{p+s-k}, & 1\leq k\leq p+s,\\ \beta(k,s,p)\overline{z}^{k-s-p}, & p+s<k\leq n,\end{cases}$$

in which $\lambda(k,s,p)$ and $\beta(k,s,p)$ are the functions with respect to s,k,p. By [6, Theorem 4.1], we know that

$$S(\overline{z}^k)=0,\quad k>p+s.$$

So in this case, $\{S(\overline{z}^k):1\le k\le n\}\subset\{\lambda(k,s,p)z^{p+s-k}:1\le k\le p+s\}$, the rank of $\{S(\overline{z}^k):1\le k\le n\}$ is at most equal to $p+s$.

Case 2. $p+s\ge n$

$$S(\overline{z}^k)=\alpha(s,k,p)z^{p+s-k},\quad 1\le k\le n,$$

in which $\alpha(s,k,p)$ is the function for s,k,p. In this case $\{S(\overline{z}^k):1\le k\le n\}\subset\{\alpha(k,s,p)z^{p+s-k}:1\le k\le n\}$, so the rank of $\{S(\overline{z}^k):1\le k\le n\}$ is at most equal to n. This completes the proof. □

Theorem 4.2 *Let $p,s\ge 0$ and at least one of them is nonzero. Let φ and ψ be two integrable radial functions on D such that $T_{e^{ip\theta}\varphi}$ and $T_{e^{-is\theta}\psi}$ are bounded operators. If the commutator $[T_{e^{ip\theta}\varphi},T_{e^{-is\theta}\psi}]$ has finite rank and $p\ge s$, then its rank is at most equal to $n+s$. If the commutator $[T_{e^{ip\theta}\varphi},T_{e^{-is\theta}\psi}]$ has finite rank and $p<s$, then its rank is at most equal to $n+p$.*

Proof Let S denote the commutator $[T_{e^{ip\theta}\varphi},T_{e^{-is\theta}\psi}]$. We will prove the case of $p\ge s$ in details.

Since $p\ge s$, by direct calculation, we have

$$\begin{aligned}&T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(z^k)\\&\quad=T_{e^{-is\theta}\psi}[2(p+k+1)\widehat{\varphi}(2k+p+2)z^{p+k}]\\&\quad=2(p+k+1)\widehat{\varphi}(2k+p+2)2(k+p-s+1)\widehat{\psi}(2k+2p-s+2)z^{k+p-s},\ k\ge 0.\end{aligned}$$

For $T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(z^k)$ we will discuss it from different cases.

Case 1. $s\ge n$,

$$T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(z^k)=\begin{cases}2(k-s+1)2(k+p-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2)z^{k+p-s}, & k\ge s,\\ 2(s-k+1)2(k+p-s+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)z^{k+p-s}, & s-n\le k<s,\\ 0, & 0\le k<s-n.\end{cases}$$

Case 2. $s<n$,

$$T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(z^k)=\begin{cases}2(k-s+1)2(k+p-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2)z^{k+p-s}, & k\ge s,\\ 2(s-k+1)2(k+p-s+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)z^{k+p-s}, & 0\le k<s.\end{cases}$$

From the above discussion, we know that if $s\ge n$, then

$$S(z^k)=\begin{cases}2(k+p-s+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2)\\ -2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)]z^{k-s+p}, & k\ge s,\\ 2(k+p-s+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\\ -2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)]z^{k+p-s}, & s-n\le k<s,\\ -2(k+p-s+1)2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)]z^{k+p-s}, & 0\le k<s-n.\end{cases}$$

If $s<n$, then

$$S(z^k)=\begin{cases}2(k+p-s+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2)\\ -2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)]z^{k-s+p}, & k\ge s,\\ 2(k+p-s+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)\\ -2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)]z^{k+p-s}, & 0\le k<s.\end{cases}$$

By [6, Theorem 4.3], we know that

$$S(z^k)=0,\quad k\ge s,$$

then we can get that

$$\{S(z^k) : k \geq 0\} \subset \{S(z^k) : 0 \leq k < s\} \subset \operatorname{span}\{z^l : p-s \leq l < p\}.$$

Next, we discuss $\{S(\overline{z}^k) : 1 \leq k \leq n\}$.

Firstly, we give the expression of $T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k)$ $(1 \leq k \leq n)$. If $p > n,\ p-s \geq n$,

$$T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k) = 2(p-k+1)2(p-k-s+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)z^{p-k-s},\ \ 1 \leq k \leq n.$$

If $p > n$, $p-s < n$, then

$$T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)2(p-k-s+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)z^{p-k-s}, & 1 \leq k \leq p-s, \\ 2(p-k+1)2(s-p+k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{k+s-p}, & p-s < k \leq n. \end{cases}$$

If $p \leq n$, then

$$T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k) = \begin{cases} 2(p-k+1)2(p-k-s+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)z^{p-k-s}, & 1 \leq k \leq p-s, \\ 2(p-k+1)2(s-p+k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{k+s-p}, & p-s < k \leq p, \\ 2(k-p+1)2(k-p+s+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+2)\overline{z}^{s+k-p}, & p < k \leq n. \end{cases}$$

Secondly, we give the expression of $T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(\overline{z}^k)$ $(1 \leq k \leq n)$.

If $s > n$, then $T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(\overline{z}^k) = 0,\ 1 \leq k \leq n$.

If $s \leq n$, $p > n$, then

$$T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(\overline{z}^k) = \begin{cases} 2(k+s+1)2(p-k-s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2)z^{p-k-s}, & 1 \leq k \leq n-s, \\ 0, & n-s < k \leq n. \end{cases}$$

If $s \leq n$, $p \leq n$, then

$$T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(\overline{z}^k) = \begin{cases} 2(k+s+1)2(p-k-s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2)z^{p-k-s}, & 1 \leq k \leq p-s, \\ 2(k+s+1)2(s-p+k+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s+p+2)\overline{z}^{k+s-p}, & p-s < k \leq n-s, \\ 0, & n-s < k \leq n. \end{cases}$$

From the above formula, we have

If $p \leq n$, $p \geq n-s$, then

$$S(\overline{z}^k) = \begin{cases} 2(p-k-s+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2) \\ -2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)]z^{p-k-s}, & 1 \leq k \leq p-s, \\ 2(k+s-p+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s+p+2) \\ -2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)]\overline{z}^{k+s-p}, & p-s < k \leq n-s, \\ -2(k+s-p+1)2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(p+2)\overline{z}^{s+k-p}, & n-s < k \leq p, \\ -2(k+s-p+1)2(k-p+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+2))\overline{z}^{s+k-p}, & p < k \leq n. \end{cases}$$

If $p \leq n$, $p < n-s$, then

$$S(\overline{z}^k) = \begin{cases} 2(p-k-s+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2) \\ -2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)]z^{p-k-s}, & 1 \leq k \leq p-s, \\ 2(k+s-p+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s+p+2) \\ -2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)]\overline{z}^{k+s-p}, & p-s < k \leq p, \\ 2(k+s-p+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s+p+2) \\ -2(k-p+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+2)\overline{z}^{s+k-p}, & p < k \leq n-s, \\ -2(k+s-p+1)2(k-p+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+2))\overline{z}^{s+k-p}, & n-s < k \leq n. \end{cases}$$

If $p > n$, $s \geq n$, $p-s \geq n$, then

$$S(\overline{z}^k) = -2(p-kps+1)2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)z^{p-s-k},\ \ 1 \leq k \leq n.$$

If $p > n$, $s \geq n$, $p-s < n$, then

$$S(\overline{z}^k) = \begin{cases} -2(p-k-s+1)2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)z^{p-k-s}, & 1 \leq k \leq p-s, \\ -2(k+s-p+1)2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{k+s-p}, & p-s < k \leq n. \end{cases}$$

If $p > n$, $s < n$, $p - s \geq n$,

$$S(\overline{z}^k) = \begin{cases} 2(p-k-s+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2) \\ -2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)]z^{p-k-s}, & 1 \leq k \leq n-s, \\ 2(p-k-s+1)2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)z^{p-k-s}, & n-s < k \leq n. \end{cases}$$

If $p > n$, $s < n$, $p - s < n$, then

$$S(\overline{z}^k) = \begin{cases} 2(p-k-s+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(p+2) \\ -2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)]z^{p-k-s}, & 1 \leq k \leq n-s, \\ -2(p-k-s+1)2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(2p-2k-s+2)z^{p-k-s}, & n-s < k \leq p-s. \\ -2(s-p+k+1)2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{s+k-p}, & p-s < k \leq n. \end{cases}$$

Then we have

(1) $p \leq n$,

$$\begin{aligned} \{S(\overline{z}^k) : 1 \leq k \leq n\} &\subseteq \{S(\overline{z}^k) : 1 \leq k < p\} \\ &\subseteq \operatorname{span}\{z^l : 0 \leq l \leq p-s-1\} \bigcup \operatorname{span}\{\overline{z}^l : 0 < l \leq n+s-p\}. \end{aligned}$$

(2) $p > n$, $s \geq n$, either

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{z^l : p-s-n \leq l \leq p-s-1\},$$

or

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{z^l : 0 \leq l \leq p-s-1\} \bigcup \operatorname{span}\{\overline{z}^l : 0 < l \leq n+s-p\}.$$

(3) $p > n$, $s < n$, either

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{z^l : p-s-n \leq l \leq p-s-1\},$$

or

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{z^l : 0 \leq l \leq p-s-1\} \bigcup \operatorname{span}\{\overline{z}^l : 0 < l \leq n+s-p\}.$$

Combining with $\{S(z^k) : k \geq 0\}$ and $\{S(\overline{z}^k) : 1 \leq k \geq n\}$, we have

$$\{S(z^k) : 0 \leq k < s\} \bigcup \{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{z^l : p-s-n \leq l < p\},$$

or

$$\{S(z^k) : 0 \leq k < s\} \bigcup \{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{z^l : 0 \leq l < p\} \bigcup \operatorname{span}\{\overline{z}^l : 0 < l < s+n-p\},$$

we can get that the rank of S is at most equal to $s + n$.

Similarly, for $p < s$ by direct calculation and [6, Theorem 4.3], we can obtain

$$S(z^k) = 0, \quad k \geq s.$$

Furthermore, we can obtain that

If $s \geq n$, then

$$\begin{aligned} &\{S(z^k) : 0 \leq k < s\} \subseteq \operatorname{span}\{z^l : 0 \leq l < p\} \bigcup \operatorname{span}\{\overline{z}^l : 0 < l \leq n\}, \\ &\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \operatorname{span}\{\overline{z}^l : s-p < l \leq n\}. \end{aligned}$$

If $s < n$, then

$$\{S(z^k) : 0 \leq k < s\} \subseteq \operatorname{span}\{z^l : 0 \leq l < p\} \bigcup \operatorname{span}\{\overline{z}^l : 0 < l \leq s\},$$

$$\{S(\overline{z}^k) : 1 \le k \le n\} \subseteq \mathrm{span}\{\overline{z}^l : 0 < l \le n\},$$

or

$$\{S(\overline{z}^k) : 1 \le k \le n\} \subseteq \mathrm{span}\{z^l : 0 \le l < p\} \bigcup \mathrm{span}\{\overline{z}^l : 0 < l \le n\}.$$

So we know that

$$\begin{aligned}&\{S(z^k) : 0 \le k < s\} \bigcup \{S(\overline{z}^k) : 1 \le k \le n\}\\ &\quad \subseteq \mathrm{span}\{z^l : 0 \le l < p\} \bigcup \mathrm{span}\{\overline{z}^l : 0 < l \le n\},\end{aligned}$$

then the rank of S is at most equal to $n + p$. This completes the proof. □

5. Finite rank semi-commutators

In this section, we will study the semi-commutators of tow Toeplitz operators with quasihomogeneous symbols.

Theorem 5.1 *Let $p, s \ge 0$ and at least one of them be nonzero. Let φ and ψ be integrable radial functions on D such that $T_{e^{ip\theta}\varphi}$ and $T_{e^{is\theta}\psi}$ are bounded operators. If $(T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi}]$ has finite rank and $p+s < n$, then its rank is at most equal to $p+s-1$. If the commutator $(T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi}]$ has finite rank and $p + s \ge n$, then its rank is at most equal to n.*

Proof Let S denote $(T_{e^{ip\theta}\varphi}, T_{e^{is\theta}\psi}]$. By Lemma 2.4, we can obtain that

$$\begin{aligned}&T_{e^{i(p+s)\theta}\varphi\psi}(z^k) = 2(p + k + s + 1)\widehat{\varphi\psi}(2k + p + s + 2)z^{k+s+p},\ k \ge 0,\\ &T_{e^{ip\theta}\varphi}T_{e^{is\theta}\psi}(z^k) = 2(k + s + 1)2(k + p + s + 1)\widehat{\psi}(2k + s + 2)\widehat{\varphi}(2k + 2s + p + 2)z^{k+p+s},\ k \ge 0.\end{aligned}$$

Then we have

$$\begin{aligned}S(z^k) =&2(p + k + s + 1)[2(k + s + 1)\widehat{\psi}(2k + s + 2)\widehat{\varphi}(2k + 2s + p + 2)-\\ &\widehat{\varphi\psi}(2k + p + s + 2)]z^{k+p+s},\quad \forall k \ge 0.\end{aligned}$$

By [5, Theorem 4], we know that

$$S(z^k) = 0,\quad \forall k \ge 0,$$

so the rank of $\{S(z^k) : k \ge 0\}$ is equal to zero.

Next, we discuss the rank of $\{S(\overline{z}^k) : 1 \le k \le n\}$. By direct calculation,

if $s \ge n$, then

$$S(\overline{z}^k) = 2(p+s-k+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k+p+2) - \widehat{\varphi\psi}(p+s+2)]z^{p+s-k},\quad 1 \le k \le n,$$

if $s < n$, $p + s < n$, then

$$S(\overline{z}^k) = \begin{cases} 2(p-k+s+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k+p+2) & \\ -\widehat{\varphi\psi}(p+s+2)]z^{p+s-k}, & 1 \le k \le s,\\ 2(p-k+s+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(p+2) & \\ -\widehat{\varphi\psi}(p+s+2)]z^{p+s-k}, & s < k \le p+s,\\ 2(k-s-p+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s-p+2) & \\ -\widehat{\varphi\psi}(2k-p-s+2)]\overline{z}^{k-p-s}, & p+s < k \le n,\end{cases}$$

if $s < n$, $p+s \geq n$, then

$$S(\overline{z}^k) = \begin{cases} 2(p-k+s+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k+p+2) \\ -\widehat{\varphi\psi}(p+s+2)]z^{p+s-k}, & 1 \leq k \leq s, \\ 2(p-k+s+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(p+2) \\ -\widehat{\varphi\psi}(p+s+2)]z^{p+s-k}, & s < k \leq n. \end{cases}$$

From the above equation, we have

If $p+s \geq n$, then

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} \subseteq \text{span}\{z^l : p+s-n \leq l \leq p+s-1\},$$

then the rank of $\{S(\overline{z}^k)\}$ is at most equal to n.

If $p+s < n$, then by [6, Theorem 5.1], we know that

$$S(\overline{z}^k) = 0, \quad k \geq p+s,$$

then

$$\{S(\overline{z}^k) : 1 \leq k \leq n\} = \{S(\overline{z}^k) : 1 \leq k < p+s\} \subseteq \text{span}\{z^l : 0 < l \leq p+s-1\},$$

the rank of $\{S(\overline{z}^k) : 1 \leq k \leq n\}$ is at most equal to $p+s-1$, the proof is completed. □

Theorem 5.2 *Let $p, s \geq 0$, $s \geq p$ and at least one of them be nonzero. Let φ and ψ be integrable radial functions on D such that $T_{e^{ip\theta}\varphi}$ and $T_{e^{-is\theta}\psi}$ are bounded operators. If $(T_{e^{ip\theta}\varphi}, T_{e^{-is\theta}\psi}]$ has finite rank and $s \leq n+p$, then the rank of it is at most equal to $n+p$. If the commutator $(T_{e^{ip\theta}\varphi}, T_{e^{-is\theta}\psi}]$ has finite rank and $s > n+p$, then its rank is at most equal to s.*

Proof Let S denote $(T_{e^{ip\theta}\varphi}, T_{e^{-is\theta}\psi}]$. We will discuss the rank of $\{S(z^k) : k \geq 0\}$ and $\{S(\overline{z}^k) : 1 \leq k \leq n\}$.

Firstly, we characterize $\{S(z^k) : k \geq 0\}$. By Lemma 2.4, we obtain the following results directly

$$T_{e^{-i(s-p)\theta}\varphi\psi}(z^k) = \begin{cases} 2(k-s+p+1)\widehat{\varphi\psi}(2k-s+p+2)z^{k-s+p}, & k \geq s-p, \\ 2(s-p-k+1)\widehat{\varphi\psi}(s-p+2)\overline{z}^{s-p-k}, & 0 \leq k < s-p. \end{cases}$$

If $p \geq n$, $s \geq n$, then

$$T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(z^k) = \begin{cases} 2(k-s+1)2(k+p-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2)z^{k-s+p}, & k \geq s, \\ 2(s-k+1)2(k+p-s+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)z^{k+p-s}, & s-n \leq k < s, \\ 0, & 0 \leq k < s-n. \end{cases}$$

If $p < n$, $s < n$, then

$$T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(z^k) = \begin{cases} 2(k-s+1)2(k+p-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2)z^{k-s+p}, & k \geq s, \\ 2(s-k+1)2(k+p-s+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)z^{k+p-s}, & s-p \leq k < s, \\ 2(s-k+1)2(s-k-p+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k-p+2)\overline{z}^{s-k-p}, & 0 \leq k < s-p. \end{cases}$$

If $p < n$, $s \geq n$, then

$$T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(z^k) = \begin{cases} 2(k-s+1)2(k+p-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2)z^{k-s+p}, & k \geq s, \\ 2(s-k+1)2(k+p-s+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2)z^{k+p-s}, & s-p \leq k < s, \\ 2(s-k+1)2(s-k-p+1)\widehat{\psi}(s+2)\widehat{\varphi}(2s-2k-p+2)\overline{z}^{s-k-p}, & s-n-p \leq k < s-p, \\ 0, & 0 \leq k < s-n-p. \end{cases}$$

From above formula, we have

If $p \geq n$, $s \geq n$, then

$$S(z^k) = \begin{cases} 2(k+p-s+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2) & \\ -\widehat{\varphi\psi}(2k+p-s+2)]z^{k-s+p} & k \geq s, \\ 2(k+p-s+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2) & \\ -\widehat{\varphi\psi}(2k-s+p-2)]z^{k+p-s}, & s-n \leq k < s, \\ -2(k-s+p+1)\widehat{\varphi\psi}(2k-s+p+2)z^{k+p-s}, & s-p \leq k < s-n, \\ -2(s-p-k+1)\widehat{\varphi\psi}(s-p+2)\overline{z}^{s-p-k}, & 0 \leq k < s-p. \end{cases}$$

If $p < n$, $s < n$, then

$$S(z^k) = \begin{cases} 2(k+p-s+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2) & \\ -\widehat{\varphi\psi}(2k+p-s+2)]z^{k-s+p}, & k \geq s, \\ 2(k+p-s+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2) & \\ -\widehat{\varphi\psi}(2k-s+p-2)]z^{k+p-s}, & s-p \leq k < s, \\ 2(s-p-k+1)[2(s-k+1)\widehat{\varphi}(s+2)\widehat{\psi}(2s-2k-p+2) & \\ -\widehat{\varphi\psi}(s-p+2)]\overline{z}^{s-p-k}, & 0 \leq k < s-p. \end{cases}$$

If $p < n$, $s \geq n$, then

$$S(z^k) = \begin{cases} 2(k+p-s+1)[2(k-s+1)\widehat{\psi}(2k-s+2)\widehat{\varphi}(2k-2s+p+2) & \\ -\widehat{\varphi\psi}(2k+p-s+2)]z^{k-s+p}, & k \geq s, \\ 2(k+p-s+1)[2(s-k+1)\widehat{\psi}(s+2)\widehat{\varphi}(p+2) & \\ -\widehat{\varphi\psi}(2k-s+p-2)]z^{k+p-s}, & s-p \leq k < s, \\ 2(s-p-k+1)[2(s-k+1)\widehat{\varphi}(s+2)\widehat{\psi}(2s-2k-p+2) & \\ -\widehat{\varphi\psi}(s-p+2)]\overline{z}^{s-p-k}, & s-n-p \leq k < s-p, \\ -2(s-p-k+1)\widehat{\varphi\psi}(s-p+2)\overline{z}^{s-p-k}, & 0 \leq k < s-n-p. \end{cases}$$

By [5, Theorem 5], we know that

$$S(z^k) = 0, \quad \forall\ k \geq s,$$

then clearly,

$$\{S(z^k) : 0 \leq k < s\} \subseteq \operatorname{span}\{z^l : 0 \leq l < p\} \bigcup \operatorname{span}\{\overline{z}^l : 0 < l \leq s-p\}.$$

Next, we discuss $\{S(\overline{z}^k) : 1 \leq k \leq n\}$. By Lemma 2.4, we have

If $s - p \geq n$, then

$$T_{e^{-i(s-p)\theta}\varphi\psi}(\overline{z}^k) = 0, \quad 1 \leq k \leq n.$$

If $s - p < n$, then

$$T_{e^{-i(s-p)\theta}\varphi\psi}(\overline{z}^k) = \begin{cases} 2(k+s-p+1)\widehat{\varphi\psi}(2k+s-p+2)\overline{z}^{k+s-p} & 1 \leq k \leq n-(s-p), \\ 0, & n-(s-p) < k \leq n. \end{cases}$$

If $s > n$, then

$$T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(\overline{z}^k) = 0, \quad 1 \leq k \leq n.$$

If $s \leq n$, $p < n$, then

$$T_{e^{ip\theta}\varphi}T_{e^{-is\theta}\psi}(\overline{z}^k) = \begin{cases} 2(k+s+1)2(k+s-p+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s+p+2)\overline{z}^{k+s-p} & 1 \leq k \leq n-s, \\ 0, & n-s < k \leq n. \end{cases}$$

Furthermore, we can obtain

If $s > n$, $s - p \geq n$, then

$$S(\overline{z}^k) = 0, \quad 1 \leq k \leq n.$$

If $s > n$, $s-p < n$, then

$$S(\overline{z}^k) = \begin{cases} 2(k+s+1)2(k+s-p+1)\widehat{\varphi\psi}(2k+s-p+2)\overline{z}^{k+s-p}, & 1 \le k \le n-(s-p), \\ 0, & n-(s-p) < k \le n. \end{cases}$$

If $s \le n$, $p < n$, then

$$S(\overline{z}^k) = \begin{cases} 2(k+s-p+1)[2(k+s+1)\widehat{\psi}(2k+s+2)\widehat{\varphi}(2k+2s+p+2) \\ \widehat{\varphi\psi}(2k+s-p+2)]\overline{z}^{k+s-p}, & 1 \le k \le n-s, \\ -2(k+s-p+1)\widehat{\varphi\psi}(2k+s-p+2)\overline{z}^{k+s-p}, & n-s < k \le n-s+p, \\ 0, & n-s+p < k \le n. \end{cases}$$

From above discuss, we have

If $s-p \ge n$, then

$$S(\overline{z}^k) = 0, \quad 1 \le k \le n.$$

If $s-p < n$, then

$$\{S(\overline{z}^k) : 1 \le k \le n\} \subseteq \operatorname{span}\{\overline{z}^l : s-p+1 \le l \le n\}.$$

Combining the above with

$$\{S(z^k) : 0 \le k < s\} \subseteq \operatorname{span}\{z^l : 0 \le l < p\} \bigcup \operatorname{span}\{\overline{z}^l : 0 < l \le s-p\},$$

we obtain that

If $s-p \ge n$, then

$$\begin{aligned} &\{S(z^k) : k \ge 0\} \bigcup \{S(\overline{z}^k) : 1 \le k \le n\} \subseteq \{S(z^k) : 0 \le k < s\} \\ &\subseteq \operatorname{span}\{z^l : 0 \le l < p\} \bigcup \operatorname{span}\{\overline{z}^l : 0 < l \le s-p\}, \end{aligned}$$

which means that the rank of S is at most equal to s.

If $s-p < n$, then

$$\begin{aligned} &\{S(z^k) : k \ge 0\} \bigcup \{S(\overline{z}^k) : 1 \le k \le n\} \\ &\subseteq \{S(z^k) : 0 \le k < s\} \bigcup \{S(\overline{z}^k) : 1 \le k \le n\} \\ &\subseteq \operatorname{span}\{z^l : 0 \le l < p\} \bigcup \operatorname{span}\{\overline{z}^l : 0 < l \le n\}, \end{aligned}$$

which means that the rank of S is at most equal to $n+p$. This completes the proof. $\square$

Theorem 5.3 *Let $p, s \ge 0$, $s \ge p$ and at least one of them be nonzero. Let φ and ψ be integrable radial functions on D such that $T_{e^{ip\theta}\varphi}$ and $T_{e^{-is\theta}\psi}$ are bounded operators. If $(T_{e^{-is\theta}\psi}, T_{e^{ip\theta}\varphi}]$ has finite rank and $s-p \le n$, then its rank is at most equal to n. If the semi-commutator $(T_{e^{-is\theta}\psi}, T_{e^{ip\theta}\varphi}]$ has finite rank and $s-p > n$, then its rank is at most equal to $s-p$.*

Proof Let S denote the semi-commutator $(T_{e^{-is\theta}\psi}, T_{e^{ip\theta}\varphi}]$. From direct calculation by Lemma 2.4, we have

$$T_{e^{-i(s-p)\theta}\varphi\psi}(z^k) = \begin{cases} 2(k-s+p+1)\widehat{\varphi\psi}(2k-s+p+2)z^{k-s+p}, & k \ge s-p, \\ 2(s-p-k+1)\widehat{\varphi\psi}(s-p+2)\overline{z}^{s-p-k}, & 0 \le k < s-p. \end{cases}$$

If $s-p>n$, then

$$T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(z^k)=\begin{cases}2(k+p+1)2(k+p-s+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)z^{k-s+p}, & k\geq s-p,\\ 2(k+p+1)2(s-k-p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k-p}, & s-p-n\leq k<s-p,\\ 0, & 0\leq k<s-p-n.\end{cases}$$

If $s-p\leq n$, then

$$T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(z^k)=\begin{cases}2(k+p+1)2(k+p-s+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)z^{k-s+p}, & k\geq s-p,\\ 2(k+p+1)2(s-k-p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\overline{z}^{s-k-p}, & 0\leq k<s-p.\end{cases}$$

From the above results, we can obtain that

If $s-p>n$, then

$$S(z^k)=\begin{cases}2(k+p-s+1)[2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)\\ -\widehat{\varphi\psi}(2k-s+p+2)]z^{k-s+p}, & k\geq s-p,\\ 2(s-k-p+1)[2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\\ -\widehat{\varphi\psi}(s-p+2)]\overline{z}^{s-k-p}, & s-p-n\leq k<s-p,\\ -2(s-k-p+1)\widehat{\varphi\psi}(s-p+2)\overline{z}^{s-k-p}, & 0\leq k<s-p-n.\end{cases}$$

If $s-p\leq n$, then

$$S(z^k)=\begin{cases}2(k+p-s+1)[2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(2k+2p-s+2)\\ -\widehat{\varphi\psi}(2k-s+p+2)]z^{k-s+p}, & k\geq s-p,\\ 2(s-k-p+1)[2(k+p+1)\widehat{\varphi}(2k+p+2)\widehat{\psi}(s+2)\\ -\widehat{\varphi\psi}(s-p+2)]\overline{z}^{s-k-p}, & 0\leq k<s-p.\end{cases}$$

By [6, Theorem 5.4], we know that

$$S(z^k)=0,\quad k\geq s-p,$$

furthermore, we derive that

$$\{S(z^k):0\leq k<s-p\}\subseteq \operatorname{span}\{\overline{z}^l:0<l\leq s-p\}.$$

Next, we characterize $\{S(\overline{z}^k):1\leq k\leq n\}$.

For $T_{e^{-i(s-p)\theta}\varphi\psi}(\overline{z}^k)$, we have

If $s-p\geq n$, then

$$T_{e^{-i(s-p)\theta}\varphi\psi}(\overline{z}^k)=0,\quad 1\leq k\leq n.$$

If $s-p<n$, then

$$T_{e^{-i(s-p)\theta}\varphi\psi}(\overline{z}^k)=\begin{cases}2(k+s-p+1)\widehat{\varphi\psi}(2k+s-p+2)\overline{z}^{k+s-p}, & 1\leq k\leq n-(s-p),\\ 0, & n-(s-p)<k\leq n.\end{cases}$$

For $T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k)$, we obtain that

If $s\geq n$, $s-p\geq n$, then

$$T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k)=0,\quad 1\leq k\leq n.$$

If $s\geq n$, $s-p<n$, then

$$T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k)=\begin{cases}2(p-k+1)2(k+s-p+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{k+s-p}, & 1\leq k\leq n-(s-p),\\ 0, & n-(s-p)<k\leq n.\end{cases}$$

If $s<n$, $p<n$, then

$$T_{e^{-is\theta}\psi}T_{e^{ip\theta}\varphi}(\overline{z}^k)=\begin{cases}2(p-k+1)2(k+s-p+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\overline{z}^{k+s-p}, & 1\leq k\leq p,\\ 2(k-p+1)2(k-p+s+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+2)\overline{z}^{s-p+k}, & p<k\leq n-s+p,\\ 0, & n-s+p<k\leq n.\end{cases}$$

From above results, we have

If $s\ge n$, $s-p\ge n$, then

$$S(\overline{z}^k)=0,\quad 1\le k\le n.$$

If $s\ge n$, $s-p<n$, then

$$S(\overline{z}^k)=\begin{cases}2(k+s-p+1)[2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\\ -\widehat{\varphi\psi}(2k+s-p+2)]\overline{z}^{k+s-p}, & 1\le k\le n-(s-p),\\ 0, & n-(s-p)<k\le n.\end{cases}$$

If $s<n$, $p<n$, $s-p<n$, then

$$S(\overline{z}^k)=\begin{cases}2(k+s-p+1)[2(p-k+1)\widehat{\varphi}(p+2)\widehat{\psi}(s+2)\\ -\widehat{\varphi\psi}(2k+s-p+2)]\overline{z}^{k+s-p}, & 1\le k\le p,\\ 2(k+s-p+1)[2(k-p+1)\widehat{\varphi}(2k-p+2)\widehat{\psi}(2k-2p+s+2)\\ -\widehat{\varphi\psi}(2k+s-p+2)]\overline{z}^{k+s-p}, & p<k\le n-s+p,\\ 0, & n-s+p<k\le n.\end{cases}$$

Then we can get

If $s-p\ge n$, then

$$\{S(\overline{z}^k):1\le k\le n\}=\phi.$$

If $s-p<n$, then

$$\{S(\overline{z}^k):1\le k\le n\}\subseteq \operatorname{span}\{\overline{z}^l: s-p<l\le n\}.$$

Combining the above with

$$\{S(z^k):k\ge 0\}\subseteq\{S(z^k):0\le k<s-p\}\subseteq \operatorname{span}\{\overline{z}^l:0\le l<s-p\},$$

we have if $s-p\ge n$, then

$$\{S(z^k):k\ge 0\}\bigcup\{S(\overline{z}^k):1\le k\le n\}\subseteq \operatorname{span}\{\overline{z}^l:0\le l<s-p\},$$

which means that the rank of S is at most equal to $s-p$.

If $s-p<n$, then

$$\begin{aligned}&\{S(z^k):k\ge 0\}\bigcup\{S(\overline{z}^k):1\le k\le n\}\\ &\quad\subseteq \operatorname{span}\{\overline{z}^l:0\le l<s-p\}\bigcup \operatorname{span}\{\overline{z}^l:s-p<l\le n\}\\ &\quad=\operatorname{span}\{\overline{z}^l:0\le l\le n\},\end{aligned}$$

which means that the rank of S is at most equal to n.

6. Examples

In this section, we give some examples with respect to the above conclusions.

Example 6.1 We give an example about Theorem 4.1. From the proof of Theorem 4.1 we know that

$$T_{e^{ip\theta}r^m}T_{e^{is\theta}f}(z^k)=T_{e^{is\theta}f}T_{e^{ip\theta}r^m}(z^k),\quad k\ge 0.$$

Next we will construct a radial function f, such that

$$T_{e^{ip\theta}r^m}T_{e^{is\theta}f}(\overline{z}^k)=T_{e^{is\theta}f}T_{e^{ip\theta}r^m}(\overline{z}^k),\quad k\ge p+s,\tag{6.1}$$

where $p > 0$, $s > 0$ and $m \geq 0$.

Eq. (6.1) implies that for $k \geq p + s$,

$$\frac{k-p+1}{2k-p+m+2}\widehat{f}(2k-2p-s+2) = \frac{k-s+1}{2k-2s-p+m+2}\widehat{f}(2k-s+2).$$

Thus for $k \geq p + s$,

$$\frac{\widehat{r^{-s-2p}f}(2k+2+2p)}{\widehat{r^{-s-2p}f}(2k+2)} = \frac{(2k+2-2p)(2k+2-p+m-2s)}{(2k+2-2s))(2k+2-p+m)}.$$

Now, using Remark 2.3, we obtain that

$$\frac{\widehat{r^{-s-2p}f}(z+2p)}{\widehat{r^{-s-2p}f}(z)} = \frac{(z-2p)(z-p+m-2s)}{(z-2s))(z-p+m)}, \tag{6.2}$$

for all $\operatorname{Re} z \geq 2p + 2s + 2$.

Let F be the analytic function defined for $\operatorname{Re} z \geq 2p + 2s$ by

$$F(z) = \frac{\Gamma(\frac{z-2p}{2p})\Gamma(\frac{z-p-2s+m}{2p})}{\Gamma(\frac{z-2s}{2p})\Gamma(\frac{z-p+m}{2p})},$$

where Γ denotes the gamma function. Then using the well-known identity $\Gamma(z+1) = z\Gamma(z)$, (6.2) implies that

$$\frac{\widehat{r^{-s-2p}f}(z+2p)}{\widehat{r^{-s-2p}f}(z)} = \frac{F(z+2p)}{F(z)}, \quad \operatorname{Re} z > 2p + 2s. \tag{6.3}$$

Eq. (6.3) combined with [13, Lemma 6] gives us that there exists a constant c such that

$$\widehat{r^{-s-2p}f}(z) = cF(z), \quad \operatorname{Re} z > 2p + 2s. \tag{6.4}$$

For a choice of $p = 2$, $s = 1$ and $m = 6$, and again using the identity $\Gamma(z+1) = z\Gamma(z)$, one can see that

$$F(z) = \frac{4(z-2)}{z(z-4)} = 2[\frac{1}{z} + \frac{1}{z-4}].$$

Since $\widehat{1}(z) = \frac{1}{z}$, $\widehat{r^{-4}}(z) = \frac{1}{z-4}$, we have

$$\widehat{r^{-5}f}(z) = c[\widehat{1}(z) + \widehat{r^{-4}}(z)], \quad \operatorname{Re} z > 6.$$

Now the preceding equation and Remark 2.3 imply that $f(r) = c(r^5 + r)$, where c is a constant. It is clear that the function f is bounded, so Toeplitz operator $T_{e^{i\theta}f}$ is bounded.

By taking the constant c to be equal to 1, the radial function $f(r) = r^5 + r$ satisfies

$$T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(z^k) = T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(z^k), \quad \forall\, k \geq 0.$$

For $n = 2$, $p = 2$, $s = 1$, $p + s = 3 > n = 2$, we have

$$T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(\overline{z}) = \frac{9}{20}z^2, \; T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(\overline{z}) = \frac{16}{25}z^2,$$

$$T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(\overline{z}^2) = \frac{32}{75}z, \; T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(\overline{z}^2) = \frac{3}{10}z.$$

Therefore, the rank of $[T_{e^{2i\theta}r^6}, T_{e^{i\theta}(r^5+r)}]$ is equal to two on b_2^2. For $n = 4$, $p = 2$, $s = 1$, $p + s = 3 < n = 4$, we have

$$T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(\overline{z}) = \frac{9}{20}z^2,\ T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(\overline{z}) = \frac{16}{25}z^2,$$

$$T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(\overline{z}^2) = \frac{32}{75}z,\ T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(\overline{z}^2) = \frac{3}{10}z,$$

$$T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(\overline{z}^3) = \frac{16}{77},\ T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(\overline{z}^3) = \frac{16}{195},$$

$$T_{e^{2i\theta}r^6}T_{e^{i\theta}(r^5+r)}(\overline{z}^4) = \frac{16}{35}\overline{z} = T_{e^{i\theta}(r^5+r)}T_{e^{2i\theta}r^6}(\overline{z}^2),$$

so the rank of $[T_{e^{2i\theta}r^6}, T_{e^{i\theta}(r^5+r)}]$ is equal to three on b_4^2.

Example 6.2 We give an example of Theorem 4.2.

Similarly to Example 6.1, there exist $\varphi = \frac{63}{4}r^{-2} - \frac{35}{2} + \frac{15}{4}r^2$, $\psi = r^6$, $p = 1$, $s = 2$ such that

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(z^k) = T_{e^{-2i\theta}r^6}T_{e^{i\theta}\varphi}(z^k), \quad k \geq 2.$$

For $n = 2$, we have

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(1) = \frac{192}{35}\overline{z}, T_{e^{-2i\theta}r^6}T_{e^{i\theta}\varphi}(1) = \frac{256}{15}\overline{z},$$

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(z) = \frac{128}{15}, T_{e^{-2i\theta}r^6}T_{e^{i\theta}\varphi}(z) = \frac{96}{35},$$

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(\overline{z}) = 0, T_{e^{-2i\theta}r^6}T_{e^{i\theta}\varphi}(\overline{z}) = \frac{512}{15}\overline{z}^2,$$

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(\overline{z}^2) = 0 = T_{e^{-2i\theta}r^6}T_{e^{i\theta}\varphi}(\overline{z}^2),$$

then the rank of $[T_{e^{i\theta}\varphi}, T_{e^{-2i\theta}r^6}(z^k)]$ is equal to $3 = n + p$ on b_2^2.

For $n = 3$, we have

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(1) = \frac{192}{35}\overline{z}, T_{e^{-2i\theta}r^6}T_{e^{i\theta}\varphi}(1) = \frac{256}{15}\overline{z},$$

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(z) = \frac{128}{15}, T_{e^{-2i\theta}r^6}T_{e^{i\theta}\varphi}(z) = \frac{96}{35},$$

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(\overline{z}) = 0, T_{e^{-2i\theta}r^6}T_{e^{i\theta}\varphi}(\overline{z}) = \frac{512}{15}\overline{z}^2,$$

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(\overline{z}^2) = 0, T_{e^{-2i\theta}r^6}T_{e^{i\theta}\varphi}(\overline{z}^2) = \frac{16}{7}\overline{z}^3,$$

$$T_{e^{i\theta}\varphi}T_{e^{-2i\theta}r^6}(\overline{z}^3) = 0 = T_{e^{-2i\theta}r^6}T_{e^{i\theta}\varphi}(\overline{z}^3),$$

so the rank of $[T_{e^{i\theta}\varphi}, T_{e^{-2i\theta}r^6}(z^k)]$ is equal to $4 = n + p$ on b_3^2.

Example 6.3 We give an example about Theorem 5.1.

As we construct Example 6.1, let $\varphi = r$, $\psi = r^6$, $p = 1$, $s = 1$. By the proof the Theorem 5.1, we know that

$$T_{e^{i\theta}r}T_{e^{i\theta}r^6}(z^k) = T_{e^{2i\theta}r^7}(z^k), \quad k \geq 0.$$

For $n=1$, $p+s=2>n=1$, we have

$$T_{e^{i\theta}r}T_{e^{i\theta}r^6}(\overline{z})=\frac{2}{9}z, T_{e^{2i\theta}r^7}(\overline{z})=\frac{4}{11}z,$$

which means that the rank of $(T_{e^{i\theta}r}, T_{e^{i\theta}r^6}]$ is equal to $1=n$ on b_1^2.

For $n=3$, $p+s=2<n=3$, we know that

$$T_{e^{i\theta}r}T_{e^{i\theta}r^6}(\overline{z})=\frac{2}{9}z, T_{e^{2i\theta}r^7}(\overline{z})=\frac{4}{11}z,$$

$$T_{e^{i\theta}r}T_{e^{i\theta}r^6}(\overline{z}^2)=\frac{2}{11}=T_{e^{2i\theta}r^7}(\overline{z}^2),$$

$$T_{e^{i\theta}r}T_{e^{i\theta}r^6}(\overline{z}^3)=\frac{8}{13}\overline{z}=T_{e^{2i\theta}r^7}(\overline{z}^3),$$

so the rank of $(T_{e^{i\theta}r}, T_{e^{i\theta}r^6}]$ is equal to $1=p+s-1$.

Example 6.4 We give an example about Theorem 5.2. As we construct Example 6.1, let $\varphi=r^5$, $\psi=r^{12}$, $p=1$, $s=4$. By the proof of Theorem 5.2, we know that

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(z^k)=T_{e^{-3i\theta}r^{17}}(z^k), \quad k\geq 4.$$

For $n=2$, $s-p=3>n=2$, by direct calculation, we have

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(1)=0, T_{e^{-3i\theta}r^{17}}(1)=\frac{4}{11}\overline{z}^3,$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(z)=\frac{2}{9}\overline{z}^2, T_{e^{-3i\theta}r^{17}}(z)=\frac{3}{10}\overline{z}^2,$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(z^2)=\frac{2}{15}\overline{z}, T_{e^{-3i\theta}r^{17}}(z^2)=\frac{1}{5}\overline{z},$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(z^3)=\frac{1}{18}, T_{e^{-3i\theta}r^{17}}(z^3)=\frac{1}{11},$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(\overline{z})=0=T_{e^{-3i\theta}r^{17}}(\overline{z}),$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(\overline{z}^2)=0=T_{e^{-3i\theta}r^{17}}(\overline{z}^2),$$

so the rank of $(T_{e^{i\theta}r^5}, T_{e^{-4i\theta}r^{12}}]$ is equal to $4=s$ at b_2^2.

For $n=4$, $s-p=3<n=4$, we have

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(1)=0, T_{e^{-3i\theta}r^{17}}(1)=\frac{4}{11}\overline{z}^3,$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(z)=\frac{2}{9}\overline{z}^2, T_{e^{-3i\theta}r^{17}}(z)=\frac{3}{10}\overline{z}^2,$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(z^2)=\frac{2}{15}\overline{z}, T_{e^{-3i\theta}r^{17}}(z^2)=\frac{1}{5}\overline{z},$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(z^3)=\frac{1}{18}, T_{e^{-3i\theta}r^{17}}(z^3)=\frac{1}{11},$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(\overline{z})=0, T_{e^{-3i\theta}r^{17}}(\overline{z})=\frac{5}{12}\overline{z}^4,$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(\overline{z}^2)=0=T_{e^{-3i\theta}r^{17}}(\overline{z}^2),$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(\overline{z}^3)=0=T_{e^{-3i\theta}r^{17}}(\overline{z}^3),$$

$$T_{e^{i\theta}r^5}T_{e^{-4i\theta}r^{12}}(\overline{z}^4) = 0 = T_{e^{-3i\theta}r^{17}}(\overline{z}^4),$$

then the rank of $(T_{e^{i\theta}r^5}, T_{e^{-4i\theta}r^{12}}]$ is equal to $5 = p + n$ at b_4^2.

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