Journal of Mathematical Research with Applications
Mar., 2020, Vol. 40, No. 2, pp. 187–195
DOI:10.3770/j.issn:2095-2651.2020.02.007
Http://jmre.dlut.edu.cn

A Global Rigidity Theorem for the Length of Concircular Curvature Tensor on the Locally Conformally Symmetric Riemannian Manifold

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Abstract In this paper, we study the locally conformally symmetric closed Riemannian manifold, and establish a global rigidity theorem for the length of the concircular curvature vector.

Keywords conformally symmetric; Riemannian curvature tensor; concircular curvature vector; Weyl tensor; Schouten tensor

MR(2010) Subject Classification 53C20; 53A30

1. Introduction and the main results

Let M be an n-dimensional compact, oriented Riemannian manifold. If n (≥ 4), it is well-known that the Riemannian curvature tensor R_m of M can be decomposed into three mutual orthogonality parts:

$$R_m = W + V + U, \tag{1.1}$$

where W denotes the Weyl conformal curvature tensor (when $n = 3$, $W \equiv 0$), V and U denote the free trace part of the Ricci curvature tensor and the scalar curvature part, respectively. The curvature tensor W, V contain rich information about M. When $W = 0$ (resp., $V = 0$), the Riemannian manifold M is locally conformally flat manifold (resp., Einstein manifold).

Let $D = W + V$ be the concircular curvature vector, which plays an important role in conformal geometry [1–3]. One can check that M is a space form if and only if $D = 0$. Thus one can think of the concircular curvature tensor as a measure of the failure of a Riemannian manifold to be of constant curvature. Mutô [4] proved that if the concircular curvature is small enough, then an Einstein manifold must be a space form. For a manifold with positive scalar curvature, Huisken [5] proved the following theorem:

Theorem 1.1 *Let $n \geq 4$. Suppose M^n is a smooth compact n-dimensional manifold with the positive scalar curvature and satisfies the pinching condition*

$$\|D\|^2 = \|W\|^2 + \|V\|^2 \leq \delta_n (1-\epsilon)^2 \|U\|^2, \tag{1.2}$$

where $\epsilon > 0$, $\delta_4 = \frac{1}{5}$, $\delta_5 = \frac{1}{10}$ and

$$\delta_6 = \frac{2}{(n-1)(n+1)}, \quad n \geq 6.$$

Received December 31, 2018; Accepted September 4, 2019
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Then M^n is diffeomorphic to the sphere S^n or a quotient space of S^n by a group of fixed point free isometries in the standard metric.

In this article, we study the geometric property of the locally conformally symmetric Riemannian manifold, and obtain a global rigidity theorem for the length of concircular curvature tensor. When the tensor W satisfies $\triangledown W = 0$, the Riemannian manifolds are called locally conformally symmetric manifolds, which include many Riemannian manifolds also, for instance, locally conformally flat Riemannian manifolds and locally symmetric Riemannian manifolds. In fact, as shown by Witold Roter [6], such manifolds are necessarily locally conformally flat or locally symmetric.

By [7, Theorem 6] we know that every compact locally conformally symmetric Riemannian manifold is a manifold with constant scalar curvature. We can only consider the case that the manifold M has constant scalar curvature $R > 0$. Moreover, we can assume, by taking a appropriately similar transformation if necessary, that $R = n(n-1)$. From now on, let $\sigma = \|D\|$ be the length of the concircular curvature tensor. In this paper, we will prove the following results.

Theorem 1.2 *Suppose M^n $(n \geq 4)$ is a closed locally conformally symmetric Riemannian manifold with scalar curvature $R = n(n-1)$. If*

$$\sigma < \frac{n}{3+\sqrt{n-2}},$$

then $\sigma = 0$, and M is a space form.

Theorem 1.3 *Suppose M^n $(n \geq 4)$ is a closed locally symmetric conformal Riemannian manifold with scalar curvature $R = n(n-1)$. Then there exists a constant $\epsilon = \epsilon(n) > 0$ depending only on n, which satisfies the following condition: if*

$$\frac{\|\sigma\|_{\frac{n}{2}}}{\min(c_1^{\frac{n}{2}}, \mathrm{vol}(M)^{\frac{n}{2}})} < \epsilon,$$

then $\sigma = 0$, and M is a space form with constant section curvature 1, where c_1 is the isoperimetric constant of M, and $\mathrm{vol}(M)$ is the volume of M.

Remark 1.4 When M^n is a locally conformally flat Riemannian manifold, then Theorems 1.3 and 1.4 hold for $n \geq 3$.

2. Preliminaries

A Riemannian manifold is closed if it is compact, without boundary. Let (M^n, g) be a closed Riemannian manifold of dimension n $(n \geq 3)$. $\{e_1, \ldots, e_n\}$ are local orthonormal frames and $\omega_1, \ldots, \omega_n$ are the dual frames, $\{\omega_{ij}\}$ are connection 1-forms. We have the following structure equations:

$$d\omega_i = -\sum_j \omega_{ij} \Lambda \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{2.1}$$

$$d\omega_{ij} = -\sum_{k} \omega_{ik}\Lambda\omega_{kj} - \frac{1}{2}\sum_{k,l} R_{ijkl}\omega_k\Lambda\omega_l. \tag{2.2}$$

Here R_{ijkl} are the components of Riemannian curvature tensor of M^n. Unless otherwise stated, we assume that the indexes range from 1 to n.

Ricci curvature and scalar curvature are defined by $R_{ij} = \sum_l R_{iljl}$ and $R = \sum_l R_{ll}$, respectively. It is well-known that

$$R_{ijkl} = -R_{jikl} = -R_{ijlk} = R_{klij}, \tag{2.3}$$

$$R_{ijkl} + R_{iljk} + R_{iklj} = 0, \tag{2.4}$$

$$R_{ijkl,m} + R_{ijmk,l} + R_{ijlm,k} = 0. \tag{2.5}$$

Under the local orthonormal frames, the components of tensor U, V in (1.1) are

$$U_{ijkl} = \frac{R}{n(n-1)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \tag{2.6}$$

$$V_{ijkl} = \frac{1}{n-2}(\delta_{ik}\hat{R}_{jl} + \delta_{jl}\hat{R}_{ik} - \delta_{il}\hat{R}_{jk} - \delta_{jk}\hat{R}_{il}), \tag{2.7}$$

where

$$\hat{R}_{ij} = R_{ij} - \frac{1}{n}R\delta_{ij}.$$

Then the components of the Weyl curvature tensor W and the concircular curvature tensor D are given as follows:

$$\begin{aligned} W_{ijkl} &= R_{ijkl} - V_{ijkl} - U_{ijkl} \\ &= R_{ijkl} - \frac{1}{n-2}(\delta_{ik}R_{jl} + \delta_{jl}R_{ik} - \delta_{il}R_{jk} - \delta_{jk}R_{il}) + \frac{R}{(n-1)(n-2)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), \end{aligned} \tag{2.8}$$

$$D_{ijkl} = R_{ijkl} - U_{ijkl} = R_{ijkl} - \frac{R}{n(n-1)}(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{2.9}$$

When M is an Einstein manifold, the concircular curvature tensor D is just the Weyl conformal curvature tensor W.

Now we define the Schouten tensor by $S = S_{ij}\omega_i \otimes \omega_j$, where

$$S_{ij} = R_{ij} - \frac{R}{2(n-1)}\delta_{ij}. \tag{2.10}$$

It is clear that $S_{ij} = S_{ji}$, and (2.8) can be rewritten

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2}(\delta_{ik}S_{jl} + \delta_{jl}S_{ik} - \delta_{il}S_{jk} - \delta_{jk}S_{il}). \tag{2.11}$$

Using the Schouten tensor, we can define the Cotten tensor by

$$B_{ijk} = S_{ik,j} - S_{ij,k}. \tag{2.12}$$

Using (2.3)–(2.5), we get

$$R_{ij,k} - R_{ik,j} = -\sum_{l} R_{lijk,l}, \tag{2.13}$$

then

$$\sum_j R_{ij,j} = \frac{1}{2} R_{,i}. \tag{2.14}$$

From (2.10) and (2.14), we get

$$\sum_j S_{ij,j} = \frac{n-2}{2(n-1)} R_{,i}. \tag{2.15}$$

From (2.10), (2.12) and (2.14), we have

$$S_{ij,k} - S_{ik,j} = -\sum_l R_{lijk,l} - \frac{1}{n-2}(\delta_{ij} \sum_l S_{kl,l} - \delta_{ik} \sum_l S_{jl,l}). \tag{2.16}$$

Substituting (2.11) into (2.16),

$$S_{ij,k} - S_{ik,j} = -\sum_l W_{lijk,l} - \frac{1}{n-2}(S_{ik,j} - S_{ij,k}). \tag{2.17}$$

If $n > 3$, then

$$B_{ikj} = S_{ij,k} - S_{ik,j} = -\frac{n-3}{n-2} \sum_l W_{lijk,l}. \tag{2.18}$$

The Weyl conformal curvature tensor W is harmonic if $\sum_l W_{lijk,l} = 0$, i.e., $B_{ikj} = 0$. We say the manifold M is a conformally symmetric Riemannian manifold if $\nabla W = 0$. Moreover, if $W = 0$, then M is a conformally flat manifold. It is well-known that M^3 is a conformally flat manifold if and only if $B_{ijk} = 0$ (see [8]), and the Schouten tensor must be the Codazzi tensor. From above argument, we obtain:

Proposition 2.1 *Let (M^n, g) be an $n(n \geq 3)$-dimensional Riemannian manifold. Then the Schouten tensor is the Codazzi tensor if and only if the Cotten tensor vanishs. When M^n $(n \geq 4)$ is a conformally symmetric Riemannian manifold, then $B_{ijk} \equiv 0$, and the Schouten tensor is always the Codazzi tensor, i.e., $S_{ik,j} = S_{ij,k}$.*

3. The proof for a global rigidity theorem

Since $R = n(n-1)$, (2.9) is rewritten

$$D_{ijkl} = R_{ijkl} - (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}). \tag{3.1}$$

One can derive the following identities

$$D_{ijkl} = -D_{jikl} = -D_{ijlk} = D_{klij}, \tag{3.2}$$

$$D_{ijkl} + D_{iljk} + D_{iklj} = 0, \tag{3.3}$$

$$D_{ijkl,m} + D_{ijmk,l} + D_{ijlm,k} = 0, \tag{3.4}$$

$$\sum_i D_{ijil} = R_{jl} - (n-1)\delta_{jl}, \tag{3.5}$$

$$\|D\|^2 = \sum_{i,j,k,l} D_{ijkl}^2 = \sum_{i,j,k,l} R_{ijkl}^2 - 2n(n-1). \tag{3.6}$$

Substituting (3.5) into (2.10), we get

$$S_{ij} = \sum_l D_{iljl} - (\frac{R}{2(n-1)} + n - 1)\delta_{ij}. \tag{3.7}$$

Using $R = n(n-1)$ again, we obtain

$$S_{ij,k} = \Big(\sum_l D_{iljl}\Big)_{,k} = \sum_l D_{iljl,k}. \tag{3.8}$$

Now we compute $\Delta\|D\|^2$. From (3.4), we have $D_{ijkl,mm} = D_{ijkm,lm} + D_{ijml,km}$. Applying the Ricci identity, we have

$$\begin{aligned}\frac{1}{2}\Delta\|D\|^2 =& \sum_{i,j,k,l,m} D^2_{ijkl,m} + \sum_{i,j,k,l,m} D_{ijkl}D_{ijkl,mm}\\ =&\|\nabla D\|^2 + 2\sum_{i,j,k,l,m} D_{ijkl}D_{ijkm,lm}\\ =&\|\nabla D\|^2 + 2\sum_{i,j,k,l,h,m} D_{ijkl}(D_{ijkm,ml} + D_{hjkm}R_{hilm}+\\ & D_{ihkm}R_{hjlm} + D_{ijhm}R_{hklm} + D_{ijkh}R_{hl}).\end{aligned} \tag{3.9}$$

From (3.2) and (3.4), we get

$$\begin{aligned}\sum_m D_{ijkm,ml} &= (\sum_m D_{ijkm,m})_{,l} = (\sum_m D_{kmij,m})_{,l} = -(\sum_m D_{kmjm,i} + \sum_m D_{kmmi,j})_{,l}\\ &= -(\sum_m D_{kmjm,i} - \sum_m D_{kmim,j})_{,l} = -(S_{kj,i} - S_{ki,j})_{,l} = 0,\end{aligned} \tag{3.10}$$

where we have used Proposition 2.1 and (3.8). Substituting $R_{ijkl} = D_{ijkl} + (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}), R_{jl} = \sum_i D_{ijil} + (n-1)\delta_{jl}$ and (3.10) into (3.9), we have

$$\begin{aligned}\frac{1}{4}\Delta\|D\|^2 =&\frac{1}{2}\|\nabla D\|^2 + \sum_{i,j,k,l,h,m} D_{ijkl}(D_{ihkm}D_{hjlm} - D_{jhkm}D_{hilm})+\\ &\sum_{i,j,k,l,h,m} D_{ijkl}D_{ijhm}D_{hklm} + 2\sum_{i,j,k,l} D_{ijkl}D_{ilkj} - \sum_{i,j,k,l} D^2_{ijkl}-\\ &2\sum_{j,k,l}(\sum_l D_{ljlk})^2 + \sum_{i,j,k,l,h,m} D_{ijkh}D_{ijkl}D_{mhml} + (n-1)\sum_{i,j,k,l} D^2_{ijkl}.\end{aligned} \tag{3.11}$$

From (3.3), we get

$$\sum_{i,j,k,l} D_{ijkl}D_{ilkj} = \sum_{i,j,k,l} D_{ijkl}(-D_{ikjl} - D_{ijlk}) = \sum_{i,j,k,l} D^2_{ijkl} - \sum_{i,j,k,l} D_{ijkl}D_{ilkj}.$$

Hence we have

$$2\sum_{i,j,k,l} D_{ijkl}D_{ilkj} = \sum_{i,j,k,l} D^2_{ijkl}. \tag{3.12}$$

Likewise

$$\sum_{i,j,k,l,h,m} D_{ijkl}D_{ijhm}D_{hklm} = -\frac{1}{2}\sum_{i,j,k,l,h,m} D_{ijkl}D_{klhm}D_{hmij}. \tag{3.13}$$

Substituting (3.12) and (3.13) into (3.11), we obtain

$$\frac{1}{4}\Delta\|D\|^2 =\frac{1}{2}\|\nabla D\|^2 + \sum_{i,j,k,l,h,m} D_{ijkl}(D_{ihkm}D_{hjlm} - D_{jhkm}D_{hilm}) - 2\sum_{jkl}(\sum_{l} D_{ljlk})^2 + \sum_{i,j,k,l,h,m} D_{ijkh}D_{ijkl}D_{mhml} - \frac{1}{2}\sum_{i,j,k,l,h,m} D_{ijkl}D_{klhm}D_{hmij} + (n-1)\|D\|^2. \tag{3.14}$$

Lemma 3.1 ([9]) *Let A, B be anti-symmetric matrixes. Then*

$$\|[A,B]\| \le \sqrt{2}\|A\|\|B\|,$$

where $[A,B] = AB - BA, \|A\|^2 = (A,A)$, and $(A,B) = -\frac{1}{2}\mathrm{tr}(AB)$.

Lemma 3.2 ([9]) *Let $\sigma = (s_{ij})$ be a symmetric $n \times n$ matrix with $s_{ij} \ge 0$ for any i, j and $s_{ii} = 0$ for any i. If $\mathrm{trace}(\sigma^2) = n(n-1)$, then*

$$\mathrm{tr}(\sigma^3) \le n(n-1)(n-2).$$

The equality holds if and only if $s_{ij} = 1 - \delta_{ij}$.

Lemma 3.3 ([10]) *Let S_{ijkl} be a $(0,4)$-tensor, $n > 2$, and S_{ijkl} satisfies the following identities*

$$S_{ijkl} = -S_{jikl} = -S_{ijlk} = S_{klij}.$$

Let $S_{ji} = \sum_i S_{ijil}, S = \sum_i S_{ii}$. Then

$$\sum_{i,j,k,l} S^2_{ijkl} \ge \frac{4}{n-2}\sum_{i,j} S^2_{ij} - \frac{2S^2}{(n-1)(n-2)}.$$

Denote by $\hat{D}_{ij} = (D_{lmij})$, then $\hat{D}_{ij}$ is an anti-symmetric matrix for every given pair (i, j). Now we have

$$\sum_{i,j,k,l,h,m} D_{ijkl}(D_{ihkm}D_{hjlm} - D_{jhkm}D_{hilm}) = 2\sum_{k,l,m}(\hat{D}_{kl}, [\hat{D}_{km}, \hat{D}_{lm}]).$$

Using Lemma 3.1, we have

$$2|\sum_{k,l,m}(\hat{D}_{kl}, [\hat{D}_{km}, \hat{D}_{lm}])| \le 2\sum_{k,l,m}|(\hat{D}_{kl}, [\hat{D}_{km}, \hat{D}_{lm}])| \le 2\sum_{k,l,m}\|\hat{D}_{kl}\| \cdot \|[\hat{D}_{km}, \hat{D}_{lm}]\| \le 2\sqrt{2}\sum_{k,l,m}\|\hat{D}_{kl}\| \cdot \|\hat{D}_{km}\| \cdot \|\hat{D}_{lm}\| = \sum_{k,l,m} s_{km}s_{ml}s_{lk},$$

where $s_{ij} = \sqrt{2}\|\hat{D}_{ij}\|$. Denote by $\sigma = (s_{ij}), t = \frac{\sqrt{n(n-1)}}{\|D\|}$, then

$$\mathrm{tr}((t\sigma)^2) = t^2\mathrm{Tr}(\sigma^2) = 2t^2\sum_{k,l}\|\hat{D}_{kl}\|^2 = t^2\sum_{i,j,k,l} D^2_{ijkl} = t^2\|D\|^2 = n(n-1).$$

Applying Lemma 3.2, we have $\mathrm{Tr}((t\sigma)^3) \le n(n-1)(n-2)$. Thus, we get

$$\sum_{i,j,k,l,h,m} D_{ijkl}(D_{ihkm}D_{hjlm} - D_{jhkm}D_{hilm}) \le 2|\sum_{k,l,m}(\hat{D}_{kl}, [\hat{D}_{km}, \hat{D}_{lm}])|$$

$$\le \frac{n-2}{\sqrt{n(n-1)}}\|D\|^3. \tag{3.15}$$

Since $\sum_{i,j} D_{ijij} = 0$, we can use Lemma 3.3 and get

$$\sum_{j,k,l}(\sum_{l} D_{ljlk})^2 \le \frac{n-2}{4}\sum_{i,j,k,l} D_{ijkl}^2 = \frac{n-2}{4}\|D\|^2, \tag{3.16}$$

$$\Big|\sum_{i,j,k,l,h,m} D_{ijkh}D_{ijkl}D_{mhml}\Big| \le \sum_{h,l}\Big(\sum_{i,j,k} D_{ijkl}^2\Big)^{\frac{1}{2}}\Big(\sum_{i,j,k} D_{ijkh}^2\Big)^{\frac{1}{2}}\sum_{m} D_{mhml}$$

$$\le \sqrt{\frac{n-2}{4}}\|D\|^3. \tag{3.17}$$

Meanwhile, we use the Cauchy inequality and get

$$\Big|\sum_{i,j,k,l,h,m} D_{ijkl}D_{klhm}D_{hmij}\Big| \le \|D\|^3. \tag{3.18}$$

Substituting (3.15),(3.16),(3.17) and (3.18) into (3.14), we have

$$\frac{1}{4}\Delta\|D\|^2 \ge \frac{1}{2}\|\nabla D\|^2 + \frac{n}{2}\|D\|^2 - (\frac{\sqrt{n-2}}{2} + \frac{1}{2} + \frac{n-2}{\sqrt{n(n-1)}})\|D\|^3$$

$$\ge \frac{n}{2}\|D\|^2 - \frac{1}{2}(\sqrt{n-2}+3)\|D\|^3. \tag{3.19}$$

Integrating over M, we have

$$\int_M (\frac{n}{2} - \frac{1}{2}(\sqrt{n-2}+3)\sigma)\sigma^2 * 1 \le 0,$$

where $\sigma = \|D\|$.

When $\sigma < \frac{n}{3+\sqrt{n-2}}$, it is easy to obtain $\sigma = \|D\| = 0$, M is a space form. This finishes the proof of Theorem 1.2. □

When the scalar curvature $R = n(n-1)$, from (3.6) we have the following corollary.

Corollary 3.4 *Assume that M^n ($n \ge 4$) is a closed locally conformally symmetric Riemannian manifold, and the scalar curvature $R = n(n-1)$. If*

$$2n(n-1) \le \sum R_{ijkl}^2 \le 2n(n-1) + (\frac{n}{3+\sqrt{n-2}})^2,$$

then $\sum R_{ijkl}^2 = 2n(n-1)$ holds, and M is a space form.

Lemma 3.5 ([11]) *Let M^n ($n \ge 3$) be a closed Riemannian manifold. If $f \in H_{1,2}(M)$, then*

$$\|f\|_{\frac{2n}{n-2}} \le k_1\Big(\int|\nabla f|^2\Big)^{\frac{1}{2}} + k_2\|f\|_2,$$

where

$$k_1 = \frac{n-1}{n-2}\cdot 2^{\frac{3n+2}{2n}}\cdot C_1^{-\frac{1}{n}},$$

$$k_2 = 2^{\frac{E(n)}{2}+\frac{n+4}{2n}},$$

$$E(3) = 1, E(n) = \frac{(n-4)(n-2)}{2}, \quad \forall n > 3,$$

$$C_1(M) = \inf_{S\subset M} \frac{(\mathrm{Area}\, S)^n}{[\min(\mathrm{vol}(M_1), \mathrm{vol}(M_2))]^{n-1}},$$

where hypersurface S divides M into two parts: $M = M_1 \bigcup M_2$.

Remark 3.6 There are some typos about this lemma in [11]. For more details we refer reader to [11].

Denote by

$$\frac{1}{2}C_0(n) = \frac{\sqrt{n-2}}{2} + \frac{1}{2} + \frac{n-2}{\sqrt{n(n-1)}}.$$

Applying the Cauchy inequality, we get

$$\|\nabla D\|^2 = \sum_{i,j,k,l,m} D^2_{ijkl,m} \geq \|\nabla\sigma\|^2.$$

Then from (3.19), we have

$$\triangle\, \sigma + C_0(n)\sigma^2 - n\sigma \geq 0. \tag{3.20}$$

Hence, we get

$$\triangle\, \sigma + C_0(n)\sigma^2 \geq n\sigma \geq 0. \tag{3.21}$$

Multiplying (3.21) by σ^α ($\alpha \geq 1$), and applying integration by parts, we get

$$C_0(n) \int \sigma^{\alpha+2} \geq \frac{4\alpha}{(\alpha+1)^2} \int |\nabla \sigma^{\frac{\alpha+1}{2}}|^2 \geq \frac{1}{\alpha} \int |\nabla \sigma^{-\frac{\alpha+1}{2}}|^2. \tag{3.22}$$

Choosing $f = \sigma^{\frac{\alpha+1}{2}}$, and applying Lemma 3.5, we obtain

$$\begin{aligned} \|\sigma^{\frac{\alpha+1}{2}}\|_{\frac{2n}{n-2}} &\leq k_1\Big(\int |\nabla f|^2\Big)^{\frac{1}{2}} + k_2\|f\|_2 \\ &\leq k_1 C_0^{\frac{1}{2}}(n)\alpha^{\frac{1}{2}}\Big(\int \sigma^{\alpha+2}\Big)^{\frac{1}{2}} + k_2\|\sigma^{\frac{\alpha+1}{2}}\|_2 \\ &\leq C_2(n)[C_1^{-\frac{1}{n}}(n)\alpha^{\frac{1}{2}}\|\sigma\cdot\sigma^{\alpha+1}\|_1^{\frac{1}{2}} + \mathrm{vol}(M)^{-\frac{1}{n}}\|\sigma^{\frac{\alpha+1}{2}}\|_2]. \end{aligned} \tag{3.23}$$

Now Choosing $\alpha + 1 = \frac{n}{2}$, and applying the Hölder inequality to $\|\sigma\cdot\sigma^{\frac{n}{2}}\|_1$, we get

$$\|\sigma^{\frac{n}{4}}\|_{\frac{2n}{n-2}} \leq C_3(n)[C_1^{-\frac{1}{n}}(n)\ \|\sigma\|^{\frac{1}{2}}_{\frac{n}{2}}\ \|\sigma^{\frac{n}{4}}\|_{\frac{2n}{n-2}} + \mathrm{vol}(M)^{-\frac{1}{n}}\|\sigma\|^{\frac{n}{4}}_{\frac{n}{2}}]. \tag{3.24}$$

Now we obtain a constant $\epsilon'(n) > 0$ which only depends on n. And for any $\epsilon' < \epsilon'(n)$, if

$$\|\sigma\|_{\frac{n}{2}} \leq C_1^{\frac{2}{n}}\epsilon', \tag{3.25}$$

then

$$\begin{aligned} \|\sigma\|_{\frac{q}{2}} = \|\sigma^{\frac{n}{4}}\|^{\frac{4}{n}}_{\frac{2n}{n-2}} &\leq C_4(n)\mathrm{vol}(M)^{-\frac{4}{n^2}}\|\sigma\|_{\frac{n}{2}} = C_4(n)\mathrm{vol}(M)^{-\frac{4}{n^2}}C_1^{\frac{2}{n}}\epsilon' \\ &\leq C_5(n)\mathrm{vol}(M)^{-\frac{4}{n^2}}C_1^{\frac{2}{n}}, \end{aligned} \tag{3.26}$$

where $q = \frac{n}{2}\frac{2n}{n-2}$.

For any $\alpha \geq 1$, using the Hölder inequality, interpolation inequality and (3.26), we have

$$\|\sigma\cdot\sigma^{\alpha+1}\|_1 \leq \|\sigma\|_{\frac{q}{2}}\ \|\sigma^{\frac{\alpha+1}{2}}\|^2_{\frac{2q}{q-2}}$$

$$\leq C_5(n)\mathrm{vol}(M)^{-\frac{4}{n^2}}C_1^{\frac{2}{n}}(\theta\|\sigma^{\frac{\alpha+1}{2}}\|_{\frac{2n}{n-2}}+\theta^{-\frac{n-2}{2}}\|\sigma^{\frac{\alpha+1}{2}}\|_2). \quad (3.27)$$

Substituting (3.27) into (3.23) gives

$$\begin{aligned}\|\sigma^{\frac{\alpha+1}{2}}\|_{\frac{2n}{n-2}} \leq & C_6(n)[\alpha^{\frac{1}{2}}\mathrm{vol}(M)^{-\frac{2}{n^2}}\theta\|\sigma^{\frac{\alpha+1}{2}}\|_{\frac{2n}{n-2}}- \\ & \alpha^{\frac{1}{2}}\mathrm{vol}(M)^{-\frac{2}{n^2}}\theta^{-\frac{n-2}{2}}+\mathrm{vol}(M)^{-\frac{1}{n}}\|\sigma^{\frac{\alpha+1}{2}}\|_2].\end{aligned} \quad (3.28)$$

Choosing $\theta=\frac{1}{2}C_6(n)^{-1}\alpha^{-\frac{1}{2}}\mathrm{vol}(M)^{\frac{2}{n^2}}$, then we have from (3.28)

$$\|\sigma^{\frac{\alpha+1}{2}}\|_{\frac{2n}{n-2}} \leq C_7(n)\alpha^{\frac{n}{4}}\mathrm{vol}(M)^{-\frac{1}{n}}\|\sigma^{\frac{\alpha+1}{2}}\|_2. \quad (3.29)$$

Writing $\chi=\frac{n}{n-2}$, and choosing $\alpha+1=\frac{n}{2}\chi^i, i\geq 0$, we have

$$\begin{aligned}\|\sigma\|_{\frac{n}{2}\chi^{i+1}} &\leq C_8(n)^{\frac{1}{\chi^i}}\chi^{\frac{1}{\chi^i}}\mathrm{vol}(M)^{-\frac{4}{n^2}\frac{1}{\chi^i}}\|\sigma\|_{\frac{n}{2}\chi^i} \\ &\leq C_8(n)^{\frac{1}{\chi^i}+\cdots+\frac{1}{\chi^0}}\chi^{\frac{1}{\chi^i}+\cdots+\frac{1}{\chi^0}}\mathrm{vol}(M)^{-\frac{4}{n^2}(\frac{1}{\chi^i}+\cdots+\frac{1}{\chi^0})}\|\sigma\|_{\frac{n}{2}}.\end{aligned}$$

Let $i\to\infty$. We get $\sigma\leq C_9(n)\mathrm{vol}(M)^{-\frac{2}{n}}\|\sigma\|_{\frac{n}{2}}$. If $\sigma<\frac{n}{3+\sqrt{n-2}}$, we obtain $\sigma=0$ from Theorem 1.2, and M is a space form. Therefore, when

$$C_9(n)\mathrm{vol}(M)^{-\frac{2}{n}}\|\sigma\|_{\frac{n}{2}} \leq \frac{n}{3+\sqrt{n-2}}$$

that is, $\|\sigma\|_{\frac{n}{2}} < C_{10}(n)\mathrm{vol}(M)^{\frac{2}{n}}=\epsilon''(n)\mathrm{vol}(M)^{\frac{2}{n}}$, we have $\sigma=0$.

Combining condition (3.25), i.e., $\|\sigma\|_{\frac{n}{2}}\leq C_1^{\frac{2}{n}}\epsilon'$, we may take $\epsilon=\min(\epsilon'(n),\epsilon''(n))$. When $\|\sigma\|_{\frac{n}{2}}\leq\epsilon\min(C_1^{\frac{2}{n}},\mathrm{vol}(M)^{\frac{2}{n}})$, it is easy to see that $\sigma=0$. The proof of Theorem 1.3 is completed. □

Acknowledgements We thank the anonymous referees for their valuable comments.

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